CHAPTER 3

SUBMANIFOLDS OF AN ALMOST $r$-PARACONTACT RIEMANNIAN MANIFOLD ENDOWED WITH A QUARTER SYMMETRIC SEMI-METRIC CONNECTION

3.1. Introduction

Almost complex and almost contact submanifolds were studied by R. S. Mishra in [9]. Lovejoy S. K. Das et al. considered submanifolds of a Riemannian manifold endowed with a quarter symmetric semi-metric connection in [6]. I. Mihai and K. Matsumoto studied submanifolds of an almost $r$-paracontact Riemannian manifold of P-Sasakian type in [8]. Hypersurfaces of an almost $r$-paracontact Riemannian manifold with quarter symmetric metric connection were studied by M. Ahmad, J. B. Jun and A. Haseeb in [1]. Also M. Ahmad, C. Ozgur and A. Haseeb studied some properties of hypersurfaces of an almost $r$-paracontact Riemannian manifold with quarter symmetric non-metric connection in [2].
Submanifolds of an almost $r$-paracontact Riemannian manifold with a semi-symmetric semi-metric connection were studied by M. Ahmad, J. B. Jun and A. Haseeb in [3]. Motivated by studies of authors in [1], [2], [3], in this chapter we study some properties of Hypersurfaces and Submanifolds of an almost $r$-paracontact Riemannian manifold with a quarter symmetric semi-metric connection.

Let $\nabla$ be a linear connection in a differentiable manifold $M^{n+1}$. The torsion tensor $T$ and the curvature tensor $R$ of $\nabla$ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

In [7], S. Golab introduced the idea of a quarter symmetric linear connection in a differentiable manifold. A linear connection is said to be quarter symmetric connection if its torsion tensor $T$ is of the form

$$T(X,Y) = u(Y)\varphi X - u(X)\varphi Y,$$

where $u$ is a 1-form and $\varphi$ is a (1,1) tensor field.

In this chapter, we study quarter symmetric semi-metric connection in an almost $r$-paracontact Riemannian manifold. We consider hypersurfaces and submanifolds of almost $r$-paracontact Riemannian manifold endowed with a quarter symmetric semi-metric connection. We also obtain Gauss and Codazzi equations for hypersurfaces and curvature tensor and Weingarten equation for submanifolds of almost $r$-paracontact Riemannian manifold with respect to quarter symmetric semi-metric connection.
3.2. Preliminaries

Let $M^{n+1}$ be an $(n+1)$-dimensional differentiable manifold of class $C^\infty$ and $M^n$ be the hypersurface immersed in $M^{n+1}$ by the differential immersion $\tau : M^n \to M^{n+1}$. The differential $d\tau$ of the immersion $\tau$ is denoted by $B$. The vector field $X$ in the tangent space of $M^n$ corresponds to a vector-field $BX$ in $M^{n+1}$. Suppose that the enveloping manifold $M^{n+1}$ is an almost $r$-paracontact Riemannian manifold with metric $\tilde{g}$. Then the hypersurface $M^n$ is also an almost $r$-paracontact Riemannian manifold with induced metric $g$ defined by

$$g(\varphi X, Y) = \tilde{g}(B\varphi X, BY),$$

where $X$ and $Y$ are the arbitrary vector fields and $\varphi$ is a tensor field of type $(1,1)$. If the Riemannian manifolds $M^{n+1}$ and $M^n$ are both orientable, we can choose a unique vector field $N$ defined along $M^n$ such that

$$\tilde{g}(B\varphi X, N) = 0$$

and

$$\tilde{g}(N, N) = 1.$$

We call this vector field $N$ the normal vector field to the hypersurface $M^n$.

