Chapter 5

EXPONENTIAL SPLINE APPROACH FOR THE SOLUTION OF SYSTEM OF FOURTH ORDER BOUNDARY VALUE PROBLEMS

5.1 Introduction

We consider the system of fourth order two point boundary value problem of the form

\[
\begin{align*}
\quad & u^{(4)}(x) = \begin{cases} 
  f(x), & a \leq x \leq c, \\
  f(x) + g(x)u(x) + r, & c \leq x \leq d, \\
  f(x), & d \leq x \leq b,
\end{cases} \\
\quad & (1.1)
\end{align*}
\]

along with the following boundary conditions

\[
\begin{align*}
\quad & u(a) = u(b) = A_1, \quad u(c) = u(d) = A_2, \\
\quad & u''(a) = u''(b) = B_1, \quad u''(c) = u''(d) = B_2, \\
\end{align*}
\]

where \( f(x) \) and \( g(x) \) are continuous functions on \([a,b]\) and \([c,d]\) respectively and \( A_1, A_2, B_1 \) and \( B_2 \) are finite real constants.

The mathematical formulation of the obstacle, unilateral, contact and equilibrium problems arising in the field of elasticity, structural analysis, transportation science, economics and optimization results in the form of system (1.1)-(1.2). Kikuchi and Oden [40] proved that the problem of equilibrium of linearly elastic bodies in contact with a rigid frictionless foundation can be studied in the framework of variational
inequalities. The general variational inequalities can be solved by penalty function and projection function methods [3-5, 8, 17, 21, 22, 40, 41, 48, 59]. Penalty function methods and Projection function methods are not very efficient, due to the instability created in the penalty function and the difficulty to find the projections in the projection function methods. In general, it is not possible to obtain the analytical solution of system (1.1)-(1.2) for arbitrary choices of \( f(x) \) and \( g(x) \).

Usmani [64-65] proposed the methods to solve the problem of bending a rectangular clamped beam of length \( l \) resting on an elastic foundation:

\[
[L + \left( \frac{K}{D} \right)]w = D^{-1} q(x), \quad \text{where} \quad L = \frac{d^4}{dx^4} \tag{1.3}
\]

with boundary conditions

\[ w(0) = w(l) = 0, \]

along with \( w'(0) = w'(l) = 0 \) or \( w''(0) = w''(l) = 0 \), \( \tag{1.4} \)

where \( w \) is the vertical deflection of the beam, \( D \) is the flexural rigidity of the beam and \( K \) is the spring constant of the elastic foundation and the load \( q(x) \) acts vertically downwards per unit length of the beam. In order to develop numerical methods for obtaining smooth, approximate solution of system of fourth order boundary value problem, we apply non-polynomial spline which has the polynomial and exponential parts. Zahra used exponential spline for nonlinear fourth order two point boundary value problems [73], for a class of two point boundary value problems [74] and for the solution of obstacle problems [75]. It belongs to quintic non-polynomial spline function space:
$T_5 = \text{span}\{1, x, x^2, x^3, e^{(kx)}, e^{-(kx)}\}$

where, $k$ is the free parameter which can be real or purely imaginary and will be used to raise the accuracy of the method.

As $k \to 0$, $T_5$ reduces to $\text{span}\{1, x, x^2, x^3, x^4, x^5\}$. (1.5)

In this chapter, we have used exponential quintic spline function to develop the new numerical method for the solution of system (1.1)-(1.2). The advantage of the new method is its higher accuracy with the same computational effort than the other known methods. Siraj-ul-Islam et al. [28] proposed a quartic non-polynomial spline solution of a system of fourth order boundary value problem at mid knots. In section 5.2, non-polynomial spline function is presented. Section 5.3 describes the numerical method and truncation errors. Section 5.4 is devoted to the matrix formulation of the method and in section 5.5, we discuss the applications and section 5.6 deals with the numerical illustration of the method which shows that our method is more efficient than other known methods.

5.2 Exponential Quintic Spline Function

To develop this method, mesh points are taken at off-step points. This approach reduces the error in the solution around the points where the solution satisfies extra continuity conditions. The interval $[a, b]$ is divided into $N+1$ equal subintervals, s.t.

$\Delta : a = x_0 < x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{\frac{N-1}{2}} < x_{\frac{N}{2}} = b,$
where \( x_{i \frac{1}{2}} = a + (i - \frac{1}{2})h \), \( i = 1, 2, \ldots N \),
\[
(2.1)
\]

and \( h = \frac{b - a}{N} \).

Without loss of generality we take,
\[
c = \frac{3a + b}{4} \quad \text{and} \quad d = \frac{a + 3b}{4}.
\]

For each of its segment the exponential quintic spline function \( P_i(x) \) has the form
\[
P_i(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i (x-x_i)^3 + d_i (x-x_i)^2 + e_i (x-x_i) + f_i,
\]
\[
i = 0, 1, 2 \ldots N.
\]
\[
(2.2)
\]

where \( a_i, b_i, c_i, d_i, e_i \) and \( f_i \) are real constants and \( k \) is a free parameter which can be real or purely imaginary. Equation (2.2) reduces to quintic spline in \([a, b]\) when \( k \to 0 \) as given in equation (1.5).

