CHAPTER 1
A NEW THEORY FOR CUP ANEMOMETERS

Abstract

A rigorous transient analysis of the response of cup anemometers is presented. It is shown that the values of speeds of rotation for various steady state wind speeds can be calculated using the theoretical formulae from the raw data about the weights and dimensions of the rotating parts along with the data regarding the aerodynamic forces that would act on stationary cups held in the path of air motion, and the data regarding the frictional constants. Since it is not possible to determine the frictional constants except in actual rotating conditions of the anemometer the theory is first verified for the case when the British cup generator anemometer is assumed to be frictionless. It is shown that the scale factors and the so-called distance constants calculated from the theoretical formulae agree with the published experimental results within a reasonable degree of accuracy. The raw data used for substitution in the theoretical formula is obtained from the weights and measurements of the rotating parts and from the curves of static forces on stationary cups published by Brevoort and Joyner.

The constant representing static friction is obtained from a knowledge of the minimum wind speed at
which the anemometer starts to rotate. The constant of dynamic friction is calculated from the theoretical formula from a knowledge of the experimentally observed speed of rotation for a wind speed approximately in the middle of the range of calibration curve. By employing the fractional constants thus obtained, it is shown that the theoretically calculated calibration curve and the time constants agree with experimental observations within about 6% at the largest wind speeds.

The theory shows that the reciprocal of the time constant for the response to a step function of wind speed is the sum of the reciprocals of a natural time constant and the ratio of the so called Distance constant to the final wind speed.

1. Introduction

Among the various meteorological parameters the most variable one is the wind, the speed and direction of which are always in a state of random fluctuation. The highest frequency of the spectrum of variation in the wind speeds in the atmosphere is yet unknown, but is perhaps of the order of about 10 to 50 cycles per second. The sensors are normally the common meteorological anemometers of the cup or vane type, pressure tube type or in some restricted applications the hot wire type. There
are various other sensors being continuously developed. But, by far, the most important sensors are, perhaps the cup anemometers and the vane anemometers. Though very simple in appearance, as these are, they are the most complicated in behaviour among all the meteorological sensors. So far, no theory has been developed to study their aerodynamic interactions with the natural wind nor any effort to understand their behaviour has been successful. This paper deals with a new theory which explains the observed characteristics of cup anemometers inside the wind tunnels quantitatively in a satisfactory manner.

2. Theory

The main approaches in trying to study the cup anemometers have been only experimental and empirical. There have been detailed studies of the response behaviour under steady state conditions obtaining in the wind tunnels and many empirical formulae have been devised or attempted to explain their behaviour.\(^1\) The individual cups of both hemispherical and conical shapes have been subjected to steady wind speeds and the forces acting on the cups have been measured.\(^2\) Certain attempts have also been made to study the transient behaviour of an anemometer for suddenly rising or falling wind speeds.\(^3,7,8\) A few of the important features of the experimental characteristics are:

1) The steady state calibration curve expressing
the speed of revolution as a function of wind speed is practically a straight line in spite of the nonlinearity of the aerodynamic torque operating on them.

ii) When the wind speed is raised suddenly (or lowered suddenly) from an initial value \( v_i \) to a final value \( v_f \), the time constant (i.e. the time taken by the anemometer to indicate a speed \( v_v + 0.63 (v_f - v_v) \)) is proportional to the reciprocal of the final wind speed.

iii) From the static curves of forces acting on the cups held rigidly, published by Brevoort and Joyner\(^2\), it is seen that all the hemispherical cups have identical curves for any given Reynolds number.

The last mentioned fact is of great significance. Brevoort and Joyner have reported on three different sizes of hemispherical cups which have identical normalized curves for the forces (acting perpendicular to the face of cups), plotted against the angle of attack of the wind. In addition, they have also published the corresponding curves for the conical cups of shapes shown in Fig 1.

The conical cups have angles 40° and 45° with
Comparison of the shape of the cup of British Cup generator anemometer with those used by Brevoort and Joyner.
the central axis. The curves for these two cups are very similar to those of hemispherical cups, but not quite identical. In this article, the curves for the 45° conical cup has been taken to represent the forces acting on the conical cups of British cup generator anemometer. The cup generator anemometer of the India Meteorological Department is identically the same as the British anemometer, both having almost the same shape as the 45° cup used by Brevoort and Joyner in their experiments (see Fig 1).

The curves of the forces acting on the cups when held rigidly in the path of air motion have been plotted by Brevoort and Joyner in terms of a normalised factor $C_N$ with respect to the angle between the normal to the face of the cup and the direction of the wind speed. The factor $C_N$ is given by

$$C_N = \frac{\text{Actual force acting normal to the cup}}{\text{Aerodynamic pressure} \times \text{Area of the cup}}$$

$$\ldots(1)$$

If an imaginary plane obstacle of area $A$ is placed normal to the direction of flow of wind, the wind force acting on the area $A$ due to a steady wind velocity $V$ is $\frac{1}{2} \rho V^2 A$. However, the actual force acting on the cup is given by $\frac{1}{2} \rho V^2 A C_N$. Thus the "effective obstacle area" which the cup presents normal to the wind is $A_{\text{eff}} = A C_N$. This effective area may also be called
as the "Capture Area" of the cup by analogy with the similar concept in connection with radio antennas. Let $\theta$ be the angle between the normal to the face of the cup and the direction of wind. We may write

\[ \text{(2)} \]

\[ \text{(2)(a)} \]

Brevoort and Joyner have published the curves of $C_N$ with respect to $\theta$ for various values of Reynold numbers from 0 to 400,000. The curves for the 45° angle cup are not very much spread out, and the mean value for the Reynold number 200,000 has been reproduced in Fig 2. Now, this curve is assumed to represent the forces acting on the cups of the British cup generator anemometer for the entire range of air velocities. When a particular cup of the anemometer is held at an angle $\theta_0$ with respect to the wind direction, and its effective capture area is $A_{\theta_0}$, the mass of air blowing on the cup per sec. is $A_{\theta_0} V$ where $V$ is the wind velocity and $\rho$ is the air density. However, if the cup is also rotating with a speed $n_{\theta}$ r.p.s. about an axis distant $R$, the air blows on the cup at the rate of the relative velocity given by

Here the velocity of wind, $V_{\theta}$ has been given the suffix $\theta$. 

....13/
Fig 2. Variation of $\frac{A_0}{A}$ and $\frac{Ag \cos \theta}{A}$ with $\theta$

For Reynolds Number 200,000.