We now define a quarter symmetric semi-metric connection $\nabla$ by

$$\nabla X Y = \tilde{\nabla} X Y - \eta^a(\tilde{X})\tilde{\varphi} Y + \tilde{g}(\tilde{\varphi} \tilde{X}, \tilde{Y})\tilde{\xi}_a$$

(3.2.1)

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ tangents to $M^{n+1}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to Riemannian metric $\tilde{g}$, $\eta^a$ is a 1-form and $\tilde{\xi}_a$ is the vector field defined by

$$\eta^a(\tilde{X}) = \tilde{g}(\tilde{\xi}_a, \tilde{X})$$

for an arbitrary vector field $\tilde{X}$ of $M^{n+1}$. Also

$$\tilde{g}(\tilde{\varphi} \tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{\varphi} \tilde{Y}).$$

Let us put
(3.2.2) \[ \xi_{\alpha} = B_{\xi_{\alpha}} + a_{\alpha}N, \]

where \( a_{\alpha} = \eta^{\alpha}(N) \), \( \xi_{\alpha} \) is a vector field and \( a_{\alpha} \) is a function on \( M^n \).

We have the following theorem:

**Theorem 3.2.1.** The connection induced on the hypersurface of an almost \( r \)-paracontact Riemannian manifold with a quarter symmetric semi-metric connection with respect to the unit normal is also quarter symmetric semi-metric connection.

**Proof.** Let \( \tilde{\nabla} \) be the induced connection from \( \tilde{\nabla} \) on the hypersurface with respect to the unit normal \( N \), then we have

(3.2.3) \[ \tilde{\nabla}_{\alpha} BY = B(\tilde{\nabla}_{\alpha} Y) + h(X,Y)N \]

for arbitrary vector fields \( X \) and \( Y \) on \( M^n \), \( h \) being a second fundamental tensor of the hypersurface \( M^n \). Similarly, let \( \nabla \) be connection induced on the hypersurface from \( \tilde{\nabla} \) with respect to the unit normal \( N \), then we have

(3.2.4) \[ \tilde{\nabla}_{\alpha} BY = B(\nabla_{\alpha} Y) + m(X,Y)N \]

for arbitrary vector fields \( X \) and \( Y \) of \( M^n \), \( m \) being a tensor field of type \((0, 2)\) on the hypersurface \( M^n \).

From equation (3.2.1), we get

\[ \tilde{\nabla}_{\alpha} BY = \tilde{\nabla}_{\alpha} BY - \eta^{\alpha}(BX)\phi BY + \tilde{g}(\phi BX, BY)\xi_{\alpha}. \]

Using (3.2.3) and (3.2.4) in the above equation, we find

(3.2.5) \[ B(\nabla_{\alpha} Y) + m(X,Y)N = B(\tilde{\nabla}_{\alpha} Y) + h(X,Y)N - \eta^{\alpha}(X)B\phi Y \]

\[ -\eta^{\alpha}(X)b(Y)N + g(X,Y)(B\xi_{\alpha} + a_{\alpha}N), \]

where

\[ \eta^{\alpha}(BX) = \eta^{\alpha}(X), \]

\[ \phi BX = \phi BX + b(X)N, \]

\[ b(X) = g(X,U), \]

\[ \phi N = BU + KN, \]
and \[ g(B\phi X, BY) = g(\phi X, Y). \]

Comparison of tangential and normal vector fields in (3.2.5) yields,

(3.2.6) \[ \nabla_x Y = \tilde{\nabla}_x Y - \eta^a(X)\phi Y + g(\phi X, Y)\xi_\alpha \]

and

(3.2.7) \[ m(X, Y) = h(X, Y) - \eta^a(X)b(Y) + a_\alpha g(\phi X, Y). \]

Thus, we have

(3.2.8) \[ \nabla_x Y - \nabla_Y X - [X, Y] = \eta^a(Y)\phi X - \eta^a(X)\phi Y. \]

Hence the connection \( \nabla \) induced on \( M^n \) is quarter-symmetric semi-metric connection.

