Chapter 2

An Adaptive Cubic Spline Approach for Solving a Singularly Perturbed Boundary Value Problem

2.1 Introduction

We consider the following self adjoint singularly perturbed boundary value problem:

\[ L u = \varepsilon u'' - q(x)u = r(x), \]

\[ u(a) = \alpha_0, \quad u(b) = \alpha_1, \]  \hspace{1cm} (1.1)

where \( q(x), r(x) \) are smooth and bounded real functions, \( \alpha_0 \) and \( \alpha_1 \) are given constants and \( \varepsilon \) is a parameter such that \( 0 < \varepsilon << 1 \). The approximate solution of self adjoint boundary value problems with a small parameter affecting the highest order derivative of the differential equation is described. It is well known fact that the solution of the singularly perturbed boundary value problem has a multi-scale character; that is, there is a thin layer where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly. Such kind of problems arise in various fields of science and engineering such as Nuclear engineering, control theory, fluid dynamics, fluid mechanics, elasticity, quantum mechanics, optical control, hydrodynamics, convection diffusion processes etc. The non self adjoint boundary value problem has a boundary layer at one end depending upon the sign of \( r(x) \), while in self adjoint boundary value problems the boundary layers occur at both the ends. The numerical treatment of self adjoint singularly perturbed boundary value problems has many computational difficulties, and to
resolve the problem various special purpose methods have been proposed to develop for accurate numerical solution. Some of these methods are collocation methods [10, 32], finite difference methods [56], finite element methods [60], spline approximations [12] etc. The survey article by Kadalbajoo and Patidar [34], gives detailed information about the various numerical methods to solve singularly perturbed two point boundary value problems. Numerical solution of singularly perturbed problems using splines has been discussed by various authors [11, 13, 29].

In this chapter, we have derived a uniformly convergent uniform mesh difference scheme using adaptive cubic spline [38] for the solution of (1.1)-(1.2). Kadalbajoo and Bawa [33] give a second order method which becomes a special case of our method for uniform mesh. The advantages of our second and fourth order methods are higher accuracy with the same computational effort. Analysis of our method shows that it has second order convergence for arbitrary $A_1, A_2, A_3, A_4$ such that $A_1 + A_2 + A_3 + A_4 = 1$ and $A_2 = A_3$ and fourth order convergence for $A_2 = A_3 = 1/12$ and $A_1 + A_4 = 10/12$.

In section (2.2), we give a brief derivation of adaptive cubic spline and some spline relations. In section (2.3) we present the formulation of our method. Section (2.4) includes convergence analysis of the method and Section (2.5) contains the numerical results and discussions that demonstrate the theoretical behaviour of our method.

2.2 Adaptive Cubic Spline and its Relations

We present the adaptive cubic spline interpolant $S(x) \in C^2[a,b]$ and derive the spline relations to be used for our method.
Consider a uniform mesh \( D = \{x_i, i = 0(1)N\} \) with mesh size \( h \).

A function \( S_\Delta(x, \omega) \) of class \( C^2[a, b] \) which interpolates \( u(x) \) at the mesh point \( x_i \) depends on the parameter \( \omega \), reduces to cubic spline \( S_\Delta(x) \) in \( [a, b] \) as \( \omega \rightarrow 0 \) is termed as parametric cubic spline function. Since the parameter \( \omega \) can occur in \( S_\Delta(x, \omega) \), in many ways, such a spline is not unique.

If the function \( S_\Delta(x, \omega) \) is a parametric cubic spline satisfying the differential equation

\[
\alpha S''_\Delta(x, \omega) - \beta S'_\Delta(x, \omega) = \frac{(x_{i+1} - x)}{h}(\alpha M_i - \beta m_i) + \frac{(x - x_i)}{h}(\alpha M_{i-1} - \beta m_{i-1}), \quad (2.1)
\]

where \( x_i \leq x \leq x_{i+1} \), \( \alpha \) and \( \beta \) are constants.

\[
S'_\Delta(x_i, \omega) = m_i, \quad S''_\Delta(x_i, \omega) = M_i, \quad h = x_{i+1} - x_i \text{ and } \omega > 0
\]

then \( S_\Delta(x, \omega) \) is termed as ‘adaptive cubic spline’.