(Data from Brevoort and Joyner)
to indicate that it is continuously varying. The second term on the right hand side gives the component of the tangential velocity of the cup in the direction of the wind. The mass of air intercepted per second is

\[ \rho A_\theta_i (N_i - 2\pi R n_i \cos \theta_i) = \rho [N_i A_{\theta_i} - 2\pi R n_i A_{\theta_i} \cos \theta_i] \]

The variation of \( A_{\theta_i}/A \) with \( \theta_i \) is plotted in Fig 2 for the cup with angle 45°. When there are three cups spaced at 120° apart from each other, the total equivalent mass of air intercepted per second by the anemometer cups providing the rotational torque is

\[ = \rho [N_i (A_{\theta_i} + A_{\theta_i+120°} + A_{\theta_i+240°})]
\]

\[ - 2\pi R n_i (A_{\theta_i} \cos \theta_i + A_{\theta_i+120°} \cos \theta_i+120°
\]

\[ + A_{\theta_i+240°} \cos \theta_i+240°)] \]

\[ = \rho [a_i N_i - 2\pi R n_i b_i] \]

where \( a_i = A_{\theta_i} + A_{\theta_i+120°} + A_{\theta_i+240°} \)

and \( b_i = A_{\theta_i} \cos \theta_i + A_{\theta_i+120°} \cos \theta_i+120°
\]

\[ + A_{\theta_i+240°} \cos \theta_i+240° \]

\( a_i \) and \( b_i \) are now normalized by taking their ratios with the actual area \( A \)

These values are tabulated in Table 1 and plotted in Fig 4.
Fig. 3-A particular orientation $\theta_l$ of the anemometer
Table 1

Calculation for $\frac{\alpha}{A}$, $\frac{\delta x}{A}$, $\frac{a_m}{A}$ and $\frac{\delta m}{A}$

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FIG. 4. VARIATION OF $\frac{a_i}{A}$ AND $\frac{b_i}{A}$ WITH $\theta$

(CALCULATED FROM FIG. 2.)
If we start following the orientation of the cup of the anemometer with respect to the wind direction, it will be observed that this orientation will be repeated after every successive motion through 120°, when the second cup will take the place of the first cup, the third cup will take the place of the second and the first cup will take the place of the third cup. We may then write the mean mass of air intercepted by the anemometer per second as

\[
\text{mean of } \left[ a_i \nu_i \frac{g}{2 \pi R} \sigma b_i \right] \rho = \left[ \frac{\sum a_i \nu_i}{\frac{R}{\rho}} - \frac{2 \pi R \sum n_i b_i}{P} \right] \rho \quad \ldots(5)
\]

The value of \( a_i \) depends only on the angle \( \theta_i \) of the cup. The instantaneous value of wind velocity is at random and is in turn not dependent on the angle \( \theta_i \). However, \( \theta_i \) is also a function of time just as \( \nu_i \) may be expressed as a function of time. The expression for \( \theta(t) \) is

\[
\theta(t_2) = \int_{t_1}^{t_2} n dt + \theta(t_1) \quad \ldots(6)
\]

where \( n \) is the speed of rotation of the cups.
\( \theta(t) \) is the angle at the instant \( t_2 \) and \( \theta(t_1) \) is the angle at the instant \( t_1 \). If we choose an instant \( t_1 \) when \( n \) was zero and count from this time, the initial value \( \theta(t_1) \) at which the cup started rotating is purely arbitrary. Thus though the value of \( \theta \) is a function of both speed of rotation \( n \) and the time \( t \), it is also dependent on the initial angle \( \theta(t_1) \) of the cup.

In addition, the averaging process, indicated in eq. 5 is to be carried out from an angle \( \theta_i \) of the cup to an angle \( \theta_i + 120^\circ \). Here again the initial angle \( \theta_i \) from which \( \theta \) is counted is arbitrary and can vary from 0 to 360°. As a result of these arguments, we can immediately realise that there is no correlation between \( \theta_i \) and \( n \) and so also between \( \theta_i \) and \( \omega \). Then we may write that the average of the product \( a_i \nu_i \) in any interval \( \theta_i \) to \( \theta_i + 120^\circ \) as the product of the average of \( a_i \) and the average of \( \nu_i \). So also the average of the product \( n_i \nu_i \) in the interval \( \theta_i \) to \( \theta_i + 120^\circ \) is the product of the average of \( n_i \) and the average of \( \nu_i \). Taking the first term,

\[
\text{the average of } a_i = \frac{\int_{\theta_i}^{\theta_i + 120^\circ} a \theta d\theta}{\int_{\theta_i}^{\theta_i + 120^\circ} d\theta} = a_m \text{ (say)}
\]

\[\cdots (7)\]
This is calculated by taking the area of the curve for 
\( a_i \) as function of \( \theta \). Then

\[
A_m = A \frac{\int_{\theta_i}^{\theta_i+120^\circ} \left( \frac{a_i}{A} - \frac{1}{A} \right) d\theta}{\int_{\theta_i}^{\theta_i+120^\circ} d\theta}
\] ....(8)

(see footnote *)

A being the actual area of the cup, the mean curve for Reynolds number 200,000 has been taken to be representative of the cup for the purposes of calculations in this paper. When the angle \( \theta \) is changing, it is not reasonable to assume that the wind velocity remains constant. However, it is possible to assume that the speed of rotation does not change very much in the interval between \( \theta_i \) and \( \theta_i+120^\circ \). This point is discussed further later. The average value of \( u_c \) between \( \theta = \theta_i \) and \( \theta = \theta_i+120^\circ \) is calculated as follows:

Let \( \theta = \theta_i \) at \( t = (t-\tau) \)

and \( \theta = \theta_i+120^\circ \) at \( t = t \) ....(9)

Since it will be found later that the change in speed

* The actual calculation of \( A_m \) in Appendix II has, however, been done using the formula

\[
\frac{A_m}{A} = \frac{1}{B} \left( \frac{a_i}{A} \right)
\]

(vide table 1) ....18/
of rotation during an interval of time for rotation through 120° is about 5% or less, we may assume that the mean value of the acceleration of wind in this interval is a constant quantity. Then, with reference to Fig 8, the average value of $N_x$ in the interval is

$$N_{aw} = V(t) - \frac{1}{2} \left[ \text{average wind acceleration} \right] T$$

$$= N(t) = \frac{1}{2} \left[ \int_{t=(t-\tau)}^{t} \frac{V'(t) \ dt}{(t-\tau)} \right] T$$

$$= \frac{1}{2} \int_{t-\tau}^{t} V'(t) \ dt \quad (10)$$

where $V'(t) = \frac{d}{dt} \left[ N(t) \right]$.

However, when calculating the average value of wind velocity $V$ in the interval between $t=(t-\tau)$ and $t=t$, we are not interested in the acceleration of wind before the instant $t=(t-\tau)$.

There is a convenient mathematical device which is frequently used in transient analyses of this nature. If we may define a unit step function $u(t-\tau)$ as
CALCULATION OF THE MEAN VELOCITY OF AIR IN AN INTERVAL OF 120° ASSUMING A CONSTANT MEAN ACCELERATION.

FIG. 5
\[
\begin{align*}
\mathcal{u}(t-T) &= 1 & \text{for } t \geq \tau \\
\text{and } \mathcal{u}(t-T) &= 0 & \text{for } t < \tau
\end{align*}
\]

(11)

This function is used to multiply the function \( u'(t) \) so that with reference to the Fig 6, it is seen that the new function \( f(t-\tau) \) is given by

\[
\begin{align*}
\mathcal{f}(t-\tau) &= u'(t) \mathcal{u}(t-\tau) \\
&= u'(t) & \text{for } (t-\tau) > 0 \\
&= 0 & \text{for } (t-\tau) < 0
\end{align*}
\]

(12)

(See foot note *)

The average acceleration can be written as

\[
\begin{align*}
\int_{t-\tau}^{t} u'(t) \, dt &= \int_{\tau}^{(t-\tau)} f(t-\tau) \, d\tau \\
&= \int_{0}^{\tau} f(t-\tau) \, d\tau \\
&= -\int_{\tau}^{0} f(t-\tau) \, d\tau \\
&= -\int_{0}^{T} f(t-\tau) \, d\tau
\end{align*}
\]

(13)

Since the function \( f(t-\tau) \) is identically zero for \(|t-\tau| < 0\) we may change the upper limit in eq. 13.

