CHAPTER 5

ANALYSIS OF DISCRETE-TIME QUEUEING SYSTEM
WITH CORRELATED ARRIVALS, NEGATIVE
CUSTOMERS AND SERVER INTERRUPTION

5.1 INTRODUCTION

This chapter analyses a discrete time infinite capacity queueing system with correlated arrival and negative customers served by two state Markovian server. Discrete time queueing system is an efficient tool to model telecommunication systems, B-ISDN networks based on ATM and mobile networks. These networks apply the slotted transportation of fixed size packets. Moreover, discrete-time models can be used to derive the results for continuous-time models but not vice versa, Tagaki (1993). In recent years, great attention is paid in studying discrete time queueing systems with different variants. For comprehensive studies on discrete time queues one may refer to Takagi (1993), Bruneel & Kim (1993) and Woodward (1994).

There are situations in which an arrival may harm the entire system and exiting customers (like virus to a computer server). In queueing terminology such customers are referred to as negative customers. Queues with negative customers are also called as G-queues. Excellent studies have been made with negative customers in the recent past. A study on G-queues was first introduced by Gelenbe (1989). For related literature, one can find the motivation by considering negative customers in Artalejo (2000) and Gelenbe (1994, 2000). Although many continuous-time queueing models
with negative arrivals have been studied, their discrete-time counterparts received considerably less attention in the literature. The early work about negative customers in discrete-time without retrials can be found in Atencia & Moreno (2004a, 2004b, 2005). They have considered single server discrete-time queue, with negative arrivals and various killing disciplines caused by the negative customers but not considering the arrival of On – Off sources.


In this chapter, the discrete time two state Markov chain generating arrival of positive customers are modelled by an On – Off source arrival which are served by a two state Markovian server. The arrival of customer alternates between On and Off states with probabilities $\alpha$ and $\beta$ respectively. When the source is On or active, one positive customer is generated and when the source is inactive or Off no positive customer is generated. This type of customer is stored in an infinite buffer and is served by a two state Markovian server. That is, server also alternates between Good and Bad states with probabilities $\gamma$ and $\sigma$ respectively. If the server is in Good state, the service of the positive customer is likely to be successful and when the server is in Bad state the service is unsuccessful. The arrival of negative customers is according to geometrical arrival process and is independent of the positive customers. The graph showing the sample path of the proposed queueing model is depicted in Figure 5.1.
5.2 MOTIVATION

The motivation of the proposed work is from a situation in mobile banking. Mobile banking system consists of a base server and a number of mobile users. Customers who request for transactions will be connected to a base server through mobile and form a queue. We say a link to be in a Bad state when the transaction is very likely to fail due to errors and in a Good state when transaction is likely to be successful. There are situation at which the negative customers will affect the queue in different ways. In the present paper, killing methodology considered is Removal of Customers at the Head (RCH killing methodology). Using Markovian processes approach, this paper investigates a discrete time infinite queueing system with On–Off source, as well as geometrically distributed service times with two states of a server namely Good and Bad and negative customers. This situation is modelled as the discrete time queueing model with correlated arrivals, negative customers and server interruption.

5.3 THE MATHEMATICAL MODEL

The mathematical model for this work describes a discrete time queueing system with correlated arrivals and negative customers. Using first order Markovian arrival process, positive customers are generated with geometrically distributed lengths of On periods and Off periods and are served by two state Markovian server which alternates between Good and Bad states
with geometrically distributed service times. Positive customers arrive in a stochastic way and wait in infinite capacity waiting room (buffer) on a first-come, first-serve (FCFS) basis and they are considered for service and leave the system. Further, the geometrically distributed arrival of negative customers removes the positive customers if any, and has no effect when the system is empty.

The analytical solution of discrete time systems consists of two possible policies namely, Late arrival system (LAS) or arrival first (AF) and Early arrival system (EAS) or departure first (DF). In AF policy, it is supposed that at first arrival of a customer happens and then the departure of another customer, whereas in DF policy, contrarily departure of customer happens first and then the arrival occurs. Depending upon the policy considered, the probabilities of system states and performance measures are affected. Late arrival system (Hunter (1983)) is considered in this model. That is, the time axis is divided into fixed length intervals called slots and are marked by 0, 1, . . . , m, . . . in order. It is always assumed that all the queueing activities such as arrivals, removals and departures occur at the slot boundaries. Potential arrivals of positive as well as negative customers occur in $\left( m^-, m \right)$ in order and the potential departure of a positive customer occurs in $\left( m, m^+ \right)$.

5.3.1 The Positive Arrival Processes

The arrival of positive customers is modelled as the first order Markovian arrival process $(A_n, n \geq 0)$ with the number of positive arrivals in the arbitrary slot as a random variable, which depends on the number of arrivals during immediately preceding time slot. A discrete time, single On – Off source arrival process which alternate between On and Off periods respectively is considered. That is exactly one positive customer is generated
in each time slot when the Markov chain is in state \textit{On} and no positive customer is generated when the Markov chain is in state \textit{Off}. The two independent parameters $\alpha$ and $\beta$ denote the probabilities that the Markov chain remains in states \textit{On} and \textit{Off} respectively. It is assumed that, the lengths of the \textit{On} – \textit{Off} periods are geometrically distributed random variables with rate $1-\alpha$ and $1-\beta$ respectively, when $0 < \alpha < 1, 0 < \beta < 1$. Thus, the positive arrival process is defined as

\[ \alpha = P(\text{a positive arrival occurs during a slot / a positive arrival occurs during previous slot}) \]

\[ \beta = P(\text{no positive arrival occurs during a slot / no positive arrival occurs during previous slot}) \]

Thus, according to Hunter (1983), the steady state probability of mean positive arrival during an arbitrary slot, that is the rare at which the customer enters into the slot and the total time spent in on periods is also called the effective arrival rate and is giving by $\lambda_1 = \frac{1-\beta}{2-\alpha-\beta}$, where $0 < \lambda_1 < 1$, which is the steady state condition of the time the chain spends in \textit{On} periods.