Solving (2.1) and using the interpolatory conditions \( S_\Delta(x_i, \omega) = u_i, \ S_\Delta(x_{i+1}, \omega) = u_{i+1}, \)

we have

\[
S_\Delta(x, \omega) = A_{i+1} + B_{i+1}e^{\omega} - \frac{h^2}{\omega^3}[p_2(\omega e^{\omega})M_{i+1} + \frac{\omega}{h}(M_i - \frac{\omega}{h} m_i)] + \frac{h^2}{\omega^3}[p_2(-\omega e^{\omega})M_i + \frac{\omega}{h}(M_i - \frac{\omega}{h} m_i)],
\]

(2.2)

where \( z = \frac{(x - x_i)}{h} \), \( \omega = \frac{\beta h}{\alpha} \), \( \bar{z} = 1 - z \), \( p_2(z) = 1 + z + \frac{z^2}{2} \),

\[
A_{i+1}(e^{\omega} - 1) = -u_{i+1} + u_i e^{\omega} - \frac{h^2}{\omega^3}[p_2(\omega)(M_{i+1} - \frac{\omega}{h} m_{i+1})] - \frac{h^2}{\omega^3}[p_2(-\omega)(M_i - \frac{\omega}{h} m_i)],
\]

\[
B_{i+1}(e^{\omega} - 1) = u_{i+1} - u_i + \frac{h^2}{\omega^3}[\left(\frac{\omega}{2} + 1\right)(M_{i+1} - \frac{\omega}{h} m_{i+1}) + \left(\frac{\omega}{2} - 1\right)(M_i - \frac{\omega}{h} m_i)].
\]
The function \( S_\omega(x, \omega) \) on the interval \([x_{i-1}, x_i] \) is obtained by replacing \( i \) with \( i-1 \) in equation (2.2). The condition of continuity of first or second order derivative of \( S_\omega(x, \omega) \) at \( x_i \) yields following equation

\[
(M_{i+1} - \frac{\omega}{h} m_{i+1})[e^{-\omega} p_2(\omega) - 1] + (M_i - \frac{\omega}{h} m_i)[e^{-\omega} \{(p_2(-\omega)e^\omega - 1) - (p_2(\omega) - e^\omega)\}]
- (M_{i-1} - \frac{\omega}{h} m_{i-1})e^{-\omega}[p_2(-\omega)e^\omega - 1] = -\frac{\omega^3}{h^2} [e^{-\omega}u_{i+1} - (1 + e^{-\omega})u_i + u_{i-1}]
\]

(2.3)

The continuity of first derivative of \( S_\omega(x, \omega) \) implies, that of second derivative and vice-versa. We obtain following relations:

(i) \( m_{i-1} = -h(A_1 M_{i-1} + A_2 M_i) + \frac{(u_i - u_{i-1})}{h} \),

(ii) \( m_i = h(A_3 M_{i-1} + A_4 M_i) + \frac{(u_i - u_{i-1})}{h} \), \hspace{1cm} (2.4)

(iii) \( \frac{\gamma h}{2\omega} M_{i-1} = -(A_1 m_{i-1} + A_2 m_i) + B_1 (u_i - u_{i-1}) \),

(iv) \( \frac{\gamma h}{2\omega} M_i = (A_1 m_{i-1} + A_4 m_i) + B_2 (u_i - u_{i-1}) \).

