CHAPTER 7

ON A DISCRETE-TIME GI/GEO/1/N QUEUE WITH STARTING FAILURES AND IMPATIENT CUSTOMERS

7.1 INTRODUCTION

This chapter studies a discrete time finite buffer queue with the server, subject to starting failures and impatient customers with discrete time renewal process arrivals. The problem of queueing systems with server failures is of continuing interest to many researchers in the literature with very few who are concerned about discrete-time repairable queueing systems. Lee et al (2011), analysed the discrete time single server queues with failure due to a disaster arrival. Tang et al (2008), considers a discrete-time batch arrival queue with unreliable server and multiple adaptive delayed vacations policy in which the vacation time, service time, repair time and the delayed time all follow arbitrary discrete distributions. Samanta et al (2007b) analysed a single server finite capacity discrete-time general input queueing system with multiple vacations. In retrial queues, starting failures have been studied by few authors; Atencia & Moreno (2006), Wang & Zhao (2007a), (2007b), Atencia et al (2009), but only little attention has been given in classical queues with starting failures and impatient customers.

The present work is the extension of Chaudhry & Gupta (1996), with the assumption that a single server finite capacity queueing system in which the server is subject to starting failures and the arrivals occur according to a discrete time renewal input process with impatient customers. It should
be noted that the memory less property of the arrival process does not always meet the need of real time applications and also it can include the special cases of geometrical, deterministic distributions, etc. Because of these reasons the general uncorrelated arrival process appears to be more appropriate and reasonable.

This chapter considers a discrete time single server finite capacity queueing system with the server subject to starting failures and impatient customers. In the proposed model after the completion of service, if the server becomes idle it may meet the unpredictable breakdown, subject to starting failure when a customer requires a service. After breakdown, the server is immediately sent to repair station and the customer who loses patience may leave the system without getting service, while those who have patience may wait for service in the finite waiting room. The graph showing the sample path of the proposed queueing model is depicted in Figure 7.1.

\[ n : \text{number of customers in the buffer} \]

**Figure 7.1 Schematic representation of the queueing model**
7.2 MOTIVATION

Motivation of the existing model comes from the situation in Optical Burst Switching (OBS) networks. In communication networks, various types of jobs like voice packets or data files have to wait in a queue if they do not get service immediately upon their arrival in the network. The jobs can become impatient due to high waiting times or due to uncertainty of receiving services and may leave the system unserved. This scenario often occurs in Optical Burst Switching (OBS) networks. The operation of an OBS controller can be seen as a queue with reneging or impatient as studied in Bocquet (2005). In OBS networks, a control packet is sent first, on a separate signalling channel, to set up a connection. It is followed by a data burst without waiting for an acknowledgment for path establishment, Qiao & Yoo (1999). In particular, when a path is not assigned, the burst control packet is accepted to the queue and is kept waiting for a path. If its delay budget is lower than the effective processing delay, the packet becomes impatient and leaves the system unserved or unable to transmit the data. This situation is modelled as discrete time GI/Geo/1/N queue with starting failures and impatient customers.

7.3 THE MATHEMATICAL MODEL

The mathematical model for this work considers a discrete time GI/Geo/1 queue with finite capacity with the server subject to starting failures and impatient customers. Arriving customers are stored in a finite waiting room in FCFS basis. The present work is discussed using Late Arrival System with delayed access (LAS).

Following are the assumptions made to derive the model
7.3.1 Arrival Time Distribution of Customers

The inter arrival times \( \{A_n, n \geq 1\} \) of customers are independent and identically distributed random variables with probability mass function

\[ a_i = P(A_n = i), i \geq 1 \]

corresponding to probability generating function

\[ A(z) = \sum_{i=1}^{\infty} a_i z^i \]

and mean inter arrival time \( \theta_a = \frac{1}{\lambda} = A'(1) \). Where \( A'(1) \) is the first derivative of \( A(z) \) with respect to \( z \) at \( z = 1 \).

7.3.2 Impatient Time Distribution of Customers

If upon arrival, the server is idle and activated successfully (with a probability \( \gamma \)), the customer begins his service immediately and leaves the system forever after completion of service; otherwise, if the server is started unsuccessfully (with a complementary probability \( \overline{\gamma} \)), the server is sent to repair station immediately and the customer who has patience must join the queue with probability \( \beta \) and impatient customers leaves the system with the component probability \( \overline{\beta} \). Customer’s patience time is also independent and geometrically distributed with probability mass function \( P(I_n = i) = \overline{\beta}^{i-1} \beta, i \geq 1 \) with mean patient time \( \frac{1}{\beta}, 0 < \beta < 1 \).