\subsection{The Removal Rule and Arrival Times of Negative Customers}

Upon arrival the negative customer removes the positive customer under service and vanishes. But, if the negative customer arrives during idle period (on empty system) it has no effect and is lost. The inter–arrival times of negative customers ($B_n, n \geq 1$) are independent and geometrically distributed random variables with parameter $\theta$. The distribution of $B_n$ is given by

\[ P(B_n = k) = \overline{\theta}^k \theta, \quad k = 1, 2, 3, \ldots \]

\[ \overline{\theta} = 1 - \theta \quad \text{and} \quad E(B_n) = \theta^{-1}. \]
5.3.3 The Service Processes

Assume that the service is also modelled as two state Markov chain, which alternates between Good and Bad states. In Bad state, the service is likely to fail due to some errors and in Good state the server is ready to serve. Assume that during a slot, if the server is in Good state, the server is ready to serve the customers, while in Bad state the server will not render service even when there are customers in the queue. Server’s state is then characterized by two independent parameters $\gamma$ and $\sigma$, which is defined as

$$\gamma = P(\text{server is in Good state during a slot / server is in Good state during the previous slot}),$$

$$\sigma = P(\text{server is in Bad state during a slot / server is in Bad state during the previous slot}).$$

During Good state, the service of one positive customer is accurately completed per slot while in Bad state, service of the positive customer will be in complete. Both the lengths of the Good and Bad states are geometrically distributed random variables with rate $1-\gamma$ and $1-\sigma$, respectively. For avoiding trivial cases, it is assumed that $0 < \gamma < 1$ and $0 < \sigma < 1$.

Similarly, the steady state probability $\mu l$ of having a Good service during an arbitrary slot is given by $\mu l = \frac{1-\sigma}{2-\gamma-\sigma}$, where $0 < \mu l < 1$, which corresponds to the overall fraction of Good service.

Positive customers are served by a two state Markovian server on the FCFS basis. Service of the positive customer takes place if the server is in Good state and therefore the service of a positive customer cannot start before
the beginning of the slot following its arrival slot. Service times \((S_n, n \geq 1)\) are independent and geometrically distributed random variable with parameter \(\mu\) and is given by

\[
P(S_n = k) = (1 - \mu)^{k-1}\mu, \quad k = 1, 2, 3, \ldots, \quad \text{and} \quad E(S_n) = \mu^{-1}, \quad S(z) = \frac{\mu z}{1 - (1 - \mu)z}.
\]

where \(\mu\) defines the probability that a positive customer concludes his service in a slot. The input such as \textit{On–Off} source, negative arrivals and \textit{Good} and \textit{Bad} state of service process are assumed to be mutually independent.

State transition diagram of the Markov chain is given in Figure 5.2 with the following notations

\[
V_1 = (1 - \alpha)(1 - \sigma)\theta; \quad V_2 = (1 - \beta)(1 - \gamma)(1 - \theta)(1 - \mu);
\]
\[
V_3 = [(1 - \beta)(1 - \gamma)(1 - \theta)\mu + (1 - \beta)(1 - \gamma)(1 - \mu)\theta]; \quad V_4 = (1 - \alpha)(1 - \sigma)(1 - \mu);
\]
\[
V_5 = (1 - \alpha)\gamma(1 - \theta)(1 - \mu); \quad V_6 = [(1 - \beta)\gamma(1 - \theta)\mu + (1 - \beta)\gamma(1 - \mu)\theta];
\]
\[
V_7 = [\beta \gamma \theta (1 - \mu) + \beta \gamma \mu (1 - \theta)]; \quad V_8 = \beta \gamma (1 - \theta)(1 - \mu);
\]
\[
V_9 = [(1 - \alpha)(1 - \gamma)(1 - \theta)\mu + (1 - \alpha)(1 - \gamma)\theta(1 - \mu)]; \quad V_{10} = \alpha(1 - \gamma)(1 - \theta)(1 - \mu);
\]
\[
V_{11} = [(1 - \alpha)(1 - \gamma)(1 - \theta)\mu + (1 - \alpha)(1 - \gamma)\theta (1 - \mu)]; \quad V_{12} = \alpha(1 - \sigma)(1 - \theta);
\]
\[
V_{13} = \alpha(1 - \sigma)\theta; \quad V_{14} = [\alpha(1 - \gamma)(1 - \theta)\mu + \alpha(1 - \gamma)\theta(1 - \mu)];
\]
\[
V_{15} = \beta(1 - \sigma)(1 - \theta); \quad V_{16} = (1 - \beta)(1 - \sigma)\theta;
\]
\[
V_{17} = (1 - \alpha)(1 - \gamma)(1 - \theta)(1 - \mu)
\]
Figure 5.2 State transition diagram
5.4 GENERATING FUNCTION ANALYSIS

We define the system by a vector process \((A(m^+), S(m^+), N(m^+)), \ m = 0, 1, 2, \ldots\) where, \(A(m^+)\) denotes the number of positive customers arriving to the system during \(m^\text{th}\) slot, \(S(m^+)\) is the state of the server in the \(m^\text{th}\) slot and \(N(m^+)\) denotes the number of positive customers in the system at \(m^+\).

Also define,

\[ Y_m(i, j, k) = \{A(m^+) = i, S(m^+) = j, N(m^+) = k\}, \ i = 0, 1; \ j = 0, 1; \ k = 0, 1, 2, \ldots \]

whose state space is given by \(S = \{(i, j, k) : \ i = 0, 1; \ j = 0, 1; \ k = 0, 1, 2, \ldots\}\).