Let \( u(x) \) be the exact solution of system (1.1) – (1.2) and let \( S_i \) be an approximation to \( u_i = u(x_i) \) obtained by a segment \( P_i(x) \) passing through \((x_i, S_i)\) and \((x_{i+1}, S_{i+1})\) with
\[
(i) \ S(x) = P_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, 2 \ldots N;
\]
\[
(ii) \ S(x) \in C^4[a, b].
\]

To derive the expressions for the coefficients in Equation (2.2), in terms of the function values \( u_{i \frac{1}{2}}, u_{i+\frac{1}{2}} \) and second and fourth order spline derivatives, we define
(i) \( P_i(x_{i+\frac{1}{2}}) = u_{i+\frac{1}{2}} \),

(ii) \( P_i(x_{i-\frac{1}{2}}) = u_{i-\frac{1}{2}} \),

(iii) \( P_i''(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}} \),

(iv) \( P_i''(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}} \),

(v) \( P_i^{(4)}(x_{i+\frac{1}{2}}) = F_{i+\frac{1}{2}} \),

(vi) \( P_i^{(4)}(x_{i-\frac{1}{2}}) = F_{i-\frac{1}{2}} \). \hspace{1cm} (2.4)

From algebraic manipulations, we get the following expressions:

\[ a_i = \frac{h^4}{32\theta^4 \sinh 2\theta} \left[ e^{\theta} F_{i+\frac{1}{2}} - e^{-\theta} F_{i-\frac{1}{2}} \right], \]

\[ b_i = \frac{h^4}{32\theta^4 \sinh 2\theta} \left[ e^{\theta} F_{i-\frac{1}{2}} - e^{-\theta} F_{i+\frac{1}{2}} \right], \]

\[ c_i = \frac{1}{6h} \left[ M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}} \right] - \frac{h}{24\theta^2} \left[ F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right], \]

\[ d_i = \frac{1}{4} \left[ M_{i+\frac{1}{2}} + M_{i-\frac{1}{2}} \right] - \frac{h^2}{16\theta^2} \left[ F_{i+\frac{1}{2}} + F_{i-\frac{1}{2}} \right], \]

\[ e_i = \frac{1}{h} \left[ u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} \right] - \frac{h}{24} \left[ M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}} \right] + \frac{h^3}{96\theta^3} \left[ F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right] (\theta^2 - 6), \hspace{1cm} (2.5) \]

\[ f_i = \frac{1}{2} \left[ u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}} \right] - \frac{h^2}{16} \left[ M_{i+\frac{1}{2}} + M_{i-\frac{1}{2}} \right] + \frac{h^4}{64\theta^4} \left[ F_{i+\frac{1}{2}} + F_{i-\frac{1}{2}} \right] (\theta^2 - 2), \]

where \( \theta = \frac{kh}{2} \) and \( i = 0, 1, 2, \ldots, N. \)

The continuity of the first derivative implies:
System of Fourth Order Boundary Value Problem

\[ M_{i+\frac{1}{2}} + 22M_{i-\frac{1}{2}} + M_{i-\frac{3}{2}} = \frac{24}{h^2} [u_{i+\frac{1}{2}} - 2u_{i-\frac{1}{2}} + u_{i-\frac{3}{2}}] \]

\[- \frac{3h^2}{2\theta^2} [(\frac{1}{\theta^2} - \frac{1}{6}) - \frac{1}{\theta\sinh\theta}][F_{i+\frac{1}{2}} + F_{i-\frac{3}{2}}] \]

\[-2(\frac{\cosh\theta}{\theta\sinh\theta} - \frac{1}{\theta^2} - \frac{11}{6})F_{i-\frac{1}{2}}, \quad (2.6) \]

\[ i = 2, 3, \ldots, N-1. \]

and the continuity of the third derivative implies

\[ M_{i+\frac{1}{2}} - 2M_{i-\frac{1}{2}} + M_{i-\frac{3}{2}} = \frac{h^2}{4\theta} [(\frac{1}{\theta} - \frac{1}{\sinh\theta})(F_{i+\frac{1}{2}} + F_{i-\frac{3}{2}}) + 2(\frac{1}{\theta} + \frac{\cosh\theta}{\sinh\theta})F_{i-\frac{1}{2}}], \quad (2.7) \]

\[ i = 2, 3, \ldots, N-1. \]

Subtracting equation (2.7) from equation (2.6) and then dividing it by 24, we obtain:

\[ M_{i+\frac{1}{2}} = \frac{1}{h^2} [(u_{i+\frac{1}{2}} - 2u_{i-\frac{1}{2}} + u_{i-\frac{3}{2}}) - \frac{h^2}{24}[(\frac{3}{2\theta^4} - \frac{3}{2\theta^3\sinh\theta} - \frac{1}{4\theta\sinh\theta})[F_{i+\frac{1}{2}} + F_{i-\frac{3}{2}}] \]

\[ \quad + (\frac{\cosh\theta}{\theta\sinh\theta} (\frac{3}{2} + \frac{1}{\theta^2} - \frac{1}{\theta^2} (\frac{3}{2} + 6))F_{i-\frac{1}{2}}], \quad (2.8) \]