7.3.3 Service Time Distribution of Servers

Arriving customers are queued in a single finite waiting room and are served in the order of their arrivals. The server can serve only one customer at a time and the service times \( \{S_n, n \geq 1\} \) are independent and geometrically distributed with probability mass function

\[ P(S_n = i) = \overline{\mu}^{i-1} \mu, i \geq 1 \]

with mean service time \( \frac{1}{\mu}, 0 < \mu < 1 \).
7.3.4 Repair Time Distribution of Servers

Repair times \(\{R_n, n \geq 1\}\) of the server are independent and geometrically distributed with probability mass function

\[P(R_n = i) = \alpha^{i-1} \theta, \quad i \geq 1\]

with mean repair time \(1/\alpha, 0 < \alpha < 1\).

The state of the server at time \(m^-\) is described as

\[J(m^-) = \begin{cases} 
0, & \text{if the server is down} \\
1, & \text{if the server is idle or busy}
\end{cases}\]

The joint probabilities are defined as follows

\[P_{n,0}(u, m^-) = P\left(N_q(m^-) = n, U(m^-) = u, J(m^-) = 0\right), \quad u \geq 0, 1 \leq n \leq N\]

\[P_{n,1}(u, m^-) = P\left(N_q(m^-) = n, U(m^-) = u, J(m^-) = 1\right), \quad u \geq 0, 0 \leq n \leq N\]

where

\[N_q(m^-) - \text{number of customers in the queue at } m^-;\]

\[U(m^-) - \text{remaining inter-arrival time for the next arrival.}\]

In steady state, the above probabilities can be defined as

\[P_{n,j}(u) = \lim_{m^- \to \infty} P_{n,j}(u, m^-), \quad j = 0, 1\]
7.4 QUEUE LENGTH DISTRIBUTION AT PRE-ARRIVAL AND ARBITRARY EPOCHS

This section gives the queue length distributions at arbitrary and pre-arrival epochs. Relationships between queue length distributions at arbitrary and pre-arrival epochs are also obtained.

7.4.1 Queue Length Distribution at Pre-arrival Epoch

This section is devoted to obtain the queue length distribution at a pre-arrival epoch by using the imbedded Markov chain technique. To achieve this, the following are defined as follows:

1. Let the probability that an inter arrival period is smaller than a down period and is denoted by \( \psi \). Then,

\[
\psi = \sum_{m=1}^{\infty} a_m \bar{\alpha}^m, \text{ where } a_m = P(A_n = m)
\]

2. Let the probability that ‘j’ customers complete service during an inter arrival period given that the server is activated successfully and is busy during this entire period is denoted by \( b_j \). Thus if the server is busy with ‘m’ slots during the inter arrival period there are ‘m’ possible service positions available. Thus, we obtain

\[
b_j = \sum_{m=\max(j,1)}^{\infty} a_m \binom{m}{j} \mu^j \bar{\mu}^{m-j}, j \geq 0
\]

3. Let the probability that ‘j’ customers complete service during an inter arrival period assuming that a server’s down period has elapsed and is denoted by \( d_j \). In LAS–DA, if an inter arrival period consisting of ‘m’ slots and a down period of length \( i + 1 \) (\( 0 \leq i \leq m - j - 1 \)) slots has elapsed then during this inter arrival
period there are \( m - i - 1 \) possible service positions available. Hence, we get
\[
d_j = \sum_{m=j+1}^{\infty} a_m \sum_{i=0}^{m-j-1} \alpha \beta^i \left( m - i - 1 \right) \mu^i \pi^{m-i-j}, \quad j \geq 0
\]

4. Let the probability that ‘\( j \)’ customers complete service during an inter arrival period assuming that a customer’s patience time has elapsed and is denoted by \( f_j \). In LAS–DA, if an inter arrival period consisting of ‘\( m \)’ slots and a patience time of length \( i+1 \) (\( 0 \leq i \leq m - j - 1 \)) slots has elapsed then during this inter arrival period, there are \( m - i - 1 \) possible service positions available. Hence, we get
\[
f_j = \sum_{m=j+1}^{\infty} a_m \sum_{i=0}^{m-j-1} \beta^i \left( m - i - 1 \right) \mu^i \pi^{m-i-j}, \quad j \geq 0
\]

In order to obtain the transition probability matrix, the following two dimensional state space of the system \( \Omega = \{(i, 1) : 0 \leq i \leq N\} \cup \{(i, 0) : 1 \leq i \leq N\} \) are defined, where \((i, \alpha)\) represents that ‘\( i \)’ customers in the system at pre-arrival epoch and \( \alpha = 0 \) (i) corresponds to the server is down (idle or busy).