After simplification, we can easily obtain

\[
h^2 [A_1 M_{i-1} + (A_1 + A_4) M_i + A_3 M_{i-1}] = \delta^2 u_i ,
\]

(2.5)

where,

\[
A_1 = \frac{1}{4} (1 + \gamma) + \frac{\gamma}{2\omega},
\]

\[
A_2 = \frac{1}{4} (1 - \gamma) - \frac{\gamma}{2\omega},
\]

\[
A_3 = \frac{1}{4} (1 + \gamma) - \frac{\gamma}{2\omega},
\]
\[ A_i = \frac{1}{4} (1 - \gamma) + \frac{\gamma}{2 \omega}, \]

\[ B_1 = \frac{1}{2} (1 - \gamma), \]

\[ B_2 = -\frac{1}{2} (1 + \gamma), \]

\[ \gamma = \coth \frac{\omega}{2} - \frac{2}{\omega} \quad \text{and} \quad \delta^2 u_i = u_{i+1} - 2u_i + u_{i-1}. \]

For \( \omega \rightarrow 0 \) (i.e. \( \frac{\beta h}{\alpha} \rightarrow 0 \)) then we have \( \gamma = 0, \frac{\gamma}{\omega} = \frac{1}{6}, A_i = A_4 = \frac{1}{3}, A_2 = A_3 = \frac{1}{6} \) and the spline function given by (2.2) reduces into cubic spline.

### 2.3 Derivation of the Method

Let \( x_0 = a, \quad x_N = b, \quad x_i = a + ih, \quad h = \frac{b - a}{N}. \)

Substituting \( \epsilon M_i q(x_i)u_i + r(x_i) \) in equation (2.5), we arrive at following linear system which may be solved to get the approximations \( u_1, u_2, u_3, \ldots, u_{N-1} \) of the solutions \( u(x) \) at \( x_1, x_2, x_3, \ldots, x_{N-1} \),

\[
(-\epsilon + h^2 A_i q_{i-1})u_{i-1} + (2\epsilon + h^2 (A_i + A_4)q_i)u_i + (-\epsilon + h^2 A_i q_{i+1})u_{i+1} =
- h^2 [A_2 r_{i+1} + (A_i + A_4)r_i + A_3 r_{i-1}],
\]

\[ i = 1 \text{(1)} N-1, \quad (3.1) \]

with \( u(a) = \alpha_0, \quad u(b) = \alpha_1, \quad q_i = q(x_i), \quad r_i = r(x_i), \quad i = 0 \text{(1)} N. \)

**Remark 1:** For \( A_i = A_4 = \frac{1}{3}, A_2 = A_3 = \frac{1}{6} \), our scheme reduces to cubic spline and hence reduces to Kadalbajoo and Bawa’s method [33] for uniform mesh.
Remark 2: For $\varepsilon = 1$ (regular problem), uniform mesh, $A_1 = A_4 = \frac{1}{3}$, $A_2 = A_3 = \frac{1}{6}$ our method reduces to the well known Bickley scheme [14] for regular problem $u'' = q(x)u + r(x)$ with $u(a) = \alpha_0$ and $u(b) = \alpha_1$.

### 2.4 Convergence of the Method

Putting the tridiagonal system (3.1) in matrix vector form

$$AU + h^2 BR = C,$$

(4.1)

where $A = (a_{i,j})$ is a tri diagonal and diagonally dominant matrix of order $N-1$, with

$$a_{i,i+1} = \text{coefficient of } u_{i+1} \text{ in (3.1)}, \quad i = 1 \ (1) \ N-2,$$

$$a_{i,i} = \text{coefficient of } u_i \text{ in (3.1)}, \quad i = 1 \ (1) \ N-1,$$

$$a_{i,i-1} = \text{coefficient of } u_{i-1} \text{ in (3.1)}, \quad i = 2 \ (1) \ N-1,$$

and $R = [r_i]^T = (r_1, r_2, r_3, \ldots, r_{N-1})^T,$

i = 1 \ (1) \ N-1.

and $U = (u_1, u_2, \ldots, u_{N-1})^T$.