* The advantage in the function \( f(t-\tau) \) being used in the place of \( u'(t) \) within the interval of time \( \tau \leq (t-\tau) \) and \( \tau = t \), is that a corresponding expression to be obtained from this, later, can be expressed as a convolution integral.
SIGNIFICANCE OF THE AUXILIARY FUNCTION

\[ f(t - T) = \nu'(t) u(t - \tau). \]

FIG. 6.
to \( \tau - t \) (vide Fig 6d) and substitute in eq. 10 to give the average velocity as

\[
\bar{v} = \bar{v}(t) + \frac{1}{2} \int_0^t f(t-\tau) \, d\tau.
\]

It is of interest to examine the amount of variation in the speed of rotation \( n \) during a change of angular position of cups through \( 120^\circ \). This change is largest during sudden increases of wind velocity by large amounts and smallest when the wind velocity drops to zero suddenly. The finite change of the mean of rotation during \( 1/3 \) of a revolution does not exceed about 5 to 6% even for the very fast rising wind velocities going up to 35 knots or so and is always less than 5% for the normal meteorological conditions. The table 2 shows \( \frac{\Delta n}{n} \) for rising wind velocities (when the fraction is at its largest values) for the British cup generator anemometer. Thus it is seen that we can ignore the change in \( n \) during an angular motion of \( 120^\circ \) and take it to be practically the same for the entire interval from \( \theta_i \) to \( \theta_i + 120^\circ \).

If the speed of rotation \( n \) is assumed as constant for the interval of time for rotation through \( 120^\circ \) we may take it out of the summation sign \( \frac{\Delta}{\varepsilon} \) in the second term of eq. 5 so that this term becomes

\[ \ldots 22/ \]
Table 2

Percentage change in \( E \) during 1/3 of revolution for the cup generator anemometer

(The basic data obtained from the Handbook of Meteorological Instruments)

<table>
<thead>
<tr>
<th>No.</th>
<th>Knots</th>
<th>Knots</th>
<th>Sec</th>
<th>RPM</th>
<th>Knots</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
<th>RPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35</td>
<td>17.5</td>
<td>0.7</td>
<td>166</td>
<td>26.5</td>
<td>275</td>
<td>221</td>
<td>3.7</td>
<td>109</td>
<td>1.82</td>
<td>2.6</td>
<td>0.23</td>
<td>6.5</td>
<td>2.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>35</td>
<td>26.0</td>
<td>0.7</td>
<td>251</td>
<td>32.0</td>
<td>306</td>
<td>278</td>
<td>4.6</td>
<td>55</td>
<td>0.92</td>
<td>1.31</td>
<td>0.99</td>
<td>1.9</td>
<td>1.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>13</td>
<td>1.1</td>
<td>130</td>
<td>21.2</td>
<td>201</td>
<td>161</td>
<td>2.7</td>
<td>81</td>
<td>1.35</td>
<td>1.23</td>
<td>0.15</td>
<td>5.6</td>
<td>2.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>26</td>
<td>17.5</td>
<td>1.3</td>
<td>166</td>
<td>22.8</td>
<td>219</td>
<td>193</td>
<td>3.6</td>
<td>53</td>
<td>0.88</td>
<td>0.68</td>
<td>0.63</td>
<td>2.0</td>
<td>1.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>17.5</td>
<td>9</td>
<td>1.7</td>
<td>79</td>
<td>14.4</td>
<td>133</td>
<td>106</td>
<td>1.8</td>
<td>54</td>
<td>0.9</td>
<td>0.53</td>
<td>0.10</td>
<td>5.6</td>
<td>1.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>17.5</td>
<td>13</td>
<td>1.85</td>
<td>120</td>
<td>15.8</td>
<td>143</td>
<td>134</td>
<td>2.2</td>
<td>28</td>
<td>0.47</td>
<td>0.21</td>
<td>0.04</td>
<td>1.7</td>
<td>1.33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>9</td>
<td>2.6</td>
<td>77</td>
<td>11.5</td>
<td>105</td>
<td>91</td>
<td>1.5</td>
<td>14</td>
<td>0.23</td>
<td>0.09</td>
<td>0.02</td>
<td>1.3</td>
<td>1.44</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
where the average of $\lambda_x$ is given by

$$\lambda_x = \frac{\int_{\theta_x}^{\theta_x+120^\circ} \lambda_x \, d\theta}{\int_{\theta_x}^{\theta_x+120^\circ} \, d\theta}$$

...(16)

This is obtained by plotting the curve $\lambda_x$ with respect to $\theta$ (vide Fig 4) (* See footnote).

From eqns. 5, 7, 14, 15 and 16 the mean mass of air intercepted by the anemometer per second

$$= \rho \left\{ \frac{a_m n(t)}{2} + \frac{a_m}{2} \int_0^t f(t - \tau) d\tau ight\} - 2\pi R \lambda_m \left( \ddot{\lambda} \right)$$

...(17)

---

* The actual calculation of $\frac{\lambda_m}{A}$ in Appendix II has, however, been done from the formula

$$\frac{\lambda_m}{A} = \frac{\lambda_e (v/A)}{p}$$

(vide table 1)
In the course of a long period of time, it is always possible to find an instant \( t = (t - \tau) \) when the speed of revolution was zero, i.e. \( n(t - \tau) = 0 \) and the anemometer again started rotating after this instant. If we count back from the instant \( t = t \) to the instant \( t = (t - \tau) \) we may write the expression for \( n(t) \) as

\[
n(t) = \int_{t-\tau}^{t} n'(t-\tau) \, d\tau
\]

where

\[
n' = \frac{d}{dt}(n)
\]

Since the rotations of the anemometer prior to \( t = (t - \tau) \) do not contribute to the speed in the interval under consideration, the upper limit of the above integral can be changed to \( \tau = t \) so that

\[
n(t) = \int_{0}^{t} n'(t-\tau) \, d\tau
\]

The mean mass of air intercepted per second by the anemometer is

\[
= \rho \left\{ a_m \nu(t) + \frac{1}{2} a_m \int_{0}^{t} \nu'(t-\tau) \, d\tau - 2\pi R \lim_{n \to \infty} \int_{0}^{t} n'(t-\tau) \, d\tau \right\} 
\]

The velocity of air blowing at any instant \( t \) is \( \nu(t) \)
Then the momentum imparted to the anemometer per second is obtained by multiplying the mean mass of air intercepted per second by the anemometer by the wind velocity.

However, when the right hand side is being multiplied by $V(t)$ it is necessary to substitute the proper time variable and include it inside the signs of the two integrals. Remembering that the momentum transferred to the anemometer per second gives the force acting on it, the aerodynamic force acting on the anemometer is given by

$$
\text{Force} = \rho \left\{ \text{am} \, V(t), \, V(t), \, + \frac{1}{2} \text{am} \int_0^t (t-\tau) V(\tau) \, d\tau \right\}
$$

$$
- 2\pi R b_m \int_0^t \omega'(t-\tau) \, V(\tau) \, d\tau
$$

...(21)

The torque responsible for rotating the anemometer is obtained by multiplying the force by $R$, the distance of the centre of the cup from the axis of rotation.