Observing the system immediately after an imbedded point, we have the transition probability matrix \( P \) with 4 block matrices of the form
\[
P = \begin{pmatrix}
\Phi_{(N+1)\times(N+1)} & \Theta_{(N+1)\times N} \\
\zeta_{N\times(N+1)} & \Gamma_{N\times N}
\end{pmatrix}_{(2N+1)\times(2N+1)}
\]

The first block \( \Phi_{(N+1)\times(N+1)} \) gives the transitions from busy state to busy state. The elements are
The second block $\Theta_{(N+1)\times N}$ gives the transitions from busy state to down state. The elements of this block is given by

$$\Phi_{1,j} = \begin{pmatrix}
  b_1 & b_0 & 0 & \ldots & 0 & 0 & 0 \\
  b_2 & b_1 & b_0 & \ldots & 0 & 0 & 0 \\
  b_3 & b_2 & b_1 & \ldots & 0 & 0 & 0 \\
  & & & \ddots & & & \\
  b_{N-1} & b_{N-2} & b_{N-3} & \ldots & b_1 & b_0 & 0 \\
  b_N & b_{N-1} & b_{N-2} & \ldots & b_2 & b_1 & b_0 \\
  b_N & b_{N-1} & b_{N-2} & \ldots & b_2 & b_1 & b_0
\end{pmatrix}$$

The third block $\zeta_{N\times(N+1)}$ gives the transitions from down state to busy state. The elements of this block is listed below:

$$\Theta_{1,j} = \begin{pmatrix}
  g_0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
  g_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
  g_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\
  & \ldots & \ldots & \ldots & & & \\
  g_{N-2} & 0 & 0 & \ldots & 0 & 0 & 0 \\
  g_{N-1} & 0 & 0 & \ldots & 0 & 0 & 0 \\
  g_N & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}$$

where

$$g_n = 1 - \sum_{i=0}^{n} b_i, \quad 0 \leq n \leq N - 1,$$

$$g_N = 1 - \sum_{i=0}^{N-1} b_i$$
Let \( \mathbf{p}^- \) be the row vector of pre-arrival epoch probabilities which can be obtained by solving \( \mathbf{p}^- \mathbf{P} = \mathbf{p}^- \) with \( \mathbf{p}^- \mathbf{e} = 1 \). Where \( \mathbf{e} \) is a column vector of 1’s with appropriate dimensions. This system of equations can be solved using the algorithm of Grassmann, Taksar and Heyman (GTH) given in Latouche & Ramaswami (1999, pp. 123).
7.4.2 Queue Length Distribution at Arbitrary Epoch

To determine the queue length distribution at arbitrary epoch and performance measures of the system, we develop the Kolmogorov difference equations using the remaining inter arrival time as the supplementary variable. Observing the states of the system at two consecutive epochs namely \( t \) and \((t + 1)\), and using the definitions and probabilities defined above, the following are obtained in steady-state for \( u \geq 1 \)

\[
P_{0,1}(u-1) = P_{0,1}(u) + \mu P_{1,1}(u) + a_u \gamma \beta P_{1,1}(u)
\]

\[
P_{1,1}(u-1) = \overline{\mu} P_{1,1}(u) + a_u \mu P_{1,1}(0) + \mu P_{2,1}(u) + \gamma \alpha P_{1,0}(u) + a_u \gamma \beta P_{1,0}(0) + a_u \gamma P_{0,1}(0)
\]

\[
P_{n,1}(u-1) = \overline{\mu} P_{n,1}(u) + a_u \mu P_{n,1}(0) + \mu P_{n+1,1}(u) + \gamma \alpha P_{n,0}(u) + a_u \gamma \beta P_{n,0}(0) + a_u \alpha \gamma P_{n,0}(0) + a_u \alpha \gamma \beta P_{n,0}(0)
\]

\[
P_{N-1,1}(u-1) = \overline{\mu} P_{N-1,1}(u) + a_u \mu P_{N-1,1}(0) + \mu P_{N,1}(u) + \gamma \alpha P_{N-1,0}(u) + a_u \mu P_{N-1,1}(0) + a_u \alpha \gamma \beta P_{N-1,0}(0) + a_u \alpha \gamma \beta P_{N-1,0}(0)
\]

\[
P_{N,1}(u-1) = \overline{\mu} P_{N,1}(u) + a_u \mu P_{N,1}(0) + a_u \mu P_{N-1,0}(0) + \mu a_u P_{N,1}(0) + \gamma \alpha P_{N,0}(u) + a_u \alpha \gamma \beta P_{N,0}(0) + a_u \alpha \gamma \beta P_{N,0}(0)
\]

\[
P_{1,0}(u-1) = \overline{\alpha} P_{1,0}(u) + \alpha \gamma P_{1,0}(u) + a_u \alpha \gamma \beta P_{1,0}(0) + a_u \alpha \gamma \beta P_{1,0}(0)
\]