The tri diagonal matrix B is given by

$$B = \begin{bmatrix}
A_1 + A_4 & A_2 & 0 \\
A_3 & A_1 + A_4 & A_2 \\
0 & A_3 & A_1 + A_4 & A_2 \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
A_3 & A_1 + A_4 & A_2 \\
0 & A_3 & A_1 + A_4
\end{bmatrix}$$
and \( C = (c_1, 0, 0, \ldots, 0, c_{N-1})^T \),

where \( c_i = (\varepsilon - h^2 A_1 q_0) \alpha_0 - h^2 A_3 r_0, \)

\( c_{N-1} = (\varepsilon - h^2 A_2 q_N) \alpha_1 - h^2 A_3 r_N \).

Also we have,

\[
A \overline{U} + h^2 BR = T(h) + C,
\]

where \( \overline{U} = (u(x_1), u(x_2), \ldots, u(x_{N-1}))^T \) denotes the exact solution vector and

\[
T(h) = (T_1(h), T_2(h), \ldots, T_{N-1}(h))^T \] is the local truncation error vector, where,

\[
T_i(h) = \varepsilon h^4 (A_2 - \frac{1}{12}) u^{(4)}(\xi_i) + \varepsilon h^6 (A_2 - \frac{1}{360}) u^{(6)}(\xi_i),
\]

\[
x_{i-1} < \xi_i < x_{i+1},
\]

(4.3)

for any choice of \( A_1, A_2, A_3, A_4 \) such that \( A_1 + A_2 + A_3 + A_4 = 1 \) and \( A_2 = A_3 \) except \( A_2 = 1/12 \).

For \( A_2 = A_3 = 1/12 \) and \( A_1 + A_4 = 10/12 \),

\[
T_i(h) = \left( \frac{\varepsilon h^6}{240} \right) u^{(6)}(\xi_i),
\]

\[
x_{i-1} < \xi_i < x_{i+1}.
\]

(4.4)

From (4.1) and (4.2) we get

\[
A(\overline{U} - U) = T(h),
\]

\[
AE = T(h),
\]

\[
E = \overline{U} - U = (e_1, e_2, \ldots, e_{N-1})^T.
\]

(4.5)

Clearly,

\[
S_1 = \sum_{j=1}^{N-1} a_{i,j} = \varepsilon + (A_i + A_4) q_i h^2 + A_2 q_2 h^2,
\]
$S_i = \sum_{j=1}^{N-1} a_{i,j} = [A_3 q_{i-1} + (A_i + A_4)q_i + A_2 q_{i+1}] h^2 = h^2 B_i, \quad i = 2 (1) N-2.$

$S_{N-1} = \sum_{j=1}^{N-1} a_{N-1,j} = \varepsilon + (A_i + A_4)q_{N-1} h^2 + A_3 q_{N-1} h^2.$

We can choose $h$ sufficiently small so that the matrix $A$ is irreducible and monotone \cite{25}. It follows that $A^{-1}$ exists and its elements are nonnegative.

Hence from equation (4.5) we have

$$E = A^{-1}T(h). \quad (4.6)$$

Also, from the theory of matrices, we have

$$\sum_{i=1}^{N-1} a_{k,i} S_i = 1, \quad k = 1 (1) N-1, \quad (4.7)$$

where $a_{k,i}$ is $(k, i)$ element of matrix $A^{-1}$.

Therefore,

$$\sum_{i=1}^{N-1} a_{k,i} \leq \frac{1}{\min_{i \leq j \leq N-1} S_j} = \frac{1}{h^2 B_{i_0}} \leq \frac{1}{h^2 |B_{i_0}|} \quad (4.8)$$

for some $i_0$ between 1 and $N-1$.

From (4.3), (4.6) and (4.7), we have

$$e_j = \sum_{i=1}^{N-1} a_{j,i} T_i(h), \quad j = 1 (1) N-1. \quad (4.9)$$

and therefore

$$|e_j| \leq \frac{Kh^2}{|B_{i_0}|}, \quad j = 1 (1) N-1. \quad (4.10)$$

where $K$ is independent of $h$.