Eliminating \( M_i \)'s between (2.7) and (2.8), we get the following relation

\[ u_{i-\frac{5}{2}} - 4u_{i-\frac{7}{2}} + 6u_{i-\frac{9}{2}} - 4u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} = h^4 [\alpha (F_{i-\frac{7}{2}} + F_{i-\frac{9}{2}}) + \beta (F_{i-\frac{9}{2}} + F_{i+\frac{1}{2}}) + \gamma F_{i+\frac{3}{2}}], \quad i = 3, 4, \ldots, N-2. \quad (2.9) \]

where \( \alpha = \frac{1}{96} \left[ \frac{6\sinh\theta - 6\theta - \theta^3}{\theta^4\sinh\theta} \right], \)
\[
\beta = \frac{1}{24} \left[ 6\theta \cosh^2 \theta + \theta^3 \cosh^2 \theta - 6 \sinh \theta - 6\theta^3 \right],
\]
\[
\gamma = \frac{1}{48} \left[ 18 \sinh \theta - 6\theta + 12 \theta \cosh \theta - \theta^3 + 22 \theta^3 \cosh 2\theta \right],
\]
where \( f_i = f(x_i) \) and \( g_i = g(x_i) \).

**Remarks:**

(i) When \((\alpha, \beta, \gamma) \to \frac{1}{384} (1,76,230)\) the relation (2.9) reduces to quartic polynomial spline [69].

(ii) When \( k \to 0 \), that is \( \theta \to 0 \) then \((\alpha, \beta, \gamma) \to \frac{1}{1920} (1,236,1446)\) and the relation (2.9) reduces to quintic polynomial spline [54].

(iii) When \((\alpha, \beta, \gamma) \to \frac{1}{360} (1,56,246)\), the relation (2.9) reduces to sextic polynomial spline at mid points.

(iv) When \((\alpha, \beta, \gamma) \to (0, \frac{62526}{375156}, \frac{250104}{375156})\), then relation (2.9) reduces to Al-Said and Noor fourth order finite difference method [9].

(v) When \((\alpha, \beta, \gamma) \to (0, \frac{1}{24}, \frac{22}{24})\), then relation (2.9) reduces to Al-Said and Noor second order finite difference method [8].
5.3 Numerical Method

Now, substituting \( u_i^{(4)} = g_iu_i + f_i + r_i \) in the spline relation (2.9), we obtain \( N-4 \) linear algebraic equations in \( N \) unknowns, \( u_{i-\frac{1}{2}} \), \( i=1, 2, \ldots, N \) as:

\[
(1 - \alpha h^4 g_{i-\frac{1}{2}}) u_{i-\frac{1}{2}} + [-4 - \beta h^4 g_{i-\frac{1}{2}}] u_{i-\frac{3}{2}} + [6 - \gamma h^4 g_{i-\frac{1}{2}}] u_{i-\frac{1}{2}} + [1 - \alpha h^4 g_{i-\frac{1}{2}}] u_{i+\frac{3}{2}} - [\alpha(f_{i-\frac{3}{2}} + f_{i-\frac{1}{2}}) + \beta(f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}) + \gamma(f_{i+\frac{1}{2}} + f_{i+\frac{3}{2}})] = h^4 [\alpha(r_{i-\frac{3}{2}} + r_{i-\frac{1}{2}}) + \beta(r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}) + \gamma r_{i+\frac{3}{2}}],
\]

\( i = 3(1) N - 2. \)  

(3.1)

To obtain the unique solution of system (3.1), we need four more equations, two at each end of the range of integration for direct computation of \( u_{i-\frac{1}{2}} \), \( i = 1, 2, \ldots, N \).

These equations are obtained by Taylor series and the method of undetermined coefficients. The boundary formula associated with boundary conditions can be determined as follows:

\[
-\frac{5}{3} u_{\frac{1}{2}} + \frac{5}{6} u_{\frac{3}{2}} - \frac{1}{6} u_{\frac{5}{2}} - h^4 \left[ d_0 u^{(4)}_0 + d_1 u^{(4)}_1 + d_2 u^{(4)}_2 + d_3 u^{(4)}_3 + d_4 u^{(4)}_4 \right] + t_1 = -u_0 + \frac{5}{24} h^2 u''_0, \tag{3.2}
\]

\[
-\frac{75}{16} u_{\frac{1}{2}} + \frac{139}{16} u_{\frac{3}{2}} - \frac{109}{16} u_{\frac{5}{2}} + \frac{29}{16} u_{\frac{7}{2}} - h^4 \left[ d_0 u^{(4)}_0 + d_1 u^{(4)}_1 + d_2 u^{(4)}_2 + d_3 u^{(4)}_3 + d_4 u^{(4)}_4 + d_5 u^{(4)}_5 + d_6 u^{(4)}_6 + d_7 u^{(4)}_7 \right] + t_2 = -u_0 - h^2 u''_0, \tag{3.3}
\]
System of Fourth Order Boundary Value Problem