### 3.3. Totally Geodesic and Totally Umbilical Hypersurfaces

We define \( \tilde{\nabla}_B \) and \( \nabla_B \) respectively by

(3.3.1) \[ (\tilde{\nabla}_B)(X, Y) = (\tilde{\nabla}_Y B)(Y) = (\tilde{\nabla}_{\tilde{\nabla}_X B}B)(Y) - B(\tilde{\nabla}_X Y) \]

and

(3.3.2) \[ (\nabla_B)(X, Y) = (\nabla_Y B)(Y) = \tilde{\nabla}_{\tilde{\nabla}_X B}B - B(\nabla_X Y), \]

where \( X \) and \( Y \) being arbitrary vector fields on \( M^n \).

Then (3.2.3) and (3.2.4) take the form

(3.3.3) \[ (\tilde{\nabla}_X B)Y = h(X, Y)N \]

and

(3.3.4) \[ (\nabla_X B)Y = m(X, Y)N. \]

These are Gauss equations with respect to induced connection \( \tilde{\nabla} \) and \( \nabla \) respectively.

Let \( X_1, X_2, X_3, X_4, \ldots, X_n \) be \( n \)-orthonormal vector fields, then the function
\[
\frac{1}{n} \sum_{i=1}^{n} h(X_i, X_i)
\]
is called the mean curvature of \(M^n\) with respect to Riemannian connection \(\nabla\) and
\[
\frac{1}{n} \sum_{i=1}^{n} m(X_i, X_i)
\]
is called the mean curvature of \(M^n\) with respect to the quarter symmetric semi-metric connection \(\nabla\).

Now, we have following definitions:

**Definition 3.3.1.** The hypersurface \(M^n\) is called totally geodesic hypersurface of \(M^{*+1}\) with respect to the Riemannian connection \(\nabla\) if \(h\) vanishes.

**Definition 3.3.2.** The hypersurface \(M^n\) is called totally umbilical with respect to connection \(\nabla\) if \(h\) is proportional to the metric tensor \(g\).

We call \(M^n\) is totally geodesic and totally umbilical with respect to quarter symmetric semi-metric connection \(\nabla\) according as the function \(m\) vanishes and proportional to the metric \(g\) respectively.

Now we have following theorems:

**Theorem 3.3.1.** Let the mean curvature of the invariant hypersurface \(M^n\) with respect to Riemannian connection \(\nabla\) coincides with that of \(M^n\) with respect to \(\nabla\). Then the vector field \(\xi^a\) is tangent to \(M^{*+1}\).

**Proof.** In view of (3.2.7) we have
\[
m(X_i, X_i) = h(X_i, X_i) - \eta^a(X_i) b(X_i) + a_\alpha g(\varphi X_i, X_i).
\]
Summing up for \(i = 1, 2, 3, \ldots, n\) and dividing by \(n\), we obtain
Theorem 3.3.2. The invariant hypersurface $M^n$ will be totally umbilical with respect to Riemannian connection $\tilde{\nabla}$, if and only if it is totally umbilical with respect to the quarter-symmetric semi-metric connection $\nabla$. 

Proof. The proof follows from (3.2.7) easily.

3.4. Gauss, Weingarten and Codazzi Equation

In this section we shall obtain Weingarten equation with respect to the quarter-symmetric semi-metric connection $\tilde{\nabla}$. For the Riemannian connection $\tilde{\nabla}$, these equations are given by

(3.4.1) \[ \tilde{\nabla}_{X} N = -BHX \]

for any vector field $X$ in $M^n$, $H$ being a tensor field of type (1,1) on $M^n$ defined by

(3.4.2) \[ g(HX,Y) = h(X,Y). \]

From equation (3.2.1) and (3.4.1), we get

(3.4.3) \[ \tilde{\nabla}_{X} N = -BMX + AXN , \]

where \[ MX = HX + \eta^n(X)U - b(X)\xi_\alpha \]
and \[ AX = a_\alpha b(X) - kn^n(X). \]