The following generating function is defined to obtain the stationary distribution of the buffer content of the system

\[ Q_{i,j}^m(z) = \sum_{k=0}^{\infty} P(A(m^+) = i, S(m^+) = j, N(m^+) = k)z^k \text{ with } i = 0, 1; \ j = 0, 1 \]

and if

\[ j = 0, \text{ if the server is in Bad state} \]
\[ = 1, \text{ if the server is in Good state} \]

Also, the stationary distribution of the system is given by

\[ \pi_{i,j,k} = \lim_{m \to \infty} P(Y_m(i, j, k)) \] with

\[ Q_{0,0}^m(z) = \lim_{m \to \infty} Q_{0,0}^m(z); \ Q_{1,0}^m(z) = \lim_{m \to \infty} Q_{1,0}^m(z) \text{ for server’s Bad state and} \]
\[ Q_{0,1}^m(z) = \lim_{m \to \infty} Q_{0,1}^m(z); \ Q_{1,1}^m(z) = \lim_{m \to \infty} Q_{1,1}^m(z) \text{ for server’s Good state.} \]

Sum of these four server’s states gives the stationary distribution of the queue length of the system. It is assumed that the condition for the system
to be stable when $\rho < 1$ and is shown in Theorem 5.1. By definition $Y_m(1,0,0) = 0$ and $Y_m(1,1,0) = 0$, because it is impossible that the system is empty in the $m$th slot with $A(m^+) = 1$, which gives $\pi_{1,0,0} = 0$ and $\pi_{1,1,0} = 0$.

Kolmogorov equations of the stationary distribution of the system are given by

$$\pi_{0,0,0} = \beta \sigma_0 \pi_{0,0,0} + \beta \sigma_0 \pi_{0,0,1} + [\beta(1-\gamma)(1-\theta)]\pi_{0,1,1} +$$
$$(1-\alpha)(1-\gamma)(1-\theta)\pi_{1,1,1} + (1-\alpha)\sigma_0 \pi_{1,0,1} +$$
$$\beta(1-\gamma)(1-\mu)\pi_{0,1,0}$$  \hspace{1cm} (5.1)

$$\pi_{0,0,k} = \beta \sigma_0 \pi_{0,0,k+1} + [\beta(1-\gamma)(1-\theta)]\pi_{0,0,k} + [(1-\alpha)(1-\gamma)(1-\theta)]\pi_{0,1,k} +$$
$$[(1-\alpha)(1-\gamma)(1-\theta)]\pi_{1,1,k} + (1-\alpha)\sigma_0 \pi_{1,0,k} +$$
$$\beta(1-\gamma)(1-\mu)\pi_{0,1,k}$$  \hspace{1cm} (5.2)

$$\pi_{1,0,1} = \alpha \sigma_0 \pi_{1,0,1} + (1-\beta)\sigma_0 \pi_{1,0,0} + [(1-\beta)(1-\gamma)(1-\theta)]\pi_{1,0,0} +$$
$$[(1-\beta)(1-\gamma)(1-\theta)]\pi_{1,0,1} + (1-\beta)\sigma_0 \pi_{0,0,1} +$$
$$\alpha(1-\gamma)(1-\theta)\pi_{1,1,1}$$  \hspace{1cm} (5.3)

$$\pi_{1,0,k} = \alpha \sigma_0 \pi_{1,0,k-1} + (1-\beta)\sigma_0 \pi_{1,0,k} + (1-\beta)(1-\gamma)(1-\theta)\pi_{1,0,k-1} +$$
$$[(1-\beta)(1-\gamma)(1-\theta)]\pi_{1,0,k-1} +$$
$$[\alpha(1-\gamma)(1-\theta)\pi_{1,1,k} + \alpha(1-\gamma)(1-\mu)\pi_{1,1,k-1} +$$
$$\alpha(1-\gamma)(1-\theta)\pi_{1,1,k} + (1-\beta)(1-\gamma)(1-\mu)\pi_{1,1,k-1} +$$
$$\alpha(1-\gamma)(1-\theta)\pi_{1,1,k}]$$  \hspace{1cm} (5.4)

$$\pi_{0,1,0} = \beta \gamma(1-\mu)\pi_{0,1,0} + [\beta(1-\gamma)(1-\theta)]\pi_{0,1,1} +$$
$$[(1-\alpha)(1-\gamma)(1-\theta)]\pi_{0,1,1} + (1-\alpha)\gamma(1-\theta)\pi_{1,1,1}$$  \hspace{1cm} (5.5)
\[ \pi_{0,1,k} = \beta(1 - \sigma)(1 - \theta)\pi_{0,0,k} + (1 - \alpha)(1 - \sigma)(1 - \theta)\pi_{1,0,k} + (1 - \alpha)(1 - \sigma)\theta\pi_{1,0,k+1} + \\
\beta\gamma(1 - \theta)(1 - \mu)\pi_{0,1,k} + [(1 - \alpha)\gamma(1 - \theta)\mu + (1 - \alpha)\gamma(1 - \mu)\theta]\pi_{1,1,k+1} + \\
(1 - \alpha)\gamma(1 - \theta)(1 - \mu)\pi_{1,1,k} + \beta(1 - \sigma)\theta\pi_{0,0,k+1} + \\
[\beta\gamma\theta(1 - \mu) + \beta\gamma\mu(1 - \theta)]\pi_{0,1,k+1} \quad k \geq 1 \]