\[
\frac{75}{16} u^{(1)}_{N} + \frac{139}{16} u^{(3)}_{N} - \frac{109}{16} u^{(5)}_{N} + \frac{29}{16} u^{(7)}_{N} - h^4 [d_1^{(4)} u^{(4)}_{N-\frac{1}{2}} + d_2^{(4)} u^{(4)}_{N-\frac{3}{2}} + d_3^{(4)} u^{(4)}_{N-\frac{5}{2}} + d_4^{(4)} u^{(4)}_{N-\frac{7}{2}} + d_5^{(4)} u^{(4)}_{N-\frac{9}{2}} + d_6^{(4)} u^{(4)}_{N-\frac{11}{2}}] + t_{N-1} = -u_{N} - h^2 u''_{N},
\]

(3.4)

\[
\frac{5}{3} u^{(1)}_{N} + \frac{5}{6} u^{(3)}_{N} - \frac{1}{6} u^{(5)}_{N} - h^4 [d_1^{(4)} u^{(4)}_{N} + d_2^{(4)} u^{(4)}_{N-\frac{1}{2}} + d_3^{(4)} u^{(4)}_{N-\frac{3}{2}} + d_4^{(4)} u^{(4)}_{N-\frac{5}{2}} + d_5^{(4)} u^{(4)}_{N-\frac{7}{2}} + d_6^{(4)} u^{(4)}_{N-\frac{9}{2}}] + t_{N} = -u_{N} + \frac{5}{24} h^2 u''_{N},
\]

(3.5)

where, \( t_1, t_2, t_{N-1} \) and \( t_N \) are local truncation errors associated with the boundary conditions (3.2)-(3.5). The local truncation error \( t_i \), associated with the scheme (3.1) is given by,

\[
t_{i} = [1 - (2\alpha + 2\beta + \gamma)]h^i u^{(4)}_{i} + \frac{1}{2} [-1 + 2\alpha + 2\beta + \gamma]h^5 u^{(5)}_{i} + \frac{1}{24} [7 - 3(34\alpha + 10\beta + \gamma)]h^6 u^{(6)}_{i} + \frac{1}{48} [-5 + 98\alpha + 26\beta + \gamma]h^7 u^{(7)}_{i} + \frac{1}{1920} [69 - 5(706\alpha + 82\beta + \gamma)]h^8 u^{(8)}_{i} + \frac{1}{11520} [-115 + 8464\alpha + 726\beta + 3\gamma]h^9 u^{(9)}_{i} + \frac{1}{967680} [2497 - 21(16354\alpha + 730\beta + \gamma)]h^{10} u^{(10)}_{i} + O(h^{11}),
\]

\[i = 3, 4, \ldots, N-2.\] (3.6)

Now the scheme (3.1 – 3.5) gives rise to the class of methods of different orders as follows:

1. **Second Order Method**

For any choice of arbitrary \( \alpha \) and \( \beta \) with \( \gamma = 1 - 2(\alpha + \beta) \) and

\[
(d_0, d_1, d_2, d_3, d_4, d_5) = (-\frac{1}{6}, \frac{77}{1152}, 0, 0, 0, 0),
\]
System of Fourth Order Boundary Value Problem

\[(d_1', d_2', d_3', d_4', d_5', d_6') = (\frac{41}{256}, \frac{1463}{768}, 0, 0, 0, 0).\]

Then local truncation errors for \((\alpha, \beta, \gamma) = \frac{1}{1920}(1, 236, 1446)\) are

\[
t_i = \begin{cases} 
\frac{-66}{2929} h^6 u_i^{(6)} + O(h^7), & i = 1, N, \\
\frac{1}{24} h^6 u_i^{(6)} + O(h^7), & i = 3, ..., N - 2, \\
\frac{560}{1919} h^6 u_i^{(6)} + O(h^7), & i = 2, N - 1.
\end{cases}
\] (3.7)

2. Fourth Order Method

For arbitrary \(\alpha\) and \(\beta = \frac{1 - 24\alpha}{6}\) with \(\gamma = 1 - 2(\alpha + \beta)\) and

\[(d_0, d_1, d_2, d_3, d_4, d_5) = (\frac{1}{320}, -\frac{241}{3456}, -\frac{239}{6912}, \frac{47}{34560}, 0, 0),\]

\[(d_1', d_2', d_3', d_4', d_5', d_6') = (\frac{653}{1463}, \frac{641}{1479}, \frac{1991}{7242}, \frac{173}{30720}, 0, 0).\]