We put $\phi_N = BU + kN$, where $U$ is a vector field in $M^n$ and $k$ is a function on $M^n$. Equation (3.4.3) is Weingarten equation with respect the quarter symmetric semi-metric connection. Let us denote the curvature tensor of $M^{n+1}$ with respect to $\tilde{\nabla}$ by $\tilde{K}$ and that of $M^n$ with respect to $\nabla$ by $K$. Thus
(3.4.4) \[ \tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_\tilde{X} \tilde{Y}_J \tilde{Z} - \tilde{\nabla}_\tilde{Y} \tilde{X}_J \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z} \]

and \[ K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \]

The equation of Gauss is given by

\[ K(X, Y, Z, U) = \tilde{K}(B_X, B_Y, B_Z, B_Z) + h(X, U)h(Y, Z) - h(Y, U)h(X, Z). \]

where

\[ \tilde{K}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{U}) \]

and similar expression for \( K(X, Y, Z, U) \) for \( M^n \).

The equation of Codazzi is given by

\[ (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z) = \tilde{K}(B_X, B_Y, B_Z, N). \]

We shall find the equation of Gauss and Codazzi with respect the quarter symmetric semi-metric connection. The curvature tensor with respect to quarter symmetric semi-metric connection \( \tilde{\nabla} \) of \( M^{n+1} \) is defined by

\[ \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_\tilde{X} \tilde{Y}_J \tilde{Z} - \tilde{\nabla}_\tilde{Y} \tilde{X}_J \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}. \]

Putting \( \tilde{X} = B_X, \tilde{Y} = B_Y \) and \( \tilde{Z} = B_Z \), we get

(3.4.5) \[ \tilde{R}(B_X, B_Y)B_Z = \tilde{\nabla}_{B_X} \tilde{Y}_J B_Z - \tilde{\nabla}_{B_Y} \tilde{X}_J B_Z - \tilde{\nabla}_{[B_X, B_Y]} B_Z. \]

By virtue of (3.2.4), (3.4.5) and (3.2.8), we find

(3.4.6) \[ \tilde{R}(B_X, B_Y)B_Z = B \left\{ R(X, Y)Z + m(X, Z)MY - m(Y, Z)MX \right\} \]

\[ +[(\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z) + m[\eta^a(X)\varphi X - \eta^a(X)\varphi Y, Z] + a_m(m(Y, Z)b(X) - m(X, Z)b(Y))]N. \]

3.5. **Submanifolds of Co-Dimension 2**

Let \( M^{n+1} \) be an \((n+1)\)-dimensional differentiable manifold of differentiability class \( C^\infty \) and \( M^{n-1} \) be \((n-1)\)-dimensional manifold immersed in \( M^{n+1} \) by immersion \( \tau: M^{n-1} \to M^{n+1} \). We denote the
differentiability of the immersion $\tau$ by $B$, so that the vector field $X$ in the tangent space of $M^{n+1}$ corresponds to a vector field $BX$ in that of $M^{n+1}$. Suppose that $M^{n+1}$ is an almost $r$-paracontact Riemannian manifold with metric tensor $\tilde{g}$. If $g$ be the induced metric on submanifold $M^{n+1}$ then

$$\tilde{g} (B\phi X, BY) = g (\phi X, Y)$$

for any arbitrary vector fields $X, Y$ in $M^{n+1}$ [4]. If the manifolds $M^{n+1}$ and $M^{n-1}$ are both orientable such that

$$\tilde{g} (B\phi X, N_1) = \tilde{g} (B\phi X, N_2) = \tilde{g} (N_1, N_2) = 0$$

and

$$\tilde{g} (N_1, N_1) = \tilde{g} (N_2, N_2) = 1$$

for arbitrary vector field $X$ in $M^{n-1}$ [5].