\[
P_{n,0}(u-1) = \overline{\alpha} P_{n,0}(u) + \alpha \gamma P_{n,0}(u) + a_u \alpha \gamma \beta P_{n,0}(0) + a_u \alpha \gamma \beta P_{n,0}(0) + a_u \overline{\alpha} \beta P_{n-1,0}(0) + a_u \overline{\alpha} \beta P_{n-1,0}(0)
\]

\[
P_{N,0}(u-1) = \overline{\alpha} P_{N,0}(u) + \alpha \gamma P_{N,0}(u) + a_u \alpha \gamma \beta P_{N,0}(0) + a_u \alpha \gamma \beta P_{N,0}(0) + a_u \overline{\alpha} \beta P_{N-1,0}(0) + a_u \overline{\alpha} \beta P_{N-1,0}(0)
\]
Now to obtain the probability generating function of the queue length distribution at arbitrary epoch, the following generating functions are defined.

\[
P_{n,j}^*(z) = \sum_{u=0}^{\infty} P_{n,j}(u)z^u, \quad 0 \leq n \leq N \text{ with } P_{n,j}(1) = P_{n,j}, \quad j = 0, 1
\]

where \( P_{n,j} \) denotes the probability of ‘n’ customers in the queue at an arbitrary epoch when the server is in state j. Multiplying equations (7.1) –(7.8) by \( z^u \) and summing over u from 1 to \( \infty \), we obtain

\[
zP_{0,1}^*(z) = P_{0,1}^*(z) - P_{0,1}(0) + \mu P_{1,1}^*(z) - \mu P_{1,1}(0) + \lambda \beta A(z) P_{0,0}^{-}
\]

(7.9)

\[
zP_{1,1}^*(z) = \mu P_{1,1}^*(z) - \mu P_{1,1}(0) + \mu A(z) P_{1,1}(0) + \mu P_{2,1}^*(z) - \mu P_{2,1}(0) + \alpha \gamma P_{1,0}^*(z) - \alpha \gamma P_{1,0}(0) + \alpha \gamma \beta A(z) P_{1,0}(0) + \gamma A(z) P_{0,1}(0)
\]

(7.10)

\[
zP_{n,1}^*(z) = \mu P_{n,1}^*(z) - \mu P_{n,1}(0) + \mu A(z) P_{n,1}(0) + \mu P_{n+1,1}^*(z) - \mu P_{n+1,1}(0) + \alpha \gamma P_{n,0}^*(z) - \alpha \gamma P_{n,0}(0) + \mu A(z) P_{n-1,1}(0) + \alpha \gamma \beta A(z) P_{n-1,0}(0) + \alpha \gamma \beta A(z) P_{n,0}(0)
\]

(7.11)

\[
zP_{N-1,1}^*(z) = \mu P_{N-1,1}^*(z) - \mu P_{N-1,1}(0) + \mu A(z) P_{N-1,1}(0) + \mu P_{N,1}^*(z) - \mu P_{N,1}(0) + \mu A(z) P_{N-2,1}(0) + \alpha \gamma P_{N-1,0}^*(z) - \alpha \gamma P_{N-1,0}(0) + \alpha \gamma \beta A(z) P_{N-2,0}(0) + \alpha \gamma \beta A(z) P_{N-1,0}(0)
\]

(7.12)

\[
zP_{N,1}^*(z) = \mu P_{N,1}^*(z) - \mu P_{N,1}(0) + \mu A(z) P_{N-1,1}(0) + \mu A(z) P_{N,1}(0) + \alpha \gamma P_{N,0}^*(z) + \alpha \gamma P_{N,0}(0) + \alpha \gamma \beta A(z) P_{N-0,0}(0) + \alpha \gamma \beta A(z) P_{N,0}(0)
\]

(7.13)

\[
zP_{1,0}^*(z) = \alpha \gamma P_{1,2}^*(z) - \alpha \gamma P_{1,2}(0) + \alpha \gamma P_{1,2}^*(z) - \alpha \gamma P_{1,2}(0) + \alpha \gamma \beta A(z) P_{1,2}(0) + \alpha \gamma \beta P_{1,2}(0)
\]

(7.14)
\[ z P_{n,0}^* (z) = \bar{\alpha} P_{n,0}^* (z) - \alpha P_{n,0}^* (0) + \alpha \bar{\gamma} P_{n,0}^* (z) - \alpha \bar{\gamma} P_{n,0}^* (0) + \bar{\alpha} \bar{\beta} A(z) P_{n,0}^* (0) + \bar{\alpha} \bar{\beta} A(z) P_{n-1,0}^* (0) + \alpha \bar{\gamma} \overline{\beta} A(z) P_{n,0}^* (0) , \quad 2 \leq n \leq N - 1 \]