Then local truncation errors for \((\alpha, \beta, \gamma) = (\frac{7}{4850}, \frac{259}{14550}, \frac{4787}{7275})\) are

\[
t_i = \begin{cases} 
\frac{-64}{124031} h^8 u_i^{(8)} + O(h^9), & i = 1, N, \\
\frac{19}{349200} h^8 u_i^{(8)} + O(h^9), & i = 3, ..., N - 2, \\
\frac{-168}{27413} h^8 u_i^{(8)} + O(h^9), & i = 2, N - 1.
\end{cases}
\] (3.8)
3. Sixth order Method

For \((\alpha, \beta, \gamma) = -\frac{1}{720}(-1,124.474)\) and

\[
(d_0, d_1, d_2, d_3, d_4, d_5) = \left(\frac{9}{51176}, -\frac{8}{127}, \frac{386}{9149}, \frac{103}{14771}, -\frac{39}{18721}, \frac{19}{65426}\right),
\]

\[
(d_1', d_2', d_3', d_4', d_5', d_6') = \left(\frac{647}{14871}, \frac{876}{631}, \frac{335}{1791}, \frac{185583}{2683}, \frac{218}{6439}, \frac{204}{43047}\right).
\]

Then local truncation errors are

\[
t_i = \begin{cases}
-\frac{28}{185583} h^{10} u_i^{(10)} + O(h^{11}), & i = 1, N, \\
\frac{1}{3024} h^{10} u_i^{(10)} + O(h^{11}), & i = 3, \ldots, N - 2, \\
-\frac{79}{20138} h^{10} u_i^{(10)} + O(h^{11}), & i = 2, N - 1.
\end{cases}
\]  

(3.9)

5.4 Spline Solutions

The scheme 3.1 along with the boundary conditions (3.2 – 3.5) gives rise to a five diagonal linear system of order \(N \times N\) and may be written in the matrix form as

\[
AU = C + T,
\]  

(4.1)

\[
A(U - \bar{U}) = 0,
\]  

(4.2)

\[
A(U - \bar{U}) = T,
\]  

(4.3)
where $U = (u_{i - \frac{1}{2}}), \bar{U} = (\bar{u}_{i - \frac{1}{2}}), T = (t_{i - \frac{1}{2}})$ and $E = (e_{i - \frac{1}{2}}) = (u_{i - \frac{1}{2}} - \bar{u}_{i - \frac{1}{2}}), \ i = 1 (1)N$, are $N$ dimensional column vectors.

Also $A = A_0 - h^4BG$, where,

\begin{align*}
A_0 &= \begin{bmatrix}
5 & 6 & 1 \\
3 & 6 & 29 \\
-16 & 139 & 109 \\
16 & 16 & 16 \\
1 & -4 & 6 \\
1 & -4 & 6 \\
. & . & . \\
1 & -4 & 6 \\
29 & 109 & 139 \\
16 & 16 & 16 \\
-1 & 5 & -5 \\
6 & 6 & 3
\end{bmatrix}\ 
(4.4)
\end{align*}

Matrix $B$ has the form

\begin{align*}
B &= \begin{bmatrix}
d_1 & d_2 & d_3 & d_4 & d_5 \\
d_1' & d_2' & d_3' & d_4' & d_5' \\
\alpha & \beta & \gamma & \beta & \alpha \\
\alpha & \beta & \gamma & \beta & \alpha \\
. & . & . & . & . \\
\alpha & \beta & \gamma & \beta & \alpha \\
d_6' & d_5' & d_4' & d_3' & d_2' & d_1' \\
d_5 & d_4 & d_3 & d_2 & d_1
\end{bmatrix}_{NN} \\
(4.6)
\end{align*}

$G = diag(g_{i - \frac{1}{2}})$ and $C = [c_1, c_2, c_3, \ldots, c_{N-2}, c_{N-1}, c_N]^T$, where $g_{i - \frac{1}{2}} \neq 0$, for $\frac{N}{4} < i \leq \frac{3N}{4}$.
System of Fourth Order Boundary Value Problem

and

\[
\begin{aligned}
-A_i + \frac{5}{24} h^2 B_1 + h^4 [d_0 (f_0 + A_i g_0) + \sum_{j=1}^{4} d_j (f^j - r_j)], & \quad i = 1, \\
-A_i - h^2 B_1 + h^4 \sum_{j=1}^{4} d_j (f^j - r_j), & \quad i = 2, \\
h^4 [\alpha (f^i, \frac{1}{i+1}) + \beta (f^i, \frac{1}{i+1}), + \gamma], & \quad 3 \leq i \leq \frac{N}{4} - 2 \text{ and } \frac{3N}{4} + 3 \leq i \leq N - 2, \\
h^4 [\alpha (f^i, \frac{1}{i+1}) + \beta (f^i, \frac{1}{i+1}), + \gamma], & \quad \frac{N}{4} - 1 \text{ and } i = \frac{3N}{4} + 2, \\
h^4 [\alpha (f^i, \frac{1}{i+1}) + \beta (f^i, \frac{1}{i+1}), + \gamma], & \quad \frac{N}{4} \text{ and } i = \frac{3N}{4} + 1, \\
h^4 [\alpha (f^i, \frac{1}{i+1}) + \beta (f^i, \frac{1}{i+1}), + \gamma], & \quad \frac{N}{4} + 1 \text{ and } i = \frac{3N}{4}, \\
h^4 [\alpha (f^i, \frac{1}{i+1}) + \beta (f^i, \frac{1}{i+1}), + \gamma], & \quad \frac{N}{4} + 2 \text{ and } i = \frac{3N}{4} - 1, \\
h^4 [\alpha (f^i, \frac{1}{i+1}) + \beta (f^i, \frac{1}{i+1}), + \gamma], & \quad \frac{N}{4} + 3 \leq i \leq \frac{3N}{4} - 2, \\
-A_i - h^2 B_1 + h^4 \sum_{j=1}^{6} d_j (f^j - r_j), & \quad i = N - 1, \\
-A_i + \frac{5}{24} h^2 B_1 + h^4 [d_0 (f_N + A_N g_N) + \sum_{j=1}^{5} d_j (f^{N, \frac{1}{2} - j} - r_j)], & \quad i = N.
\end{aligned}
\]