This aerodynamic torque is balanced by the inertia and friction. The torque due to inertia is

$$
2\pi I \frac{d\omega}{dt}
$$

where $I$ is the moment of inertia of the rotating parts. The friction may be split into two separate quantities:

...26/
(i) $B_0$, the static frictional torque which is independent of $n$ and (ii) $B_1$, a constant of dynamic frictional torque and magnetic drag, if any, the net torque being given by $B_1n$ for a speed of rotation $n$.

Then the integrodifferential equation which represents the various torques acting on the anemometer is

$$2\pi I \frac{d}{dt} [n(t)] + B_0 + B_1n(t) = \rho R \left\{ a_m \omega(t) \omega(t) + \frac{1}{2} a_m \int_0^t f(t-\tau) \omega(\tau) d\tau \right\} - 2\pi R l_m \int_0^t n'(t-\tau) \omega(\tau) d\tau \right\}$$

or

$$\nu \frac{d}{dt} [n(t)] + B_0 + B_1n(t) = 2D \omega(t) \omega(t) + D \int_0^t f(t-\tau) \omega(\tau) d\tau - c \int_0^t n'(t-\tau) \omega(\tau) d\tau \right\}$$

where

$$\nu = 2\pi I$$

$$D = \frac{1}{2} \rho R a_m$$

and

$$c = 2\pi \rho R^2 l_m.$$

This equation is quite general for cup anemometers. The only assumptions are that the variation or change of speed in a period of $1/3$ of a
revolution is negligible compared to the actual speed, and consequently that the acceleration of wind can be taken to be uniform within this period.

Eq. 22 is a non-linear integrodifferential equation. This equation is capable of an analytical solution only for a step function change of wind speed. In order to solve this equation exactly, it becomes necessary to take into account the initial conditions of the equation which are mathematically the same as the "boundary conditions". The necessity to take into consideration the initial conditions suggests the solutions of the equation by the method of Laplace Transforms.

The eq. 22 is solved in Appendix I for the case of step wind velocity function given by

\[
\begin{align*}
\nu(t) &= \nu_0 \quad \text{for } t < 0^+ \\
\nu(t) &= \nu_1 \quad \text{for } t \geq 0^+ (23)
\end{align*}
\]

This form of wind variation can be assumed to take place in an "Ideal Wind Tunnel" the time taken to change over from the initial wind velocity \( \nu_0 \) to the final velocity \( \nu_1 \) being negligible. The solution of eq. 22 giving the speed of rotation of the anemometer for

is given by

\[
\eta(t) = \frac{(D \nu^2 - B_0)(1 - e^{-t/T})}{B_1 + e \nu_1} - \frac{(D \nu_1 - C \eta_0)}{B_1 + e \nu_1} \eta(1 - e^{-t/T}) + \eta_0 e^{-t/T} \]

\[\eta(t) \quad \text{for } 0 \leq t \leq T\]
Fig. 7 - Step function of velocity $v(t)$
where \( n(t) = n_0 \) and \( v(t) = v_0 \) for \( t < 0^+ \)
and \( v(t) = v, \) for \( t > 0^+ \)

The exponentials in eq. 24 have "Time constant" \( T \) which
is similar to the time constants in ordinary linear mechanics.
However, in the nonlinear aerodynamics of the anemometer
the "time constants" are no more constants but are
themselves functions of wind speeds. The "time constant"
\( T \) occurring in eq. 24 is given by

\[
T = \frac{\gamma}{B_1 + C_0}
\]  

(25)

Thus the "time constant" is larger the smaller the wind speed.
Taking reciprocals,

\[
\frac{1}{T} = \frac{B_1}{\gamma} + \frac{C_0}{\gamma} = \frac{1}{\tau_n} + \frac{v_0}{(\gamma/c)} = \frac{1}{\tau_n} + \frac{v_1}{(\gamma/c)}
\]  

(26)

where \( \tau_n = \frac{B_1}{B_1 - \frac{2\pi}{\gamma}} \) may be called as the "natural
time constant" of the anemometer and \( \tau = \gamma/c \) is the
so called Distance Constant. It is, perhaps, relevant
to define an "aerodynamic time constant" \( T_a \) given by

\[
T_a = \frac{\tau}{\gamma/c}
\]  

(27)

3. Evaluation of steady state calibration curve

In the simplest case, we may assume that the
anemometer is frictionless, so that both the constants \( B_0, \)
and \( B \), are zero. Then the speed of rotation \( n(t) \) at an instant \( t \) is given by

\[
n(t) = n_0 - \frac{D}{c} \left( \nu_i - \nu_o \right) \left( 1 - e^{-t/T} \right)
\]

If \( \nu_o = 0 \) and \( n_o = 0 \),

\[
n(t) = \frac{D}{c} \nu_i \left( 1 - e^{-t/T} \right)
\]

and as steady state is reached \( t \to \infty \) and \( n(t) \to n_i \),

\[
n_i = \frac{D}{c} \nu_i
\]

Thus the steady state scale factor for the anemometer is

\[
\gamma = \frac{D}{c}
\]

It is evident that since the initial conditions in eq. 34 were assumed to be steady before \( t = 0 \), the relation between \( n_o \) and \( \nu_o \) is given by

\[
n_o = \frac{D}{c} \nu_o
\]

and this can be directly substituted in eq. 28 so that

\[
n(t) = \frac{D}{c} \left[ \nu_o e^{-t/T} + \nu_i \left( 1 - e^{-t/T} \right) \right]
\]

\[
= n_o e^{-t/T} + n_i \left( 1 - e^{-t/T} \right)
\]

(See Fig 3)

or multiplying both sides by \( \frac{c}{\gamma} \) indicated wind velocity is given by

\[\ldots\ldots 30/\]
Fig. 8. Response of cup anemometers to a step function velocity.
From eq. 31, it is seen that the scale factor for a frictionless anemometer is a constant \( y = \frac{D}{c} \) given purely by the aerodynamic character of the cups.

The effect of static friction is to limit the starting of response of the anemometer to a certain minimum wind speed. The anemometer operating inside a wind tunnel would not rotate for wind speeds less than this value. This minimum wind speed is obtained from the eq. 24 as follows:

Since the anemometer is at rest initially
\[ u_0 = 0 \quad \text{and} \quad \bar{B}_t = 0 \]

and since only steady state conditions are being examined the exponential terms are zero. Then at the wind speed at which the anemometer will begin to rotate, i.e. the left hand side begins to have a value greater than zero, the wind speed is related to \( \bar{B}_c \) by

\[ \frac{D \bar{u}}{c} > \frac{\bar{B}_c}{c \bar{u}} \]  \hspace{1cm} (35)

or the minimum wind speed for starting is

\[ U_{\text{min}} = \frac{\bar{B}_c}{D} \]  \hspace{1cm} (36)
From the specifications for the British cup generator anemometer, it is seen that the wind speed at the start of response should not exceed $3 \frac{1}{2}$ knots. On the average, it is about 2 knots corresponding to nearly 1 meter per second. Substituting this value and that of $D$ obtained from Brevoort curve of Fig 2, we get the value for the constant $B_c$ as

$$B_c = 1.78 \times 10^{-4} \text{ dyne-cm}. \quad (37)$$

It is important to recognise that the constant $B_c$ is very much dependent on each individual anemometer, its maintenance, lubrication or wear of its ball bearings, etc.