We suppose that the enveloping manifold $M^{n+1}$ admits a quarter symmetric semi-metric connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \tilde{\nabla}_X Y - \tilde{\eta}^a (X) \phi Y + \tilde{g} (\phi \tilde{X}, \tilde{Y}) \tilde{\xi}_a$$

for arbitrary vector fields $\tilde{X}, \tilde{Y}$ in $M^{n+1}$. $\tilde{\nabla}$ denotes the Levi–Civita connection with respect to the Riemannian metric $\tilde{g}$ and $\tilde{\eta}^a$ is a 1-form.

Let us now put

\[(3.5.1) \quad \tilde{\xi}_a = B\xi_a + a_a N_1 + b_a N_2,\]

$\xi_a$ being a vector field in the tangent space on $M^{n-1}$, and $a_a, b_a$ are functions on $M^{n-1}$ defined by

$$\eta^a (N_1) = a_a, \quad \eta^a (N_2) = b_a$$

**Theorem 3.5.1.** The connection induced on the submanifold $M^{n-1}$ of co-dimension two of almost $r$-paracontact Riemannian manifold $M^{n+1}$ with quarter symmetric semi-metric connection $\nabla$ is also quarter symmetric semi-metric connection.
**Proof.** Let $\tilde{\nabla}$ be the connection induced on the submanifolds $M^{n-1}$ from the connection $\nabla$ on the enveloping manifold with respect to unit normals $N_1$ and $N_2$, then we have [1]

\[(3.5.2) \quad \tilde{\nabla}_{\alpha\beta}BY = B(\nabla_X Y) + h(X,Y)N_1 + k(X,Y)N_2\]

for arbitrary vector fields $X, Y$ of $M^{n-1}$, where $h$ and $k$ are second fundamental tensors of $M^{n-1}$. Similarly, if $\nabla$ be connection induced on $M^{n-1}$ from the quarter symmetric semi-metric connection $\tilde{\nabla}$ on $M^{n-1}$, then

\[(3.5.3) \quad \tilde{\nabla}_{\alpha\beta}BY = B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2,\]

$m$ and $n$ being tensor fields of type $(0, 2)$ of the submanifold $M^{n-1}$.

In view of equation (3.2.1), we have

\[\tilde{\nabla}_{\alpha\beta}BY = \tilde{\nabla}_{\alpha\beta}BY - \eta^a(X)B\phi Y + \tilde{g}(\phi BX, BY)\xi_a.\]

In view of (3.5.1), (3.5.2) and (3.5.3), we get

\[(3.5.4) \quad B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2 = B(\tilde{\nabla}_X Y) + h(X,Y)N_1 + k(X,Y)N_2 - \eta^a(X)B\phi Y - \eta^a(X)a(Y)N_1 - \eta^a(X)b(Y)N_2 + g(\phi X,Y)\xi_a + g(\phi X,Y)a_\alpha N_1 + g(\phi X,Y)b_\alpha N_2,\]

where

\[a(X) = g(X,U), \quad b(X) = g(X,V).\]

Comparing tangential and normal vector fields to $M^{n-1}$, we obtain

\[(3.5.5) \quad \nabla_X Y = \tilde{\nabla}_X Y - \eta^a(X)\phi Y + g(\phi X,Y)\xi_a,\]

where $a_\alpha$ and $b_\alpha$ are chosen such that

\[(3.5.6) \quad (a) \quad m(X,Y) = h(X,Y) - \eta^a(X)a(Y) + a_\alpha g(\phi X,Y), \]

\[ (b) \quad n(X,Y) = k(X,Y) - \eta^a(X)b(Y) + b_\alpha g(\phi X,Y).\]

Thus, we find
(3.5.7) \[ \nabla_Y X - \nabla_X Y = \sum_{a} (\alpha^a(Y)\varphi X - \alpha^a(X)\varphi Y). \]

Hence the connection \( \nabla \) induced on \( M^{n-1} \) is quarter symmetric semi-metric connection.