(7.15)

\[ z P_{N,0}^* (z) = \bar{\alpha} P_{N,0}^* (z) - \alpha P_{N,0}^* (0) + \alpha \bar{\gamma} P_{N,0}^* (z) - \alpha \bar{\gamma} P_{N,0}^* (0) + \bar{\alpha} \bar{\beta} A(z) P_{N-1,0}^* (0) + \alpha \bar{\gamma} \overline{\beta} A(z) P_{N-1,0}^* (0) + \alpha \bar{\gamma} \overline{\beta} A(z) P_{N,0}^* (0) \]

(7.16)

Adding (7.9) – (7.16), we obtain

\[
\sum_{n=0}^{N} P_{n,1}^* (z) + \sum_{n=0}^{N} P_{n,0}^* (z) = \frac{A(z)-1}{z-1} \left( \sum_{n=0}^{N} P_{n,1}^* (0) + \sum_{n=0}^{N} P_{n,0}^* (0) \right)
\]

Taking limit as \( z \to 1 \) in the above equations and using the normalisation condition \( \sum_{n=0}^{N} P_{n,1} + \sum_{n=1}^{N} P_{n,0} = 1 \), the arrival rate is obtained as

\[
\sum_{n=0}^{N} P_{n,1}^* (0) + \sum_{n=0}^{N} P_{n,0}^* (0) = \lambda
\]

(7.17)

### 7.4.3 Relationship between Pre-arrival and Arbitrary Epoch Probabilities

Let the pre-arrival epoch probabilities \( P_{n,1}^- \), \( 0 \leq n \leq N \) and \( P_{n,0}^- \), \( 1 \leq n \leq N \) denotes an arrival sees \( i \) customers in the system at an arrival epoch when the server is busy and down, respectively. Applying Bayes’ theorem, we obtain

\[ P_{n,j}^- = P(\text{\( n \) customers in the queue prior to an arrival epoch when the server is in state \( j \) / the server is either busy or down prior to an arrival epoch}) \]
Further, using the equation (7.17), we have

\[ P_{n,1} = \frac{P_{n,1}(0)}{\lambda}, \quad 0 \leq n \leq N \]

\[ P_{n,0} = \frac{P_{n,0}(0)}{\lambda}, \quad 1 \leq n \leq N \] (7.18)

By setting \( z = 1 \) in equations (7.9) – (7.16) and using (7.17), after some simplifications, we can get the following relations between pre-arrival and arbitrary epoch probabilities

\[ P_{1,0} = \frac{\lambda \beta}{\alpha \gamma} (1 - \alpha \gamma) P_{1,0}^- \] (7.19)

\[ P_{n,0} = \frac{\lambda \beta}{\alpha \gamma} \left\{ (\alpha \gamma - 1) \left( P_{n,0}^- - P_{n-1,0}^- \right) \right\}, \quad n = 2, 3, 4, ..., N - 1 \] (7.20)

\[ P_{N,0} = \frac{\beta \lambda}{\alpha \gamma} (1 - \alpha \gamma) P_{N-1,0}^- \] (7.21)

\[ P_{0,1} = \frac{\lambda}{\mu} \left\{ P_{0,1}^- + \mu P_{1,1}^- \right\} \] (7.22)

\[ P_{1,1} = \frac{1}{\mu} \left\{ \mu P_{0,1}^- + \lambda \bar{\mu} P_{0,1}^- + \lambda \mu P_{0,1}^- - \gamma \bar{\beta} \lambda P_{0,1}^- \right\} \] (7.23)
\[
\begin{align*}
P_{n+1,1} &= \frac{1}{\mu} \left\{ \mu P_{n,1} + \lambda \beta P_{n+1,1} + \lambda \mu P_{n+1,1} - \lambda \alpha \gamma P_{n,0} + \alpha \gamma \lambda P_{n,0} - \lambda \beta P_{n+1,1} - \lambda \alpha \gamma P_{n-1,0} \right\} \\
&\quad \text{for } n = 1, 2, 3, \ldots, N - 2
\end{align*}
\]

(7.24)

\[
P_{N,1} = \frac{\lambda}{\mu} \left\{ \mu P_{N-1,1} + \alpha \gamma P_{N,0} + \alpha \gamma \beta P_{N-1,0} \right\}
\]

(7.25)

It is well known that, the arbitrary time epoch probabilities can be obtained easily from the equations (7.19) – (7.25) and the pre-arrival epoch probabilities obtained in section 7.4.1.