\[ \pi_{1,1,l} = (1 - \beta)(1 - \sigma)(1 - \theta)\pi_{0,0,0} + (1 - \beta)(1 - \sigma)\theta\pi_{0,0,1} + \alpha(1 - \sigma)\theta\pi_{1,0,1} + \\
(1 - \beta)\gamma(1 - \mu)(1 - \theta)\pi_{0,1,0} + [\alpha\gamma(1 - \theta)\mu + \alpha\gamma(1 - \mu)\theta]\pi_{1,1,1} + \\
[ (1 - \beta)\gamma(1 - \theta)\mu + (1 - \beta)\gamma\theta(1 - \mu)]\pi_{0,1,l} \]

\[ \pi_{1,l,k} = (1 - \beta)(1 - \sigma)\theta\pi_{0,0,k} + \alpha(1 - \sigma)(1 - \theta)\pi_{1,0,k-1} + (1 - \beta)(1 - \sigma)(1 - \theta)\pi_{0,0,k-1} + \\
[\alpha\gamma(1 - \theta)\mu + \alpha\gamma\theta(1 - \mu)]\pi_{1,1,k} + [ (1 - \beta)\gamma(1 - \theta)\mu + (1 - \beta)\gamma\theta(1 - \mu)]\pi_{0,1,k} + \\
(1 - \beta)\gamma(1 - \mu)(1 - \theta)\pi_{0,1,k-1} + \alpha\gamma(1 - \mu)(1 - \theta)\pi_{1,1,k-1} + \alpha(1 - \sigma)\theta\pi_{0,0,k} \quad k \geq 2 \]

5.5 **STeady State Buffer CONTENT**

This section is devoted to find the stationary distribution of the steady state buffer contents \( Q(z) \) at the boundary of an arbitrary slot and the expected number of customers in the system.

**Theorem 5.1**

In steady state, the stationary distribution of the buffer content at the boundary of an arbitrary slot is given by

\[
Q(z) = \frac{T_{0,0}(z) + T_{0,1}(z) + T_{1,0}(z) + T_{1,1}(z)}{T(z)}
\]

where

\[
T_{0,0}(z) = \begin{vmatrix}
 z\beta\sigma\theta\pi_{0,0,0} + z(1 - \gamma)(1 - \mu)\theta\pi_{0,1,0} & a_{12} & a_{13} & a_{14} \\
 z(1 - \beta)\sigma\theta\pi_{0,0,0} + z(1 - \gamma)(1 - \mu)\theta\pi_{0,1,0} & a_{22} & a_{23} & a_{24} \\
 z\beta(1 - \sigma)\theta\pi_{0,0,0} + z\beta\gamma(1 - \mu)\theta\pi_{0,1,0} & a_{32} & a_{33} & a_{34} \\
 z(1 - \beta)(1 - \sigma)\theta\pi_{0,0,0} + z(1 - \beta)\gamma(1 - \mu)\theta\pi_{0,1,0} & a_{42} & a_{43} & a_{44}
\end{vmatrix}
\]
with the stability condition

$$\rho = \frac{1 - \beta}{(2 - \alpha - \beta)(1 - (1 - \mu)(1 - \theta)(1 - \sigma))}$$

and is obtained in Section 5.6.

**Proof:**

Multiplying equations (5.1) - (8.7) by $Z^k$ and summing over k, these equations becomes

$$[z - \beta \sigma(\theta + z(1 - \theta))]Q_{0,0}(z) = z \beta \sigma \theta \tau_{0,0,0} + z \beta (1 - \gamma)(1 - \mu) \theta \tau_{0,1,0} +$$

$$\beta (1 - \gamma)h(z)Q_{0,1}(z) + (1 - \alpha)\sigma(\theta + z(1 - \theta))]Q_{1,0}(z) + (1 - \alpha)(1 - \gamma)h(z)Q_{1,1}(z)$$

(5.9)

$$[1 - \alpha \sigma(\theta + z(1 - \theta))]Q_{1,0}(z) = z(1 - \beta)\sigma \theta \tau_{0,0,0} + z(1 - \beta)(1 - \gamma)(1 - \mu) \theta \tau_{0,1,0} +$$

$$(1 - \beta)\sigma(\theta + z(1 - \theta)]Q_{0,0}(z) + (1 - \beta)(1 - \gamma)h(z)Q_{0,1}(z) + \alpha(1 - \gamma)h(z)Q_{1,1}(z)$$

(5.10)
where $h(z) = (1 - \theta)\mu + \theta(1 - \mu) + z(1 - \theta)(1 - \mu)$

Equations (5.9)-(5.12) can be written as the following system of equations

\[
[z - \beta\gamma h(z)]Q_{0,1}(z) = z\beta(1 - \sigma)\theta\pi_{0,0,0} + z\beta(1 - \mu)\theta\pi_{0,1,0} + \\
\beta(1 - \sigma)[\theta + z(1 - \theta)]Q_{0,0}(z) + (1 - \alpha)(1 - \sigma)[\theta + z(1 - \theta)]Q_{1,0}(z) + \\
(1 - \alpha)\gamma h(z)Q_{1,1}(z)
\]

(5.11)

\[
[1 - \alpha\gamma h(z)]Q_{1,1}(z) = z(1 - \beta)(1 - \sigma)\theta\pi_{0,0,0} + z(1 - \beta)\gamma(1 - \mu)\theta\pi_{0,1,0} + \\
(1 - \beta)(1 - \sigma)[\theta + z(1 - \theta)]Q_{0,0}(z) + \alpha(1 - \sigma)[\theta + z(1 - \theta)]Q_{1,0}(z) + \\
(1 - \beta)\gamma h(z)Q_{0,1}(z)
\]

(5.12)

\[
\text{Where } h(z) = (1 - \theta)\mu + \theta(1 - \mu) + z(1 - \theta)(1 - \mu)
\]