### 3.6. Totally Geodesic and Totally Umbilical Submanifolds

Let \( X_1, X_2, X_3, \ldots, X_{n-1} \) be \( (n-1) \) orthonormal vector fields on the submanifold \( M^{n-1} \). Then the function

\[
\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left\{ h(X_i, X_i) + k(X_i, X_i) \right\}
\]

is mean curvature of \( M^{n-1} \) with respect to the Riemannian connection \( \tilde{\nabla} \) and

\[
\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left\{ m(X_i, X_i) + n(X_i, X_i) \right\}
\]

is the mean curvature of \( M^{n-1} \) with respect to \( \nabla \) [5].

Now we have the following definitions:

**Definition 3.6.1.** If \( h \) and \( k \) vanish separately, the submanifold \( M^{n-1} \) is called totally geodesic with respect to the Riemannian connection \( \tilde{\nabla} \).

**Definition 3.6.2.** The submanifold \( M^{n-1} \) is called totally umbilical with respect to the connection \( \tilde{\nabla} \) if \( h \) and \( k \) are proportional to the metric tensor \( g \).

We call \( M^{n-1} \) is totally geodesic and totally umbilical with respect to the quarter symmetric semi-metric connection \( \nabla \) according as the functions \( m \) and \( n \) vanish separately and are proportional to metric tensor \( g \) respectively.
Theorem 3.6.1. The mean curvature of $M^{n-1}$ with respect to the Riemannian connection $\hat{\nabla}$ coincides with that of $M^{n-1}$ with respect to the connection $\nabla$, it is necessary and sufficient that $\xi_\alpha$ is in the tangent space of $M^{n-1}$.

Proof. In view of (3.5.6) we have

$$m(X_i,X_i)+n(X_i,X_i)=h(X_i,X_i)+k(X_i,X_i)-\eta^a(X_i)a(X_i)-\eta^a(X_i)b(X_i)$$

$$+(a_a+b_a)g(\phi X_i,X_i).$$

Summing up for $i=1,2,3,...,(n-1)$ and dividing by $2(n-1)$, we get

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left( m(X_i,X_i)+n(X_i,X_i) \right) = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left( h(X_i,X_i)+k(X_i,X_i) \right)$$

if and only if $a_a=b_a=0$ and $a=b=0$.

Hence in consequence of (3.2.2) it follows that $\vec{\xi}_\alpha = B\xi_\alpha$. Thus the vector field $\xi_\alpha$ is in the tangent space of $M^{n-1}$.

Thus our assertion is proved.

Theorem 3.6.2. The submanifold $M^{n-1}$ is totally umbilical with respect to the Riemannian connection $\hat{\nabla}$ if and only if it is totally umbilical with respect to the quarter symmetric semi-metric connection $\nabla$.

Proof: The proof follows easily from equations ((3.5.6) (a) and (b)).

3.7. Curvature Tensor and Weingarten Equations

For the Riemannian connection $\hat{\nabla}$, the Weingarten equations are given by [9]

$$\hat{\nabla}_{\hat{X}} N_1 = -BH N_1 + 1(X) N_2$$

and

$$\hat{\nabla}_{\hat{X}} N_2 = BH N_2 + 1(X) N_1$$
where \( H \) and \( K \) are tensor fields of type (1,1) such that

\[
(3.7.2) \quad \begin{align*}
(a) \quad g(HX, Y) &= h(X, Y), \\
(b) \quad g(KX, Y) &= K(X, Y).
\end{align*}
\]

Also, using (3.2.1) and (3.7.1) (a), we get

\[
\tilde{\nabla}_{\alpha} N_1 = -B(HX - a(X)\xi_{\alpha} + \eta^a(X)U) + (1(X) - \eta^a(X)K_2 + b_\alpha a(X))N_2 + a_\alpha a(X)N_1,
\]

which implies that

\[
(3.7.3) \quad \tilde{\nabla}_{\alpha} N_1 = -BM_1X + L_1(X)N_2 + L_2(X)N_1,
\]

where

\[
M_1X = HX - a(X)\xi_{\alpha} + \eta^a(X)U,
\]

\[
L_1(X) = 1(X) - \eta^a(X)K_2 + b_\alpha a(X) \quad \text{and} \quad L_2(X) = a_\alpha a(X).
\]