So also, the constant $B_1$ representing the dynamic friction is a quantity pertaining to each individual anemometer. It is also not correct to assume that the value of $B_1$ remains the same for the entire range of calibration of the anemometer. This is due to the following reasons:

i) The frictional resistance under actual conditions of operation is actually a nonlinear function of the speed of rotation.

ii) Secondly, the constant $B_1$ is also representative of the magnetic drag on the rotor due to the sensing mechanism which acts superposed on the nonlinear type of friction that the bearings present to the rotor.
The average value of \( B_i \) for a particular instrument is obtained from the observed speed of rotation for a wind speed in the middle of range of calibration. This may be assumed to represent complete instrumental constant for the entire range. Thus, from the table of speed of rotation with respect to wind speed published in the Handbook, it is seen that at a wind speed of 40 knots the speed of rotation is 385 RPM. This gives a value

\[ B_i = 9.5 \times 10^4 \text{ C.G.S. units} \quad (39) \]

Thus, using the values of \( B_0 \) and \( B_i \), as given by eq. 37 and eq. 38, the speed of rotation for the anemometer for the entire range of operating conditions can be obtained from the relation (for the steady state conditions).

\[ n = \frac{D \omega^2 - B_0}{C \omega + B_i} \quad (39) \]

Eq. 39 is obtained from eq. 34 after proper substitutions for the steady state assumptions.

The values of speed of rotation obtained from eq. 39 for various wind speeds are tabulated in table 3 and plotted in Fig 9. The published experimental values of the speeds of rotation are also drawn in Fig. 9. The
Table 3
Calculated speeds of rotation and the calibration data for the British cup Generator Anemometer for different wind speeds

<table>
<thead>
<tr>
<th>Wind speed m/sec</th>
<th>Theoretical RPM</th>
<th>Experimental RPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>73</td>
<td>90</td>
</tr>
<tr>
<td>7.5</td>
<td>123</td>
<td>135</td>
</tr>
<tr>
<td>10</td>
<td>173</td>
<td>136</td>
</tr>
<tr>
<td>15</td>
<td>274</td>
<td>277</td>
</tr>
<tr>
<td>20</td>
<td>376</td>
<td>376</td>
</tr>
<tr>
<td>25</td>
<td>478</td>
<td>467</td>
</tr>
<tr>
<td>30</td>
<td>580</td>
<td>555</td>
</tr>
<tr>
<td>35</td>
<td>682</td>
<td>645</td>
</tr>
<tr>
<td>40</td>
<td>784</td>
<td>735</td>
</tr>
<tr>
<td>45</td>
<td>887</td>
<td>825</td>
</tr>
</tbody>
</table>
Figure 9. Calculated values of speeds of rotation and the calibration values for steady state wind speeds.

1. Theoretical frictionless anemometer
2. Theoretical with friction
3. Experimental values (as published in the handbook)
straight line calibration curve representing an ideal frictionless anemometer is also drawn in Fig 9.

It would be evident that whereas the calibration curve would have been a straight line in the case of a frictionless anemometer, the effect of frictional constants is to displace the line slightly to the right and give a slight curvature which becomes more prominent at both the lowest and the highest wind speeds of the range. The agreement in the curves 2 and 3 in Fig 9 proves the validity of eq. 39.

4. Transient characteristics of anemometers inside a wind tunnel

When discussing the transient characteristics of anemometers, it is important to recognise the fact that the transient response of an anemometer inside a wind tunnel is likely to be different from the transient response in natural atmosphere. This difference comes out due to the complications introduced by the totally random and transient character of the atmospheric wind speed coupled with the fact that the aerodynamic force acting on a moving obstacle is essentially nonlinear. The transient response of anemometers in natural atmosphere is dealt with in chapter 3 of this thesis. Here we are
only concerned with their transient response inside an "ideal wind tunnel" where a facility to obtain sudden changes of wind speed in accordance with the step function of eq. 23 (see also Fig 7) are available.

The time constant \( T \) for the response of anemometer is given by

\[
\frac{1}{T} = \frac{1}{T_n} + \frac{1}{T_a}
\]

where \( T_n \) is the "natural time constant" determined by

\[
T_n = \frac{2\pi I}{B_i} = \frac{I}{(\frac{B_i \omega_i}{2\pi})}
\]

and \( T_a \) is the "aerodynamic time constant" given by

\[
T_a = \frac{2\pi I}{c \nu_i} = \frac{\nu}{\nu_i}
\]

and \( \nu = \frac{2\pi I}{c} \). The natural time constant refers to the character of the anemometer as a mechanical instrument and gives the response time of the instrument when left inside a vacuum with a certain initial speed of rotation. For an ideal frictionless anemometer the natural time constant would be infinity and the instrument would behave as an ideal frictionless flywheel. It would continue to rotate inside the vacuum for ever*. The aerodynamic time constant

---

* If the wind speed acting on a frictionless anemometer drops to zero suddenly, the anemometer would continue to rotate indefinitely. This is almost similar to the case of a circulating current in a loop of superconducting wire excited once under suitable conditions. The main idea to be obtained from this fact is that the "response" of the sensor can be maintained indefinitely even after the removal of the forcing function (provided that there is no dissipation of energy due to damping losses) or in other words, the response is no more in proportion to the driving function. (See Chapter 3).
refers to the interactions established by the anemometer with the wind in motion. With reference to eq. 2.2, it is seen that this is the consequence of reduction of the aerodynamic torque acting on the instrument over an ideal torque that would operate on an ideal instrument. Such an ideal instrument would have infinite aerodynamic time constant or in other words, it would not offer any aerodynamic resistance to the wind motion. (This statement means that the constant \( C \) in eq. 2.3 for such an ideal instrument would be zero). We may consider a simple analogy to illustrate these effects.

If an electrical source with an internal resistance \( R \) is suddenly connected to a coil of inductance \( L \) the time constant for the establishment of the final value of current is given by

\[
T = \frac{L}{R} = \frac{\text{Inductance}}{\text{Internal Resistance of Source}}
\]

In a similar way, when the wind speed is suddenly raised to new value, the time constant for an ideal frictionless anemometer (i.e. \( T_\infty = \infty \)) would be given by

\[
T = T_a = \frac{2\pi I}{c \omega_i} = \frac{I}{\left( \frac{c \omega_i}{2\pi} \right)}
\]

\begin{align*}
&= \text{Moment of Inertia} \\
&\quad \text{Aerodynamic Resistance (44)}
\end{align*}
In eq. 44a, we define a new term, "aerodynamic resistance" of an anemometer given by

$$\text{aerodynamic resistance} = \frac{C \Omega}{2 \pi T}$$

(see footnote *)

It is seen from eq. 42 that the aerodynamic time constant is inversely proportional to the final wind speed of the step function. We may then define a distance constant $D$ such that

$$D = \frac{2 \pi T}{c}$$

This constant is incidentally the same as that described by the term "synchronisation length" of the anemometer. However, the term distance constant can have a meaning only for an ideal frictionless anemometer, and only for a step function change of windspeed as will be seen later.