The following assumption is made for further development.
\begin{align*}
a_{11} &= z - \beta \sigma(\theta + z(1 - \theta)), & a_{12} &= -(1 - \alpha)\sigma(\theta + z(1 - \theta)) \\
a_{13} &= -\beta(1 - \gamma)h(z), & a_{14} &= -(1 - \alpha)(1 - \gamma)h(z) \\
a_{21} &= -(1 - \beta)\sigma(\theta + z(1 - \theta)), & a_{22} &= 1 - \alpha\sigma(\theta + z(1 - \theta)) \\
a_{23} &= -(1 - \beta)(1 - \gamma)h(z), & a_{24} &= -\alpha(1 - \gamma)h(z) \\
a_{31} &= -\beta(1 - \sigma)(\theta + z(1 - \theta)), & a_{32} &= -(1 - \alpha)(1 - \sigma)(\theta + z(1 - \theta)) \\
a_{33} &= z - \beta\gamma h(z), & a_{34} &= -(1 - \alpha)\gamma h(z) \\
a_{41} &= -(1 - \beta)(1 - \sigma)(\theta + z(1 - \theta)), & a_{42} &= -\alpha(1 - \sigma)(\theta + z(1 - \theta)) \\
a_{43} &= -(1 - \beta)\gamma h(z), & a_{44} &= 1 - \alpha\gamma h(z)
\end{align*}

Using these assumptions, the equations (5.13) to (5.16) can be written as the matrix form $AX = B$, where

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
 a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
 a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}, \quad X = \begin{bmatrix} Q_{0,0}(z) \\
Q_{1,0}(z) \\
Q_{0,1}(z) \\
Q_{1,1}(z)\end{bmatrix}
\]

\[
B = \begin{bmatrix}
z\beta\sigma\theta\pi_{0,0,0} + z\beta(1 - \gamma)(1 - \mu)\theta\pi_{0,1,0} \\
z(1 - \beta)\sigma\theta\pi_{0,0,0} + z(1 - \beta)(1 - \gamma)(1 - \mu)\theta\pi_{0,1,0} \\
z\beta(1 - \sigma)\theta\pi_{0,0,0} + z\beta\gamma(1 - \mu)\theta\pi_{0,1,0} \\
z(1 - \beta)(1 - \sigma)\theta\pi_{0,0,0} + z(1 - \beta)\gamma(1 - \mu)\theta\pi_{0,1,0}
\end{bmatrix}
\]

The above system can be solved using Cramer’s rule. The unknowns $Q_{0,0}(z)$, $Q_{1,0}(z)$, $Q_{0,1}(z)$, $Q_{1,1}(z)$ give the generating functions of server’s states, which are given by
\[ Q_{0,0}(z) = \frac{T_{0,0}(z)}{T(z)}, \quad Q_{1,0}(z) = \frac{T_{1,0}(z)}{T(z)}, \quad Q_{0,1}(z) = \frac{T_{0,1}(z)}{T(z)}, \quad Q_{1,1}(z) = \frac{T_{1,1}(z)}{T(z)}, \]

where

\[ T_{0,0}(z) = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ z\beta(1-\gamma)(1-\mu)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z(1-\beta)\sigma \theta \pi_{0,0,0} + z(1-\beta)(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z\beta(1-\sigma)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \end{vmatrix} \]

\[ T_{1,0}(z) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ z\beta(1-\gamma)(1-\mu)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z(1-\beta)\sigma \theta \pi_{0,0,0} + z(1-\beta)(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z\beta(1-\sigma)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \end{vmatrix} \]

\[ T_{0,1}(z) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ z\beta(1-\gamma)(1-\mu)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z(1-\beta)\sigma \theta \pi_{0,0,0} + z(1-\beta)(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z\beta(1-\sigma)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \end{vmatrix} \]

\[ T_{1,1}(z) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ z\beta(1-\gamma)(1-\mu)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z(1-\beta)\sigma \theta \pi_{0,0,0} + z(1-\beta)(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \\ z\beta(1-\sigma)\theta \pi_{0,0,0} + z\beta(1-\gamma)(1-\mu)\theta \pi_{0,1,0} \end{vmatrix} \]

\[ T(z) = |A| \]

Now, the Stationary distribution of the steady state buffer contents at the boundary of an arbitrary slot is given by

\[ Q(z) = Q_{0,0}(z) + Q_{0,1}(z) + Q_{1,0}(z) + Q_{1,1}(z) \]

\[ Q(z) = \frac{T_{0,0}(z) + T_{0,1}(z) + T_{1,0}(z) + T_{1,1}(z)}{T(z)} \]
By substituting the equations (5.17) – (5.21) in (5.22), the stationary
distribution of the steady state buffer content at the boundary of an arbitrary
slot is obtained.

where

\[ T_{0,0}(z) + T_{0,1}(z) + T_{1,0}(z) + T_{1,1}(z) = \]

\[ s1 \times [1 + \alpha^2 + \beta^2 - 2\alpha - 2\beta + 2\alpha\beta] \times \]

\[ \left\{ \left[ \theta + z(1-\theta) \right] h^2(z) \left( \gamma - \gamma^2 - \sigma\gamma \right) + \left[ \theta + z(1-\theta) \right]^2 h(z) \left( -1 + \gamma - \sigma^2 - \sigma\gamma + 2\gamma \right) \right\} + \]

\[ s2 \times [1 + \alpha^2 + \beta^2 - 2\alpha - 2\beta + 2\alpha\beta] \times \]

\[ \left\{ \left[ \theta + z(1-\theta) \right] h^2(z) \left( -1 + \gamma + 2\gamma - \gamma^2 - \sigma\gamma \right) + \left[ \theta + z(1-\theta) \right]^2 h(z) \left( \gamma - \sigma^2 - \sigma\gamma \right) \right\} + \]