(4.7)

where

\[
\begin{aligned}
r_i = \begin{cases} 
    r, & \frac{N}{4} < i \leq \frac{3N}{4}, \\
    0, & \text{elsewhere.}
\end{cases}
\end{aligned}
\]

5.5 Applications

Problem (1.1) along with the boundary conditions (1.2) can be studied under the framework of variational inequalities. Khan et al. [37] considered, the linear fourth order boundary value problem describing the equilibrium configuration of an elastic
beam, pulled at the ends and lying over an elastic obstacle of constant height \(1/4\) and unit rigidity of the type,

\[
\begin{align*}
  u^{(4)} &\geq f(x), \\
  u &\geq \psi(x), \\
  [u^{(4)} - f(x)][u - \psi(x)] &= 0, \\
  u(-1) &= u(1) = u^{''}(-1) = u^{''}(1) = 0,
\end{align*}
\]

where, \(f\) is the given force acting on the beam string and \(\psi(x)\) is the elastic obstacle.

We study the variational inequality formulation of the problem (5.1). For this, we define the set \(K\) as

\[
K = \{ v : v \in H_0^2(\Omega) : v \geq \psi, \text{on } \Omega \}
\]

which is a closed convex set in \(H_0^2(\Omega)\), where \(H_0^2(\Omega)\) is a Sobolev space \([17, 22, 40]\), which is basically a Hilbert space.

Now, from Kikuchi and Oden technique \([40]\), we can show that the energy functional associated with the obstacle problem (5.1) is

\[
I[v] = \int_{-1}^{1} \left\{ \frac{d^4 v}{dx^4} - 2 f(x) \right\} v(x) dx, \quad \forall v \in H_0^2(\Omega),
\]

\[
= \int_{-1}^{1} \left( d^2 v / dx^2 \right)^2 dx - 2 \int_{-1}^{1} f(x) v(x) dx,
\]

\[
= a(v, v) - 2 \langle f, v \rangle,
\]

where

\[
a(u, v) = \int_{-1}^{1} \left( d^2 u / dx^2 \right) \left( d^2 v / dx^2 \right) dx,
\]

and

\[
\langle f, v \rangle = \int_{-1}^{1} f(x) v(x) dx.
\]
It can be proved that the form \( a(u,v) \) defined by (5.4) is bilinear, positive and symmetric. Also, the functional \( f \) defined by (5.5) is linear and continuous. The minimum \( u \) of the functional \( I[v] \) defined by (5.3) on the closed convex set \( K \) in \( H^2_0(\Omega) \) can be characterized by the variational inequality [17, 22, 40]

\[
a(u,v-u) \geq \langle f, v-u \rangle, \quad \forall v \in K,\tag{5.6}
\]

Hence, the obstacle problem (5.1) is equivalent to solving the variational inequality problem (5.6). The equivalence has been used to study the existence of a unique solution of (5.1): see for example Ref. [17, 22]. We can rewrite problem (5.6) using Lewy and Stampacchia technique [41],

\[
\begin{align*}
\frac{d^4}{dx^4} u + \mu (u - \psi)(u - \psi) &= f, & -1 < x < 1, \\
u(-1) &= u(1) = 0, & u'(1) = \ddot{u}'(1) = \varepsilon.
\end{align*}
\tag{5.7}
\]

where, \( \varepsilon \) is a small positive constant and \( \psi \) is the obstacle function and \( \mu(t) \) is the penalty function defined by,

\[
\mu(t) = \begin{cases} 
4, & t \geq 0, \\
0, & t < 0.
\end{cases}
\tag{5.9}
\]

Equation (5.6) describes the equilibrium configuration of an elastic beam, pulled at the ends and lying over an elastic obstacle of constant height 1/4. Since the obstacle function \( \psi \) is known, it is possible to find the exact solution of the problem in the interval \( -1/2 \leq x \leq 1/2 \).