**Remarks 7.1**

When the inter arrival time is geometrically distributed with \(a_i = \bar{\lambda}^{i-1} \lambda, i \geq 1\), then the expressions for \(\psi, b_j, d_j\) and \(f_j\) can be simplified for LAS-DA and are given by

\[
\psi = \frac{\lambda \bar{\alpha}}{\alpha + \lambda \bar{\alpha}}, \quad b_0 = \frac{\lambda \bar{\mu}}{\mu + \lambda \bar{\mu}}, \quad b_j = \frac{\lambda (\mu)^j \bar{\lambda}^{j-1}}{(\mu + \lambda \bar{\mu})^{j+1}}, j \geq 1
\]

\[
d_j = \frac{\lambda \alpha (\mu \bar{\lambda})^j}{(\alpha + \lambda \bar{\alpha})(\mu + \lambda \bar{\mu})^{j+1}}, j \geq 0
\]

\[
f_j = \frac{\lambda (\mu \bar{\lambda})^j}{(\beta + \lambda \bar{\beta})(\mu + \lambda \bar{\mu})^{j+1}}, j \geq 0
\]

7.5 **OUTSIDE OBSERVER’S DISTRIBUTION**

Using Little’s formula, one can obtain the average sojourn time in the queue which is the distribution of queue size at outside observer’s
observation epoch. The outside observer’s distributions in LAS-DA have been obtained by considering arbitrary and outside observer’s observation epochs.

The outside observer’s observation epoch in LAS-DA falls in a time interval after a potential departure and before a potential arrival. According to this, the probability \( P_{n,1}^0 \) (or \( P_{n,0}^0 \)) that outside observer sees \( n \) customers in the queue and server busy (on down) is same as \( P_{n,1} \) (or \( P_{n,0} \)).

7.6 WAITING TIME ANALYSIS

This section is devoted to obtain the probability generating function of actual waiting time in the queue of an accepted customer under the FCFS queueing discipline.

Let the probability that the actual waiting time in the queue of an accepted customer in \( m \) slots is denoted by \( W_q(m) \) and the corresponding probability generating function be \( W_q(z) \). By memoryless property of the service and down time distributions, we may have the following two cases:

**Case 1:** If at an arrival epoch there are \( n \) (1 ≤ \( n \) ≤ \( N \)) customers in the system and the server is busy, then the customer will have to wait for either service time of \( n \) customers with probability \( \mu \), or service time of \( n - 1 \) customers with probability \( \mu \).

**Case 2:** If at an arrival epoch there are \( n \) (1 ≤ \( n \) ≤ \( N - 1 \)) customers in the system and the server is on down, then the customer who have patience will have to wait for either service time of \( n \) customers with probability \( \alpha \), or the sum of a remaining down time and service time of \( n \) customers with probability \( \alpha \).
Combining the above two cases, we obtain the probability generating function of actual waiting time in the queue which is given by

\[
W_q(z) = \frac{1}{1 - \text{PBL}} \left[ \sum_{n=0}^{N-1} P_{n,1} \left\{ \mu \left( \frac{\mu z}{1 - \mu z} \right)^n + \mu \left( \frac{\mu z}{1 - \mu z} \right)^{n-1} \right\} \right] + \sum_{n=1}^{N-1} P_{n,0} \left\{ \alpha \left( \frac{\mu z}{1 - \mu z} \right)^n + \alpha \left( \frac{\mu z}{1 - \mu z} \right)^n \right\}
\]

where PBL represents the blocking probability of the customers which is obtained by using \( \text{PBL} = P_{N,1}^+ + P_{N,0}^- \). Thus the expected actual waiting time is given by

\[
W_q = \frac{1}{1 - \text{PBL}} \left[ \sum_{n=0}^{N-1} \left( \frac{n - \mu}{\mu} \right) P_{n,1}^- + \sum_{n=1}^{N-1} \left( \frac{n - \alpha}{\alpha} \right) P_{n,0}^- \right]
\]

### 7.7 PARTICULAR CASE

When \( \alpha = 1 \) and \( \beta = 1 \), that is server never fails and no impatient of customers, then the equations (7.1) – (7.8) and (7.9) – (7.16) and waiting time distribution are coinciding with the equations in model GI/Geo/1/N discussed by Chaudhry & Gupta (1996).