\[ s3 \times [\theta + z(1-\theta)] h(z) \times \]

\[ \left\{ (1-\alpha - \beta)(-1 - \gamma - \sigma) - (1 - 2\alpha - \beta + \alpha^2 + \alpha\beta)(z - \sigma z) - 2\sigma \gamma z(1 - \alpha - \beta) + \right\} \]

\[ \left\{ \gamma^2 z \left( 2 - 3\alpha - 2\beta + \alpha^2 + \alpha\beta \right) \right\} + \]

\[ s4 \times [\theta + z(1-\theta)] h(z) \times \]

\[ \left\{ (\alpha - \alpha^2 - \alpha\beta)(z - \sigma z - \gamma z) - (1 - 2\alpha - \beta + \beta^2 + \alpha\beta) + \sigma(2 - 2\alpha - 3\beta + \beta^2 + \alpha\beta) \right\} + \]

\( (s4 + z \times s3) \times \]

\[ \left\{ z - \gamma^2 h^2(z)(1 - \alpha - \beta) - [\theta + z(1-\theta)] \sigma^2 (1 - \alpha - \beta) - [\theta + z(1-\theta)](az + \beta) \sigma \right\} - \]

\( (s1 + z \times s2) \times \gamma \beta \times h^2(z)(\alpha z + \beta) - s2 \times (\beta + z(1-\beta)) \times (2\alpha z + \beta) \times (\alpha z + \beta) + \]

\( (1 - \alpha - \beta) \left\{ [\theta + z(1-\theta)] \sigma (\beta + z(1-\beta)) \times s1 + h^2(z) \gamma \right\} + \]

\[ \left\{ s1 \times [\theta + z(1-\theta)] + s2 \times h(z) \right\} \times \left\{ (\beta + z - \beta z) \times \beta + (z + az^2 - az) \times (1 - \beta) \right\}, \]

(5.23)

where

\[ s1 = \left( z\sigma \theta \pi_{0,0,0} + z(1 - \gamma)(1 - \mu) \theta \pi_{0,1,0} \right), \]

\[ s2 = \left( z(1 - \sigma) \theta \pi_{0,0,0} + z\gamma(1 - \mu) \theta \pi_{0,1,0} \right), \]

\[ s3 = \left( z(1 - \beta) \theta \pi_{0,0,0} + z(1 - \beta)(1 - \mu) \theta \pi_{0,1,0} \right), \]

\[ s4 = \left( z\beta \theta \pi_{0,0,0} + z\beta(1 - \mu) \theta \pi_{0,1,0} \right). \]

and
Using the condition \( Q(z) = 1 \) in equation (5.22), we get

\[
T_{0,0}(1) + T_{0,1}(1) + T_{1,0}(1) + T_{1,1}(1) = T(1).
\]

On further simplification, it is obtained that

\[
\pi_{0,0,0} + \pi_{0,1,0} = 1 - \frac{1 - \beta}{(2 - \alpha - \beta)(1 - (1 - \mu)(1 - \theta)(1 - \sigma))}.
\]

which is required for the derivation of the stability condition.
Remark 5.1

Using the normalisation condition $\lim_{z \to 1} Q(z) = 1$ and the roots of $\nabla(z) = 0$, one can find the unknowns $\pi_{0,0,0}$ and $\pi_{0,1,0}$. Since the system contains too many parameters the expressions for the unknown probabilities $\pi_{0,0,0}$ and $\pi_{0,1,0}$ are not given explicitly, but the values are tabulated in section 11 using MATLAB with the specific values for $\alpha$, $\beta$, $\gamma$, $\sigma$, $\theta$ and $\mu$.

5.6 MODIFIED SERVICE TIME

The actual service time that a positive customer receives before departing the system either by service completion or by a negative arrival can be defined as modified service time $S^*$. Its probability mass function is given by

$$P(S^* = k) = P(S_n = k)(1 - \theta)^k(1 - \sigma)^k + P(S \geq k)\theta(1 - \theta)^{k-1}(1 - \sigma)^k, k = 1, 2, \ldots,$$

The probability generating function of the modified service time $S^*(z)$ is derived as

$$S^*(z) = \sum_{k=1}^{\infty} P(S^* = k)z^k$$

$$= \sum_{k=1}^{\infty} \left( P(S_n = k)(1 - \theta)^k(1 - \sigma)^k + P(S \geq k)\theta(1 - \theta)^{k-1}(1 - \sigma)^k \right) z^k$$

$$= \sum_{k=1}^{\infty} (1 - \mu)^{k-1}\mu (1 - \theta)^k(1 - \sigma)^k z^k + \sum_{k=1}^{\infty} (1 - \mu)^{k-1}\theta(1 - \theta)^{k-1}(1 - \sigma)^k z^k$$

$$S^*(z) = \frac{[1-(1-\mu)(1-\theta)(1-\sigma)]z}{1-(1-\mu)(1-\theta)(1-\sigma)z}$$
That is the modified service time has geometric distribution with parameter
\[ \tau = [1 - (1 - \mu)(1 - \theta)(1 - \sigma)] , \]
which means that the positive customer will leave the system with probability \( \tau \) (effective service rate) and stay in the system with the complementary probability \( 1 - \tau \) in each slot.