---

* The internal resistance of the source in the analogy drops the available voltage at the terminals of the coil by virtue of the voltage developed by the current flowing across the same. In the same way the aerodynamic resistance reduces the available aerodynamic torque, the reduction of torque being proportional to the speed of rotation. (Refer to the second term on the right hand side of eq. 22).
For practical anemometers, the product of the time constant and the final speed is only an approximate constant.

If the British cup generator anemometer were assumed frictionless, the distance constant would be 18.2 meters according to value calculated from eq. 45. The product of the actual time constants and the final speeds as presented in col. 5 of Table 4 for the data obtained from the Handbook is always less than 18.2 meters. The col. 6 of Table 4 gives the distance constant obtained from the same experimental data, however, after correcting for the existence of a natural time constant of 10 seconds (See eq. 70, p.55). It is seen that these "corrected values" of the distance constant in this column are in good agreement with the theoretical value for the frictionless anemometer. In addition, the time constants for the various wind speeds calculated from the theoretical equation 25 are presented in Col. 4 of this table as against the published experimental data in Col. 3. Here again, the agreement is good. This proves the validity of eq. 25.

In Table XXVII on page 194 of the Handbook
Table 4

Calculated values of time constants and distance constant and the corresponding values obtained from experimental data. (Col. 6 shows the distance constant from experimental data after neglecting the effect of natural time constant and Col. 7 those "corrected" for natural time constant).

<table>
<thead>
<tr>
<th>Final velocity knots</th>
<th>Initial velocity knots</th>
<th>Time constant from experimental values sec.</th>
<th>Time constant calculated from eq. 25. sec.</th>
<th>Distance constant meters</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>17.5</td>
<td>0.7</td>
<td>0.9</td>
<td>12.9</td>
</tr>
<tr>
<td>35</td>
<td>26</td>
<td>0.7</td>
<td>0.9</td>
<td>12.9</td>
</tr>
<tr>
<td>26</td>
<td>13</td>
<td>1.1</td>
<td>1.36</td>
<td>14.6</td>
</tr>
<tr>
<td>26</td>
<td>17.5</td>
<td>1.3</td>
<td>1.36</td>
<td>17.4</td>
</tr>
<tr>
<td>17.5</td>
<td>9</td>
<td>1.7</td>
<td>1.66</td>
<td>15.3</td>
</tr>
<tr>
<td>17.5</td>
<td>13</td>
<td>1.35</td>
<td>1.66</td>
<td>17.5</td>
</tr>
<tr>
<td>13</td>
<td>9</td>
<td>2.6</td>
<td>2.26</td>
<td>17.4</td>
</tr>
</tbody>
</table>

\[T_n = 10\text{ sec}\]
another set of measurements is presented in which the time constants are obtained for the case when the anemometer started from rest. It may be noted that these time constants are considerably shorter than the values which would be obtained by the eq. 25. It is now necessary to examine the situation when the anemometer (which is at rest initially) is suddenly exposed to a strong wind velocity. At the moment of the sudden exposure, a strong impulse of momentum is offered to the particular cup which faces the wind direction and the other cups are practically indifferent. The anemometer picks up the response very much faster due to this initial impulse. The formula of eq. 26 for the time constant and the theoretical development leading to the formula have to be modified for the new condition. This is because the original assumptions regarding the averaging process for the effective cross section of all the three cups and that of the constancy of the speed of rotation during a motion through 120° are no more representative of the situation when an anemometer at rest is suddenly exposed to a strong wind velocity. It is in fact not possible to define a time constant for an anemometer in these conditions. Any measurement of the time constant would
invariably depend on the orientation of the anemometer cups at the moment of exposure. The same difficulty was also pointed out by Schubauer and Adams who originally developed the idea of the distance constant. It is due to this reason the time constant becomes very low when the anemometer which is initially at rest is suddenly exposed to a strong wind. The occurrence of the very low value of time constant of 0.7 sec. for the large wind velocity of 35 knots (see table 4) may also be explained by a similar reasoning. From the curves of static forces of Brevoort and Joyner it is seen that for low Reynolds numbers, there is some what a large deviation from the mean curve for the Reynolds number 200,000. This deviation, not having been taken into account in the calculations of this paper, may perhaps also account for differences observed in the calculated and the observed values of the time constants presented on the table 4. In addition, it is practically impossible to simulate an ideal step function of wind speed in a practical tunnel.

5. Conclusions

A rigorous transient analysis for the cup anemometer has been presented in this paper and the results are verified on the basis of the experimental data which are already available in the published form.
The fact that the cup and vane anemometers have a constant called the Distance constant was first recognised by Schubaur and Adams who have published the relevant data for the American anemometers, Yaglom had simultaneously proposed a term called the "synchronisation length" to describe the response behaviour of the anemometers. However, though the fact is recognised that the time constants are inversely proportional to the wind velocities, the important result of this Chapter that they are also controlled by the natural time constant in addition to the so-called distance constant has not been recognised so far. An explanation, based on a transient theory, for the observed study state characteristics inside the wind tunnel has been presented for the first time. It is now possible to calculate beforehand an approximate steady state calibration curve for any design of the cup anemometer on the basis of inertia and the steady state aerodynamic forces measured with stationary cups, the extent of departure of the actual curve from the theoretical curve depending on the amount of friction.
Fig. 10. Cup generator anemometer installed along with Dines pressure tube anemometer.
Appendix I

Solution of the nonlinear integrodifferential equation (eq. 22)

The definition of the step function wind speed is given by eq. 23 and illustrated in Fig 7. Let the Laplace transforms of various quantities be given by

\[ N = N(s) = L[N(t)] \]
\[ V = V(s) = L[V(t)] \]
\[ and \ F = F(s) = L[F(t)] \]

The Laplace transform of the product \( V(t) \cdot \Phi(t) \) is obtained by substituting \( \Phi(t) = \Phi_1(t) \cdot \Phi_2(t) \) in the formula

\[
L \left[ \Phi_1(t) \cdot \Phi_2(t) \right] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Phi(s) \cdot F_2(s) \cdot \sigma \, ds
\]

where \( F_2(s) = L[\Phi_2(t)] \) and \( \Phi_1(t) = \Phi(t) \).

The Laplace transform of a derivative is obtained from the formula

\[
L \left[ \frac{d}{dt} \Phi(t) \right] = S \Phi(s) - \Phi(0)
\]

where \( \Phi(s) = L[\Phi(t)] \).