### 7.8 PERFORMANCE MEASURES

As the queue length distributions at various epochs are known, the following performance measures are obtained for LAS-DA

(a) Average number of customers in the queue at outside observer’s observation epoch

\[
L_s^0 = \sum_{n=1}^{N-1} n P_{n,1}^0 + \sum_{n=1}^{N} n P_{n,0}^0
\]
(b) Average number of customers in the queue at outside observer’s observation epoch \( L_0^q = \sum_{n=1}^{N+1} (n-1) P_{n,1}^0 + \sum_{n=1}^{N} n P_{n,0}^0 \)

(c) Average number of customers in the queue at an arbitrary epoch which is obtained by replacing \( P_{n,1}^0 \) and \( P_{n,0}^0 \) by \( P_{n,1} \) and \( P_{n,0} \) respectively, and is given by \( L_0 = \sum_{n=1}^{N+1} n P_{n,1} + \sum_{n=1}^{N} n P_{n,0} \)

(d) Average number of customers in the queue at an arbitrary epoch which is obtained by replacing \( P_{n,1}^0 \) and \( P_{n,2}^0 \) by \( P_{n,1} \) and \( P_{n,0} \) respectively, and is given by \( L_0 = \sum_{n=1}^{N+1} (n-1) P_{n,1} + \sum_{n=1}^{N} n P_{n,0} \)

(e) Average waiting time in the system is \( W_s = L_0^q / \lambda' \), where \( \lambda' = \lambda (1 - PBL) \)

(f) Average waiting time in the queue is \( W_q = L_0^q / \lambda' \)

7.9 NUMERICAL ILLUSTRATION

To demonstrate the applicability of the model in the previous sections, some numerical results in the form of self explanatory tables and graphs are presented. Various performance measures such as probability of blocking, average system (queue) length and average waiting-time in the system (queue) are given. Numerical works have been carried out using the software MATLAB. To study the effect of probability of blocking, server’s busy period and average repair time on buffer size, mean queue length and average waiting time, the following are the notations used:
Arrival rate \( \lambda \)
Service rate \( \mu \)
Repair rate \( \alpha \)
Probability of server activation \( \gamma \)
Patient rate \( \beta \)
Busy period \( \rho \)
Arrival rate \( a_n \)
Average inter arrival time \( \theta_a \)
Maximum buffer content \( N \)

Distributions of the number of customers in the queue at various epochs for LAS-DA model with inter arrival time follows geometric distribution with \( \lambda = 0.4, \mu = 0.6, \alpha = 0.3, \beta = 0.7, \gamma = 0.7, \rho = 0.666666 \), \( N = 25 \) is listed in Table 7.1.

Figure 7.2 compares the buffer size versus probability of blocking for various inter arrival times with same mean with \( \theta_a = 8, \mu = 0.2, \alpha = 0.4, \beta = 0.6, \gamma = 0.5 \) and the inter arrival time distributions are taken as geometric with \( \lambda = 0.125 \), deterministic with \( a_8 = 1 \) and arbitrary with \( a_2 = 0.7, a_3 = 0.2, a_{30} = 0.1 \).

It is observed that

- The loss probability in geometric distribution is very low when compared to arbitrary and deterministic distributions.
Figures 7.3 and 7.4 gives the effect of loss probability when the inter arrival follows geometric distribution with the following input parameters $\lambda = 0.4, \mu = 0.6, \alpha = 0.3, \beta = 0.7, \gamma = 0.7$ for Figure 7.3 and $\lambda = 0.125, \mu = 0.6, \alpha = 0.4, \beta = 0.2, \gamma = 0.5$ for Figure 7.4, respectively.

It is observed that

- The loss probability decreases when busy period increases.

Figure 7.5 gives the relationship between server’s busy period and average waiting time in the queue with the input parameters $\mu = 0.7, \alpha = 0.4, \beta = 0.6, \gamma = 0.6, N = 40$ when the inter arrival time follows geometric distribution.

It is observed that

- Average waiting time in the queue ($W_q$) increases initially as server busy period $\rho$ increases up to the level 0.4 and decreases upto 0.6 and stabilizes further.

Figure 7.6 depicts the effect of average waiting time in the queue and the average repair time of the server with the capacity of the buffer $N = 25$ for the fixed input parameters $\beta = 0.6, \mu = 0.4, \gamma = 0.6$ and various inter arrival time distributions. The input parameter for the various inter arrival distribution is as follows: geometric distribution with $\lambda = 0.125$, deterministic distribution with $a_8 = 1$ and arbitrary distribution with $a_2 = 0.5, a_5 = 0.2, a_{20} = 0.3$.

It is observed that

- Average waiting time in the queue increases when average repair time increases.
Average waiting time in the queue is high in geometric distribution than in deterministic and arbitrary distribution.

Figure 7.7 compares the loss probability with average repair time of the server with the capacity of the buffer $N = 25$ for the fixed input parameters $\beta = 0.6, \mu = 0.4, \gamma = 0.6$ and various inter arrival time distributions. The input parameter for the various inter arrival distribution is as follows: geometric distribution with $\lambda = 0.125$, deterministic distribution with $a_8 = 1$ and arbitrary distribution with $a_2 = 0.5, a_5 = 0.2, a_{20} = 0.3$.

It is observed that

- Probability of loss increases with the increase of average repair time.
- Probability of loss is higher in geometric distribution than in deterministic and arbitrary distribution.