The average occupancy of the server is defined as the traffic intensity \( (\rho) \) of the system and is obtained as

\[
\rho = \frac{\lambda I}{\tau} = \frac{1 - \beta}{(2 - \alpha - \beta)(1 - (1 - \mu)(1 - \theta)(1 - \sigma))}
\]  

(5.25)

### 5.7 Unfinished Work

The remaining number of slots needed to serve all positive customers present in the system is known as the unfinished work \( (W) \) of the queueing system in steady state. According to Wang et al (2011), the unfinished work \( W \) is given by

\[ W = (C - 1)S + R \]

where \( C \) is the stationary queue length including the customer being served if any, \( S \) is the service time of an arbitrary positive customer and \( R \) is the remainder service time of the positive customer which is being served in server. The probability generating function of \( W \) is obtained using the memory less property of geometrically distributed service time and is given by

\[
W(z) = \frac{Q_{0,0}(S^*(z)) + Q_{0,1}(S^*(z)) + Q_{1,0}(S^*(z)) + Q_{1,1}(S^*(z))}{S^*(z)}
\]

\[
W(z) = \frac{T_{0,0}(S^*(z)) + T_{0,1}(S^*(z)) + T_{1,0}(S^*(z)) + T_{1,1}(S^*(z))}{T(S^*(z))S^*(z)}
\]

(5.26)
5.8 STATIONARY SOJOURN TIME DISTRIBUTION

The sojourn time of an arbitrary positive customer is defined as the total number of slots between the boundary of the arrival slot of a tagged positive customer and the departure instant of this positive customer. In the case of MMBP/G/1 queueing system without negative customers, the probability generating function (PGF) of the stationary sojourn time distribution of the positive customers $D(z)$ is known as (Zhou & Wang, 2008),

$$D(z) = \lim_{n \to \infty} \sum_{j=1}^{\infty} \frac{P(W_n = 1, A_n = 1)}{P(A_n = 1)} z^j$$

where $W_n$ is the unfinished work in the $n^{th}$ slot. In steady state, $\lim_{n \to \infty} W_n = CS + R$ and $\lim_{n \to \infty} P(A_n = 1) = \lambda l$.

Using this, PGF of the stationary sojourn time distribution of the positive customers when the server alternate between good and bad state is derived by

$$D(z) = \frac{Q_{1,0}(S^*(z)) + Q_{1,1}(S^*(z))}{\lambda l}$$

$$D(z) = \frac{Q_{1,0}(S^*(z)) + Q_{1,1}(S^*(z))}{\lambda l} = \frac{1}{\lambda l} \frac{T_{1,0}(S^*(z)) + T_{1,1}(S^*(z))}{T(S^*(z))}$$ (5.27)

5.9 PARTICULAR CASES

1. When $\sigma = 0$, the server is always in Good state. Then, the model becomes a discrete-time single-server infinite-capacity queueing system with correlated arrivals,
geometrically distributed service times and negative customers. Substituting $\sigma = 0$ in equation (5.25) it reduces to

$$\rho = \frac{1 - \beta}{(2 - \alpha - \beta)(1 - (1 - \mu)(1 - \theta))}. \quad (5.28)$$


Also, when the server is always in Good state, that is when $\sigma = 0$, one can get $Q_{1,0}(z) + Q_{1,1}(z) = Q_1(z)$ and $Q_{0,0}(z) + Q_{0,1}(z) = Q_0(z)$. Now equation (5.26) reduces to

$$W(z) = \frac{Q_0(S^*(z)) + Q_1(S^*(z))}{S^*(z)}, \quad (5.29)$$

which coincides with the PGF of the unfinished work of Wang et al (2011).

Similarly, when $\sigma = 0$ equation (5.27) reduces to

$$D(z) = \frac{Q_1(S^*(z))}{\lambda l} \quad \frac{1}{\lambda l} \frac{S^*(z)(1 - \beta)(S^*(z) - h(S^*(z))) \pi_{0,0,0}}{S^*(z) - h(S^*(z))(\beta + \beta S^*(z)) - (1 - \alpha - \beta)h^2(S^*(z))} \quad (5.30)$$

which coincides with the equation of Wang et al (2011).

2. When there is no negative arrival that is if $\theta = 0$, the server is always in Good state. Then the model becomes a discrete time queueing model with first order Markovian arrival process and geometrically distributed service times.
Substituting $\theta = 0$ and $\sigma = 0$ in equation (5.25) it reduces to

$$\rho = \frac{1 - \beta}{(2 - \alpha - \beta)\mu} \quad (5.31)$$

Equation (5.31) is consistent with corresponding traffic intensity of Zhou & Wang (2008).

### 5.10 PERFORMANCE MEASURES

Some performance measures of the proposed model are listed below:

(a) Server’s idle period

Server’s idle period is given by $\pi_{0,0,0} + \pi_{0,1,0}$.

(b) Mean Queue length

Expected number of customers in the queue is obtained by $E(Q) = Q'(1)$.

(c) Average Packet delay

The average packet delay of a positive customer can be obtained using sojourn time distribution, given by $E(D) = \lim_{z \to 1} \frac{d(D(z))}{dz}$.

### 5.11 NUMERICAL ILLUSTRATION

The impact on the main parameters of the performance measures are presented with numerical examples. To study the effect of the arrival of negative customers with the parameter $\theta$, effective service rate $\tau$ and effective arrival rate $\lambda_1$ on mean queue length and mean packet delay, the following are the notations used:
Probability of positive arrival $\alpha$

Probability of no positive arrival $\beta$

Probability of *Good* service $\gamma$

Probability of *Bad* service $\sigma$

Arrival of negative customer is geometric with parameter $\theta$

Service times are geometric with parameter $\mu$

*Good* service rate during an arbitrary slot $\mu l$

Using the software MATLAB, the values of the unknown constants $\pi_{0,0,0}$, $\pi_{0,1,0}$ are obtained. The mean queue length and mean packet delay for different values of the parameter $\mu$ are obtained using the following parameters $\alpha = 0.4$, $\beta = 0.3$, $\gamma = 0.5$, $\sigma = 0.4$, $\theta = 0.2$ and are tabulated in Table 5.1.