The Laplace transforms of the integrals occurring on the right hand side of eq. 22 are given by convolution theorem.
Then the eq. 22 becomes, after taking the Laplace transforms,

\[ \mathcal{L} \left[ \int_0^t b_1(t-\tau) b_2(t-\tau) d\tau \right] = F_1(s) F_2(s) \]  

\[ 49 \]

This equation gives the Laplace transform of \( \eta(t) \) as a function of the initial wind velocity \( v_o \), the initial rotational speed of the anemometer \( \eta_0 \) and the Laplace transform of the wind velocity \( \nu(t) \). This is a complicated relation. This complication is further increased when we subject the anemometer to natural random wind speeds. The natural wind velocity is always a
random variable and the Laplace Transform of a random variable is yet another random variable. It is perhaps simpler to study the response of the anemometer to conditions existing inside the wind tunnel. The development leading to the study of response in the natural environment is considered later in Chapter 3. In an ideal case, let us assume that a steady wind speed $v_0$ exists initially; Let this speed be arbitrarily increased to $v_i$ at an instant $t = 0$ and thereafter kept steady at the new value $v_i$. Let us further assume that the change over from the initial speed $v_0$ to the final speed $v_i$ occurs instantly at $t = 0$ without involving any appreciable duration. Such a function of wind speed is described by eq. 23.

$$
\begin{align*}
v(t) &= v_0, \quad t < 0 \\
v(t) &= v_i, \quad t > 0
\end{align*}
$$

and is commonly known as the step function. The Laplace transform for this function is given by

$$
V(s) = V = \int [v(t) e^{-st}] = \frac{v_i}{s} \tag{52}
$$

The Laplace transform representing the acceleration of wind is

$$
v'(s) = V' = sV - v_0 = v_i - v_0
$$
The Laplace transform of the product function

\[
\mathcal{L}[v(t)w(t)] = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} \frac{\mathcal{L}[v(t)]}{\mathcal{L}[w(t)]} \mathcal{L}[w(t)] d\sigma \quad \text{--- (54)}
\]

By using partial fraction,

\[
\mathcal{L}[v(t)w(t)] = \frac{1}{2\pi j} \frac{\mathcal{L}[v(t)]^2}{\mathcal{L}[w(t)]} \left[ \int \frac{d\sigma}{s-\sigma} + \int \frac{d\sigma}{\sigma} \right]_{-j\infty}^{j\infty}
\]

\[
= \frac{1}{2\pi j} \frac{\mathcal{L}[v(t)]^2}{\mathcal{L}[w(t)]} \log \left( \frac{s-\sigma}{\sigma} \right)_{-j\infty}^{j\infty}
\]

\[
= \frac{1}{2\pi j} \frac{\mathcal{L}[v(t)]^2}{\mathcal{L}[w(t)]} \log \left( \frac{s}{s-\sigma} \right)_{-j\infty}^{j\infty}
\]

\[
= \frac{1}{2\pi j} \frac{\mathcal{L}[v(t)]^2}{\mathcal{L}[w(t)]} \log \left( \frac{s}{s-\sigma} \right)
\]

\[
= 0 \quad \text{--- (55)}
\]

The Laplace transform of the function \( f(t-r) \) is given by

\[
F(s) = \mathcal{L}[f(t-r)] = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} \mathcal{L}[f(t-r)] d\sigma
\]

From eq. (5.2)

\[
\mathcal{L}[f(t-r)] = \frac{\mathcal{L}[f(t)]}{s-\sigma} \quad \text{--- (56)}
\]

\[
\mathcal{L}[f(t-r)] = \frac{\mathcal{L}[f(t)]}{s-\sigma} \quad \text{--- (57)}
\]
From eq. (49)

\[ W'(s - \sigma) = (s - \sigma) W(s - \sigma) - W_0 \]

\[ = (s - \sigma) \frac{U_1 - U_0}{\sigma} - W_0 \]

\[ = U_1 - U_0 \]

...(53)

Now the Laplace Transform of \( u(t - \tau) \)

\[ L[u(t - \tau)] = u(s) = \frac{e^{-s\tau}}{s} \]

or \( u(s) = \frac{e^{-s\tau}}{s} \)

...(59)

From eq. 46, 47, 58 and 60

\[ F(s) = L[-f(t - \tau)] \]

\[ = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\frac{-e^{-s\tau}}{s}}{s} \, ds \]

\[ = \left[ \frac{1}{2\pi j} \right] \int_{c-j\infty}^{c+j\infty} \frac{e^{-s\tau}}{s} \, ds \]

\[ = (U_1 - U_0) \]

\[ \ldots (61) \]

However, the expression within the square brackets is identically equal to \( u(t - \tau) \)

or

\[ F(s) = (U_1 - U_0) u(t - \tau) \]

\[ \ldots (63) \]
From eq. (12), the condition for the function \( f(t-\tau) \) to exist is that \( (t-\tau) \geq 0 \). Under this condition

\[ \tau(t-\tau) = 1 \]

or

\[ F = F(s) = (\nu_1 - \nu_0) \quad .. (64) \]

Substituting eq. 64 and 52 in eq. 51

\[ N = \frac{D [\nu_1 - \nu_0]}{s} \frac{\nu_1}{s} + \frac{c_1 \nu_1}{s} \nu_0 - \frac{B_0}{s} + \frac{\gamma \nu_0}{s} \left[ \gamma + \frac{c_1 \nu_1}{s} \right] s + B_1 \]

\[ = \frac{D \nu_1}{(B_1 + c_1 \nu_1) s (1 + Ts)} - \frac{(D \nu_0 - c_1 \nu_0) \nu_1}{(B_1 + c_1 \nu_1) S(1 + Ts)} \]

\[ - \frac{B_0}{(B_1 + c_1 \nu_1) S(1 + Ts)} + \frac{T \nu_0}{1 + Ts} \quad .. (65) \]

where

\[ T = \frac{\gamma}{B_1 + c_1 \nu_1} \]

\[ ..(25) \]
Taking inverse Laplace transform, the solution for the step function wind speed is given by

\[
\tau(t) = \frac{D \nu_r e^{-(t/\tau)}}{B_1 + C \nu_r} - \frac{(D \nu_0 - C \nu) e^{-(t/\tau)}}{C (B_1 + C \nu_r)} - B_0 \frac{e^{-t/\tau}}{B_1 + C \nu_r} + n_0 e^{-t/\tau} - 2A
\]

where \( \tau(t) = n_0 \) at \( t = 0^- \)
and \( \tau(t) = \nu_0 \)

and \( \nu(t) = \nu_1 \) for \( t > 0^+ \)
(a) CONICAL SECTION OF THE CUP

$R_C = 2.63\text{ cm}$.
(b) ELEVATION SKETCH

(C) PLAN OF SECTION PP IN FIG.(IIa)

CALCULATION OF MOMENT OF INERTIA

By PRP

FIG.11
Appendix II
Details of calculation

1) Calculation of moment of inertia of rotating parts of the anemometers

The exact shape of cup of the anemometer is shown in Fig 1. To a first approximation, we may assume that it is conical. Fig 1la gives the conical section assumed for the cup and the Fig 1lb is the elevation diagram showing the position of the cup with respect to the axis of rotation.

Let us take the section $P-P$ in Fig 1la at a depth $l$ from the vertex along the surface. Then

$$l \cos \theta = r \quad \text{or} \quad l = \frac{r}{\cos \theta}$$

since $\theta = 45^\circ$, $r$ being the radius of the 'ring' section at $P-P$. The circle (or 'ring') shown in Fig 1lc gives the section of the cone $P-P$.