We assume that the obstacle function \( \psi \) is defined by

\[
\psi(x) = \begin{cases} 
-1/4, & -1 \leq x \leq -1/2, \\
-1/2, & 1/2 \leq x \leq 1,
\end{cases}
\tag{5.10}
\]

From Equations (5.6) - (5.9), the following system of equations can be obtained as
System of Fourth Order Boundary Value Problem

\[ u^{(4)} = \begin{cases} f, & -1 \leq x \leq -1/2, \quad 1/2 \leq x \leq 1, \\ 1 - 4u + f, & -1/2 \leq x \leq 1/2. \end{cases} \quad (5.11) \]

with the following boundary conditions

\begin{align*}
    u(-1) &= u(-1/2) = u(1/2) = u(1) = 0, \\
    u'(1) &= u'(-1/2) = u'(1/2) = u'(1) = 0, \\
    \end{align*} \quad (5.12)

and the conditions of continuity for \( u \) and \( u' \) at \( x = -1/2 \) and \( 1/2 \).

5.6 Numerical Illustrations

We have implemented our method on two examples of the system of fourth order boundary value problems and the maximum absolute errors are listed in tables 1-3. We also compared our results with the existing methods in the references \([5-7, \ 9, \ 36, \ 37, \ 46, \ 55, \ 57, \ 59]\) to illustrate the comparative performance of our method over other existing methods.

A numerical study of the problem on the interval \([-\frac{1}{2}, \ 1]\) is given in \([47]\) using Pade’ approximants together with a finite difference technique. Here, we present the numerical study over the whole interval \([-1, \ 1]\).

Example 1:

Consider the system of linear fourth order boundary value problem

\[ u^{(4)}(x) = \begin{cases} 1, & -1 \leq x \leq -\frac{1}{2}, \quad \frac{1}{2} \leq x \leq 1, \\ 2 - 4u, & -\frac{1}{2} \leq x \leq \frac{1}{2}. \end{cases} \]

with the boundary conditions
System of Fourth Order Boundary Value Problem

\[
u(-1) = u\left(-\frac{1}{2}\right) = u\left(\frac{1}{2}\right) = u(1) = 0, \\
u'(-1) = u'\left(-\frac{1}{2}\right) = u'\left(\frac{1}{2}\right) = u'(1) = 0.
\]

The analytical solution of this problem is

\[
u(x) = \begin{cases} 
\frac{1}{24}x^4 + \frac{1}{8}(x^3 + x^2) + \frac{3}{64}x + \frac{1}{192}, & \text{for } -1 \leq x \leq -\frac{1}{2}, \\
\frac{1}{2} - \frac{1}{\beta_3}[\beta_1 \sin x \sinh x + \beta_2 \cos x \cosh x], & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\
\frac{1}{24}x^4 - \frac{1}{8}(x^3 - x^2) - \frac{3}{64}x + \frac{1}{192}, & \text{for } \frac{1}{2} \leq x \leq 1,
\end{cases}
\]

with \(u(-1) = u\left(-\frac{1}{2}\right) = u''(-1) = u'\left(-\frac{1}{2}\right) = 0.

where

\[
\beta_1 = \sin\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2}\right), \quad \beta_2 = \cos\left(\frac{1}{2}\right) \cosh\left(\frac{1}{2}\right), \quad \beta_3 = \cos(1) + \cosh(1).
\]

This was solved by Al-said and Noor [5] using finite difference scheme based on

\[h^4u_i^4 = u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} - \frac{h^6}{6}u_i^6\]

and Khalifa and Noor [36] using collocation method with quintic B-spline as basis function. It is clear from the tables 1-3 that our methods are better than the other existing methods. The Results of our methods are better than those of 12th order method of Al-said and Noor [6] based on (0, 8) Pade’ Approximants.
Table 1: Observed Maximum absolute errors for Example 1

<table>
<thead>
<tr>
<th>h</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Our Sixth order method</td>
<td>1.04 E-10</td>
<td>3.88 E-11</td>
<td>9.82 E-12</td>
<td>1.35 E-12</td>
</tr>
<tr>
<td>Our Fourth order method</td>
<td>1.71 E-08</td>
<td>8.48 E-10</td>
<td>1.67 E-10</td>
<td>4.11 E-11</td>
</tr>
<tr>
<td>Our Second order method</td>
<td>1.35 E-06</td>
<td>7.69 E-07</td>
<td>2.87 E-07</td>
<td>8.53 E-08</td>
</tr>
<tr>
<td>In[7]</td>
<td>2.53 E-05</td>
<td>6.38 E-06</td>
<td>1.66 E-06</td>
<td>4.29 E-07</td>
</tr>
<tr>
<td>In[9]</td>
<td>2.4 E-08</td>
<td>1.5 E-09</td>
<td>9.5 E-11</td>
<td>-</td>
</tr>
<tr>
<td>In[46]</td>
<td>1.3 E-06</td>
<td>8.7 E-08</td>
<td>6.8 E-09</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2: Observed Maximum absolute errors for Example 1