It is observed that

- When the service rate increases mean queue length and mean packet delay decreases.

Table 5.2 gives the effect of the parameter $\beta$ (Probability of no positive arrival) on the mean queue length, for different values of the parameter $\alpha$ (Probability of positive arrival). The remaining parameters chosen here is $\gamma = 0.5, \sigma = 0.4, \theta = 0.1, \mu = 0.4$.

From the table, it is observed that

- If the probability of positive arrival $\alpha$ increases, then the mean queue length increases
If the probability of no positive arrival $\beta$ increases, then the mean queue length decreases.

Figure 5.3 and Figure 5.4 gives the impact of the arrival of negative customers with the parameter $\theta$ on mean queue length and mean packet delay for different values of effective arrival rate $\lambda_1$ with the parameters $\beta = 0.4, \sigma = 0.5, \tau = 0.82$.

From the figures it is observed that

- Increase of the negative arrival rate $\theta$ will decrease the mean queue length and mean packet delay
- Increase of effective arrival rate $\lambda_1$ will increase the mean queue length and mean packet delay.

The relationship between the mean queue length (Mean packet delay) and the effective service rate $\tau$ for different values of the effective arrival rate $\lambda_1$ with the fixed parameters $\beta = 0.4, \sigma = 0.5$ are illustrated in Figure 5.5 and Figure 5.6.

It is observed that

- Increase of the effective service rate $\tau$ will decrease the mean queue length and mean packet delay
- Increase of effective arrival rate $\lambda_1$ will increase the mean queue length and mean packet delay.

The impact of effective arrival rate $\lambda_1$ and the probability of no positive arrival $\beta$ on mean queue length and mean packet delay with the fixed parameters $\tau = 0.82, \mu_1 = 0.56$ are shown in Figure 5.7 and Figure 5.8.
It is observed that

- If the probability of no positive arrival $\beta$ increases, then the mean queue length and mean packet delay decrease.

The relationship between the mean queue length (mean packet delay) and effective arrival rate for different values of effective service rate with the fixed parameters $\beta = 0.7, \gamma = 0.6, \sigma = 0.5, \theta = 0.2, \mu_1 = 0.5556$ is given in Figure 5.9 and Figure 5.10.

From the Figures, it is observed that

- The mean queue length and mean packet delay are increasing with the increasing values of effective arrival rate $\lambda_1$.
- The mean queue length and mean packet delay are decreasing with the increasing values of effective service rate $\tau$.

### 5.12 CONCLUSION

In this chapter, a discrete time infinite queueing system with On – Off source, as well as geometrically distributed service times with two states of a server namely Good and Bad and negative customers are analysed using Markovian process approach. Closed-form expressions for stationary distribution of the steady state buffer contents at the boundary of an arbitrary slot, unfinished work and stationary sojourn time distribution are obtained. Particular case of the proposed model has also been discussed. Thereafter the measures of interest are also evaluated with numerical illustration.
Table 5.1  Mean queue length and mean packet delay versus service rate $\mu$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\pi_{0,0,0}$</th>
<th>$\pi_{0,1,0}$</th>
<th>Mean queue length</th>
<th>Mean packet delay</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0262</td>
<td>0.0316</td>
<td>28.1457</td>
<td>50.1813</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0554</td>
<td>0.0674</td>
<td>12.6088</td>
<td>21.0517</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0883</td>
<td>0.1084</td>
<td>7.4339</td>
<td>11.7396</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1256</td>
<td>0.1558</td>
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<td>7.3095</td>
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<td>0.5</td>
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<td>3.2861</td>
<td>4.8046</td>
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<tr>
<td>0.6</td>
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<td>0.3585</td>
<td>1.5006</td>
<td>2.1852</td>
</tr>
<tr>
<td>0.8</td>
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<td>0.4596</td>
<td>0.9385</td>
<td>1.4500</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4390</td>
<td>0.5910</td>
<td>0.4975</td>
<td>0.9164</td>
</tr>
</tbody>
</table>

Table 5.2  Mean queue length versus $\alpha$ (probability of positive arrival) and $\beta$ (probability of no positive arrival)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha$</th>
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<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
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<td>8.6319</td>
<td>10.3710</td>
<td>12.4148</td>
<td>14.8581</td>
<td>17.8389</td>
<td>21.5355</td>
<td>26.2551</td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td>3.8517</td>
<td>5.0943</td>
<td>6.5416</td>
<td>8.2485</td>
<td>10.2854</td>
<td>12.7634</td>
<td>15.8507</td>
<td>19.7848</td>
<td>24.9824</td>
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<td>1.5577</td>
<td>2.7148</td>
<td>4.0795</td>
<td>5.7093</td>
<td>7.6941</td>
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<td>23.2224</td>
</tr>
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<td>0</td>
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<td>2.6222</td>
<td>4.4726</td>
<td>6.8369</td>
<td>9.9576</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>1.9190</td>
<td>8.4474</td>
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Figure 5.3 Mean queue length versus negative arrival rate $\theta$

Figure 5.4 Mean packet delay versus negative arrival rate $\theta$
Figure 5.5 Mean queue length versus effective service rate $\tau$

Figure 5.6 Mean packet delay versus effective service rate $\tau$
Figure 5.7  Mean queue length versus effective arrival rate and $\beta$ (probability of no positive arrival)

Figure 5.8  Mean packet delay versus effective arrival rate and $\beta$ (probability of no positive arrival)
Figure 5.9 Mean queue length versus effective arrival rate

Figure 5.10 Mean packet delay versus effective arrival rate