Therefore,

$$\|E\| = O(h^2). \quad (4.11)$$

However, for the choice of parameter $A_2 = A_3 = 1/12$ and $A_1 + A_4 = 10/12$, \ldots
\[ |e_j| \leq \frac{Kh^4}{B_B}, \quad j = 1 \text{ to } N-1. \tag{4.12} \]

Therefore,
\[ \|E\| = O(h^4). \tag{4.13} \]

We summarize the above results in the following theorem:

**Theorem 4.1:**

The method given by equation (3.1) for solving the boundary value problem (1.1)-
(1.2) for \( q(x)>0 \) and sufficiently small \( h \) and \( \varepsilon \), gives a second order convergent solution for arbitrary \( A_1, A_2, A_3, A_4 \) such that \( A_1 + A_2 + A_3 + A_4 = 1 \) and \( A_2 = A_3 \) and a fourth order convergent solution for \( A_2 = A_3 = 1/12 \) and \( A_1 + A_4 = 10/12 \).

**2.5 Numerical Illustrations and Discussion**

We have implemented our method on one problem which supports the theoretical analysis for second and fourth - order convergence. The maximum error at the nodal points, \( \max |u(x_i) - u| \) is tabulated in Tables 1 and 2 for different values of the parameters \( \varepsilon, N, A_1, A_2, A_3 \) and \( A_4 \).

**Example 5.1 (Doolan et. al.) [18]**

\[
e u'' = u + \cos^2(\pi x) + 2\varepsilon \pi^2 \cos(2\pi x),
\]

\[ u(0) = u(1) = 0. \]

The exact solution is given by
\[ u(x) = \frac{\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})}{1 + \exp(-1/\sqrt{\varepsilon})} \left[ 1 + \cos^2(\pi x) \right]. \]

Since \( q=1>0 \), the boundary layer exists at both ends.
# An Adaptive Cubic Spline Approach

Table 1: Maximum absolute errors for Second order method

<table>
<thead>
<tr>
<th>(A_2 = A_3)</th>
<th>(A_1 + A_4)</th>
<th>(\varepsilon = 10^{-5})</th>
<th>(\varepsilon = 10^{-8})</th>
<th>(\varepsilon = 10^{-10})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(N = 100)</td>
<td>(N = 200)</td>
<td>(N = 250)</td>
<td></td>
</tr>
<tr>
<td>1/18</td>
<td>8/9</td>
<td>1.444E-03</td>
<td>6.2234E-02</td>
<td>6.2738E-02</td>
</tr>
<tr>
<td>1/19</td>
<td>17/19</td>
<td>1.0213E-03</td>
<td>5.852E-02</td>
<td>5.9020E-02</td>
</tr>
<tr>
<td>1/14</td>
<td>6/7</td>
<td>1.5282E-02</td>
<td>8.3364E-02</td>
<td>8.3911E-02</td>
</tr>
<tr>
<td>1/6[33]</td>
<td>2/3</td>
<td>1.197E-01</td>
<td>2.6683E-01</td>
<td>2.6793E-01</td>
</tr>
<tr>
<td>1/24</td>
<td>22/24</td>
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<td>4.507E-02</td>
<td>4.554E-02</td>
</tr>
<tr>
<td>1/30</td>
<td>28/30</td>
<td>1.669E-02</td>
<td>3.529E-02</td>
<td>3.575E-02</td>
</tr>
</tbody>
</table>

Table 2: Maximum absolute errors for Fourth order method

<table>
<thead>
<tr>
<th>(A_1 + A_4 = 10/12,) (A_2 = A_3 = 1/12)</th>
<th>(\varepsilon = 10^{-3})</th>
<th>(\varepsilon = 10^{-4})</th>
<th>(\varepsilon = 10^{-5})</th>
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<tr>
<td>(N = 50)</td>
<td>1.1727E-04</td>
<td>8.318E-03</td>
<td>6.81E-02</td>
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<tr>
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<td>7.4138E-04</td>
<td>2.61E-02</td>
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<td>(N = 200)</td>
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<td>4.744E-05</td>
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<tr>
<td>(N = 250)</td>
<td>1.960E-07</td>
<td>1.915E-05</td>
<td>1.814E-03</td>
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