Similarly, from (3.2.1) and (3.7.1) (b), we find

\[
(3.7.4) \quad \tilde{\nabla}_{\alpha} N_2 = -B(HX - b(X)\xi_{\alpha} + \eta^a(X)V) + (1(X) - \eta^a(X)K_1 + a_\alpha b(X))N_2 + b_\alpha b(X)N_1,
\]

where

\[
\begin{align*}
\tilde{\nabla}_{\alpha} N_2 &= -BM_2X + Q_1(X)N_2 + Q_2(X)N_1, \\
M_2X &= HX - b(X)\xi_{\alpha} + \eta^a(X)V, \\
Q_1(X) &= 1(X) - \eta^a(X)K_1 + a_\alpha b(X)
\end{align*}
\]

and

\[
Q_2(X) = b_\alpha b(X).
\]

Equations (3.7.3) and (3.7.4) are Weingarten equations with respect to the quarter symmetric semi-metric connection \( \tilde{\nabla} \).
3.8. Riemannian Curvature Tensor for Quarter Symmetric Semi-Metric Connection

Let \( \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} \) be the Riemannian curvature tensor of the enveloping manifold \( M^{n+1} \) with respect to the quarter symmetric semi-metric connection \( \tilde{\nabla} \), then we have

\[
\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_\tilde{X}\tilde{\nabla}_\tilde{Y}\tilde{Z} - \tilde{\nabla}_\tilde{Y}\tilde{\nabla}_\tilde{X}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}.
\]

Replacing \( \tilde{X} \) by \( BX \) and \( \tilde{Y} \) by \( BY \) and \( \tilde{Z} \) by \( BZ \), we get

\[
\tilde{R}(BX,BY)BZ = \tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ - \tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ - \tilde{\nabla}_{[BX,BY]}BZ.
\]

Using (3.7.3), (3.7.4) and (3.5.3) and (3.5.7), we obtain

\[
\tilde{R}(BX,BY)BZ = B[R(X,Y)Z - m(Y,Z)M_{1}X - n(Y,Z)M_{2}Y + m(X,Z)M_{1}Y
\]

\[
+ n(X,Z)M_{2}Y + \{(\tilde{\nabla}_\tilde{X}m)(Y,Z) - (\tilde{\nabla}_\tilde{Y}m)(X,Z)\}N_{1}
\]

\[
+ \{(\tilde{\nabla}_\tilde{X}n)(Y,Z) - (\tilde{\nabla}_\tilde{Y}n)(X,Z)\}N_{2} + m\eta^{\alpha}(Y)\phi X
\]

\[
- \eta^{\alpha}(X)\phi Y,Z)N_{1} + n\{\eta^{\alpha}(Y)\phi X - \eta^{\alpha}(X)\phi Y,Z\}N_{2}
\]

\[
+ m(Y,Z)L_{1}(X)N_{2} + m(Y,Z)L_{2}(X)N_{2} + n(Y,Z)Q_{1}(X)N_{1}
\]

\[
+ n(Y,Z)Q_{2}(X)N_{2} - m(X,Z)L_{1}(Y)N_{2} - m(X,Z)L_{2}(Y)N_{1}
\]

\[
- n(X,Z)Q_{1}(Y)N_{1} - n(X,Z)Q_{2}(Y)N_{2},
\]

where \( R(X,Y,Z) \) being the Riemannian curvature tensor of the submanifold with respect to the connection \( \nabla \).
References


5. CHEN, B.Y., (1973), ‘Geometry of submanifold’, Marcel Dekker, New York,