Let QQ be a diameter of the ring parallel to the axis of rotation and also parallel to Q'Q' of Fig 1lb which is a diameter of the base. The ring of Fig 1lc has a radius $r$ and an element of length $ds$ along the ring is $rd\phi$ where $\phi$ is the angle as shown in the Figure, and $d\phi$ is the angle subtended by the element $ds$ at the centre of the ring. If $\rho_r$ refers to the mass per unit length of the ring, mass of an element $ds$ is given by $\rho_r ds = \rho_r rd\phi$
mass of the total ring is \[ m = 2\pi \rho \tau \]

moment of inertia of the element \( ds \) about the diameter \( QQ \) is
\[ \rho \tau \sin \phi \left( \tau \sin \phi \right)^2 = \rho \tau^3 \sin^2 \phi \ d\phi \]

Total moment of inertia of the ring about \( QQ \) is
\[ = 4\rho \tau^3 \int_0^{\pi/2} \sin^2 \phi \ d\phi = \frac{\tau^3 \pi \rho}{2} \sin \alpha \ m = 2\pi \rho \tau \]

The moment of inertia of the ring about the parallel axis \( QQ' \) at the base is \[ m \left[ \frac{\tau^2}{2} + \zeta^2 \right] \] where \( \zeta \) is the distance of the plane of the ring \( PP \) from the plane of the base. If \( R_c \) is the radius of the base, the height of the cone is also \( R_c \) and the distance of the plane of the ring \( PP \) from the vertex is \( \tau \), so that
\[ \zeta = R_c - \tau \]

Moment of inertia of the ring \( PP \) about the axis \( QQ' \) is
\[ = m \left[ \frac{\tau^2}{2} + (R_c - \tau)^2 \right] = m \left[ \frac{\tau^2}{2} - 2R_c \tau + R_c^2 \right] \]

where \( m \) is the mass of the ring.

The circumference of the ring \( = 2\pi \tau \)

the width of the ring \( = \tau \cos \theta \) \( \text{d}r \)

area of the ring along the surface \( = 2\pi \sqrt{\rho} \text{d}r \)

If \( \rho_a \) is the mass per unit area of the material of the cup (assumed to be uniform), the mass of the ring \( PP \) is
\[ 2\pi \sqrt{\rho} \tau \text{d}r \rho_a = m \]

...69/
Moment of inertia of the ring about the axis is $Q'Q'$
\[ m \left[ \frac{3}{2} r^2 - 2R_c r + R_c^2 \right] = 2\pi \sqrt{2} \left[ \frac{3}{2} r^2 - 2R_c r + R_c^2 \right] \]

Total M.I. of the cone about $Q'Q'$
\[ 2\pi \sqrt{2} \rho \left\{ \frac{3}{2} \int_0^{R_c} r^3 dr - 2R_c \int_0^{R_c} r^2 dr + \frac{R_c^2}{2} \right\} \]
\[ = \frac{5\pi}{24} \sqrt{2} R_c^4 \rho \]

The mass of the cup
\[ = \int_0^{R_c} 2\pi \sqrt{2} \rho \frac{r^2}{2} \] (C = M) say

The M.I. of the cup about $Q'Q'$ = \( \frac{5}{12} M R_c^2 \)

The M.I. of the cup about the axis of rotation (shaft) =
\[ = \text{M.I. of the cup about } Q'Q' + MR^2 \]
\[ = M \left[ \frac{5}{12} R_c^2 + R^2 \right] \] (66)

$R$ being the distance of the cone $Q'Q'$ from the axis of rotation.

Mass of the cup = 160 gm. Radius of the base $R_c = 2.612''$

The distance between $Q'Q'$ and the axis = 6.5''

Therefore M.I. of each cup about the axis = $4.64 \times 10^4$ gm cm$^2$.

M.I. of 3 cups = $1.4 \times 10^5$ gm cm$^2$

Mass of the connecting rod = 55 gm.

'Effective length' in which the mass is distributed = 6'' (approx.)

Moment of inertia of each rod = $4.26 \times 10^3$

M.I. of 3 rods = $1.28 \times 10^4$

This is small compared to that of the cups. Similarly M.I. of the shaft itself can be shown to be negligible in comparison with that of the cups. Hence M.I. of the rotating system = $1.4 \times 10^5 + 0.13 \times 10^5 = 1.53 \times 10^5$ gm cm$^2$
11) Calculation of Anemometer characteristics

1. \( \gamma = 2n^2 \) = 9.5 \times 10^5 \text{ COS Units} 

2. \( \frac{(am/A)}{c} = 1.25 \text{ where } A = \frac{\pi \times 5.26^2 \times 2.54^2}{4} \text{ sq cm} \)

3. \( a_m = 1.74 \times 10^{-6} \text{ sq cm} \)

4. \( B_m = 2.35 \times 10^{-3} \text{ sq cm} \)

5. \( c = 2 \pi \times R \times b = 3 \pi \times 1.29 \times 10^{-3} \times 6.5 \times 2.54 \times 2.35 \times 10^{-3} \)

6. \( D = 1.78 \times 10^4 \text{ dynes cm} \)

7. Scale factor \( = \frac{D}{c} = 3.4 \times 10^{-3} \text{ rps per cm per sec.} \)

8. The minimum velocity at which anemometer will rotate = 1 m/sec.

9. The speed of rotation for 40 knots = 326 rpm (vide Handbook)

10. Natural time constant \( T_n = \frac{2\pi}{\gamma} = 10 \text{ seconds} \)

11. Distance constant \( = \frac{T}{c} = \frac{9.5 \times 10^4}{521} = 18.2 \text{ meters} \)

iii) Evaluation of the calibration curve from the theoretical formula of eq. 47

Example:

Let the wind speed be 5 m/sec.

\[ n = \frac{1.78 \times 500^2 - 1.78 \times 10^4}{9.5 \times 10^4 + 521 \times 500} \times 60 = 73 \text{ rpm} - 67 \]

....54
The results obtained for various wind speeds in this way are tabulated in Table 3 and plotted in Fig 3. The experimental calibration curve from the data published in the Handbook is also drawn in Fig 3 for comparison. The two curves agree within 5% for all the velocities.

iv) Calculation of time constants

1. Natural time constant

\[ T_n = \frac{\gamma}{B_1} = \frac{9.5 \times 10^5}{9.5 \times 10^4} = 10 \text{ sec.} \quad \ldots \ldots \ldots \ldots \ldots (6) \]

2. Time constant for any final speed is

\[ T = \frac{T_a T_n}{T_a + T_n} \quad \ldots \ldots \ldots \ldots \ldots (69) \]

where

\[ T_a = \frac{\text{Distance constant}}{\text{Final wind speed}} = \frac{b}{V_f} \]

and \( b \) = Distance constant = \( \gamma / c \)

Example: (Let the speed be 36 knots = 36 \times 51.5 \text{ cm/sec})

\[ T_a = \frac{18.2 \times 100}{35 \times 51.5} = 1.0 \text{ sec} \]

\[ T = \frac{T_a T_n}{T_a + T_n} = \frac{1.0 \times 10}{1 + 10} = 0.9 \text{ sec} \]

The time constants for speeds of 26, 17.5 and 13 knots have been calculated and tabulated along with the experimental data from the Handbook in Table 4. There is a large measure of agreement and the probable reasons for the discrepancies have already been discussed.
v) Calculation of distance constant for an ideal frictionless anemometer from the experimental data of an actual anemometer

For the frictionless anemometer, the time constant for a particular wind speed is

\[ T_{\text{ideal}} = T_a \]  \hspace{1cm} (69a)

For the actual anemometer, the time constant for the same wind speed is

\[ T = \frac{Ta T_n}{Ta + T_n} \]  \hspace{1cm} (69)

Using eq. (69) and (69a)

\[ T_{\text{ideal}} > T_a > \frac{T T_n}{T_n - T} \]

the distance constant for the ideal anemometer from the experimental data is

\[ \zeta = T_{\text{ideal}} \times \frac{T T_n}{T_n - T} \]  \hspace{1cm} (70)

This equation is used to calculate the figures under column 6 of Table 4.
References