<table>
<thead>
<tr>
<th>h</th>
<th>1/12</th>
<th>1/24</th>
<th>1/48</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Our Sixth order method</td>
<td>6.78 E-11</td>
<td>1.74 E-11</td>
<td>3.92 E-12</td>
</tr>
<tr>
<td>Our Fourth order method</td>
<td>2.36 E-09</td>
<td>3.06 E-10</td>
<td>7.53 E-11</td>
</tr>
<tr>
<td>Our Second order method</td>
<td>1.01 E-06</td>
<td>4.50 E-07</td>
<td>1.43 E-07</td>
</tr>
<tr>
<td>In[5]</td>
<td>1.83 E-04</td>
<td>7.17 E-05</td>
<td>4.59 E-05</td>
</tr>
<tr>
<td>In[37]</td>
<td>8.42 E-06</td>
<td>2.16 E-06</td>
<td>5.40 E-07</td>
</tr>
<tr>
<td>In[57]</td>
<td>2.19 E-06</td>
<td>5.47 E-07</td>
<td>1.48 E-07</td>
</tr>
<tr>
<td>In[59]</td>
<td>2.19 E-06</td>
<td>5.47 E-07</td>
<td>1.48 E-07</td>
</tr>
<tr>
<td>In[55]</td>
<td>1.16 E-08</td>
<td>3.31 E-09</td>
<td>4.55 E-10</td>
</tr>
</tbody>
</table>
Example 2:

Consider the system of linear fourth order boundary value problem

\[
 u^{(4)}(x) = \begin{cases} 
 0, & -1 \leq x \leq -\frac{1}{2}, \\
 1 - 4u, & -\frac{1}{2} \leq x \leq \frac{1}{2},
\end{cases}
\]

with the boundary conditions

\[ u(-1) = u(-\frac{1}{2}) = u(\frac{1}{2}) = u(1) = 0, \]
\[ u'(-1) = -u'(-\frac{1}{2}) = u'(\frac{1}{2}) = -u'(1) = \varepsilon, \quad \text{where} \quad \varepsilon \to 0. \]

The theoretical solution of this problem is

\[
 u(x) = \begin{cases} 
 -\left(\frac{2}{3}x^3 + \frac{3}{2}x^2 + \frac{13}{12}x + \frac{1}{4}\right)\varepsilon, & \text{for } -1 \leq x \leq -\frac{1}{2}, \\
 0.25 - \frac{1}{2\beta_3} [\beta_1 \sin x \sinh x + \beta_2 \cos x \cosh x], & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\
 -\left(\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{13}{12}x + \frac{1}{4}\right)\varepsilon, & \text{for } \frac{1}{2} \leq x \leq 1,
\end{cases}
\]

with \( u(-\frac{1}{2}) = u(\frac{1}{2}) = u''(-\frac{1}{2}) = u''(\frac{1}{2}) = 0. \)

\[
 u(\frac{1}{2}) = u(1) = 0, \quad u''(\frac{1}{2}) = u''(1) = \varepsilon, \quad \varepsilon \to 0,
\]

where

\[
 \beta_1 = \sin \left(\frac{1}{2}\right) \sinh \left(\frac{1}{2}\right), \quad \beta_2 = \cos \left(\frac{1}{2}\right) \cosh \left(\frac{1}{2}\right), \quad \beta_3 = \cos (1) + \cosh (1).
\]
Table 3: Observed Maximum absolute errors for Example 2 with $\varepsilon = 10^{-16}$

<table>
<thead>
<tr>
<th>$h$</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our Six order method</td>
<td>5.212E-11</td>
<td>1.940 E-11</td>
<td>4.91 E-12</td>
<td>6.79 E-13</td>
</tr>
<tr>
<td>Our Fourth order method</td>
<td>8.57 E-09</td>
<td>4.24 E-10</td>
<td>8.39 E-11</td>
<td>2.059 E-11</td>
</tr>
<tr>
<td>Our Second order method</td>
<td>6.75 E-07</td>
<td>3.84 E-07</td>
<td>1.43 E-07</td>
<td>4.26 E-08</td>
</tr>
<tr>
<td>In[5]</td>
<td>1.4 E-04</td>
<td>3.6 E-05</td>
<td>8.9 E-06</td>
<td>-</td>
</tr>
<tr>
<td>In[6]</td>
<td>1.9 E-05</td>
<td>4.8 E-06</td>
<td>1.2 E-06</td>
<td>-</td>
</tr>
<tr>
<td>In[7]</td>
<td>1.3 E-05</td>
<td>3.2 E-06</td>
<td>8.1 E-07</td>
<td>-</td>
</tr>
<tr>
<td>In[36]</td>
<td>3.0 E-04</td>
<td>7.0 E-05</td>
<td>1.4 E-05</td>
<td>-</td>
</tr>
<tr>
<td>In[37]</td>
<td>1.89 E-05</td>
<td>4.78 E-06</td>
<td>1.18 E-06</td>
<td>-</td>
</tr>
<tr>
<td>In[55]</td>
<td>9.09 E-08</td>
<td>5.38 E-09</td>
<td>3.15 E-10</td>
<td>2.77 E-11</td>
</tr>
</tbody>
</table>