Figure 7.8 conveys the value of mean queue length with average repair time of the server with the capacity of the buffer $N = 25$ for the fixed input parameters $\beta = 0.6, \mu = 0.4, \gamma = 0.6$ and various inter arrival time distributions. The input parameter for the various inter arrival distribution is as follows: geometric distribution with $\lambda = 0.125$, deterministic distribution with $a_8 = 1$ and arbitrary distribution with $a_2 = 0.5, a_5 = 0.2, a_{20} = 0.3$.

It is observed that

- Mean queue length increases with the increase of average repair time.
- Mean queue length converges with the largest average repair time for arbitrary, geometric and deterministic distributions.
7.10 CONCLUSION

This chapter gives the study of a discrete time finite buffer queue with the server, subject to starting failures and impatient customers with discrete time renewal process arrivals. Using supplementary variable and imbedded Markov chain techniques, the steady state queue length distributions at pre-arrival, arbitrary and outside observer’s observation epochs under the late arrival system with delayed access is carried out. Various performance measures are also obtained. More specifically this chapter discusses starting failures with impatient customers in classical queues.
Table 7.1  Distribution of customers in the queue at various epochs for LAS-DA

<table>
<thead>
<tr>
<th>n</th>
<th>( P_{n,0} )</th>
<th>( P_{n,1} )</th>
<th>( P_{n,0} )</th>
<th>( P_{n,1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.08143721</td>
<td>0.07714582</td>
<td>0.04767829</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.05000536</td>
<td>0.06776517</td>
<td>0.05267231</td>
<td>0.09448008</td>
</tr>
<tr>
<td>3</td>
<td>0.02175214</td>
<td>0.03398391</td>
<td>0.02976006</td>
<td>0.05695044</td>
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<tr>
<td>4</td>
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<td>0.01332120</td>
<td>0.01459622</td>
<td>0.02649218</td>
</tr>
<tr>
<td>5</td>
<td>0.00259208</td>
<td>0.00456579</td>
<td>0.00558571</td>
<td>0.01101328</td>
</tr>
<tr>
<td>6</td>
<td>0.00080337</td>
<td>0.00145093</td>
<td>0.00188410</td>
<td>0.00495788</td>
</tr>
<tr>
<td>7</td>
<td>0.00024052</td>
<td>0.00044105</td>
<td>0.00059287</td>
<td>0.00288857</td>
</tr>
<tr>
<td>8</td>
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<td>0.00013046</td>
<td>0.00017912</td>
<td>0.00223237</td>
</tr>
<tr>
<td>9</td>
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<td>0.00005278</td>
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</tr>
<tr>
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<td>0.00197430</td>
</tr>
<tr>
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</tr>
<tr>
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<td>0.00000088</td>
<td>0.00000125</td>
<td>0.00195228</td>
</tr>
<tr>
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<td>0.00000025</td>
<td>0.00000041</td>
<td>0.00195148</td>
</tr>
<tr>
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<tr>
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<td>0.02349607</td>
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<td>0.03205629</td>
</tr>
<tr>
<td>17</td>
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<td>0.00431825</td>
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<tr>
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<td>0.00261491</td>
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<tr>
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<tr>
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<td>0.00196579</td>
</tr>
<tr>
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<td>0.00000458</td>
<td>0.00195526</td>
</tr>
<tr>
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<td>0.00000090</td>
<td>0.00000129</td>
<td>0.00195231</td>
</tr>
<tr>
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<td>0.00000013</td>
<td>0.00000025</td>
<td>0.00000036</td>
<td>0.00195148</td>
</tr>
<tr>
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<td>0.00000008</td>
<td>0.00000007</td>
<td>0.00000004</td>
<td>0.00195125</td>
</tr>
<tr>
<td>Sum</td>
<td>0.38480574</td>
<td>0.61519426</td>
<td>0.4388706</td>
<td>0.5511192</td>
</tr>
</tbody>
</table>

\( \lambda' = 0.3999999 \), PBL = 0.00000015, \( L_q = 9.876573263 \),
\( L_s = 10.42769248 \), \( W_q = 40.07935715 \), \( W_s = 43.13847022 \)
Figure 7.2 Buffer size versus probability of blocking

Figure 7.3 Buffer size versus PBL with $\rho = 0.666667$ using geometric distribution
Figure 7.4 Buffer size versus PBL with $\rho = 0.625$ using geometric distribution

Figure 7.5 Average waiting time in the queue versus server’s busy period using geometric distribution
Figure 7.6  Average waiting time in the queue versus average repair time when N=25

Figure 7.7 Blocking probability versus average repair time when N=25
Figure 7.8 Mean queue length versus average repair time when N=25