3.1. Introduction

Let us again consider the problem defined in chapter 2 of allocating the sample to various strata for a given budget when several characters are under study. The variances of various characters are minimized under the condition of fixed budget.

So we consider the following $p$ convex programming problems given in (2.8) of chapter 2.

\[
\begin{align*}
\text{Min.} & \quad V = \sum_{i=1}^{L} \frac{W_i^2 S_{ij}^2}{n_i}, \quad j = 1,...,p \\
\text{Subject to} & \quad \sum_{i=1}^{L} c_i n_i \leq C \\
& \quad 2 \leq n_i \leq N_i, \quad n_i \in I, \quad i = 1,...,L
\end{align*}
\]

By putting $n_i = \frac{1}{x_i}, \quad i = 1,...,p$ and $a_{ij} = W_i^2 S_{ij}^2$, the problems (3.1) are transformed the following $p$ problems where all the objective functions are linear:

\[
\begin{align*}
\text{Min} & \quad \sum_{i=1}^{L} a_{ij} x_i, \quad j = 1,...,p, \quad (i) \\
\text{Subject to} & \quad \sum_{i=1}^{L} \frac{c_i}{x_i} \leq C \quad (ii) \\
& \quad \frac{1}{N} \leq x_i \leq \frac{1}{2}, \quad i = 1,...,L \quad (iii)
\end{align*}
\]
If the costs $c_i$ in the various strata are assumed random with independent normal distributions, the problem (3.2) are transformed the following chance constrained programming form:

$$
\begin{aligned}
\text{Min} & \sum_{i=1}^{L} a_j x_j, \quad j = 1, \ldots, p \quad (i) \\
\text{Subject to} & \\
\mathbb{P}\left(\sum_{i=1}^{L} \frac{c_i}{x_i} \leq C\right) & \geq p_0 \quad (ii) \\
2 \leq n_i \leq N_i, \; i = 1, \ldots, L, \; n_i \in N & \quad (iii)
\end{aligned}
$$

where $p_0$, $0 \leq p_0 \leq 1$ is a specified probability.

### 3.2 Deterministic equivalent using Chance Constrained Programming

We have assumed that the costs $c_i$, $i = 1, \ldots, L$ in the constraint functions 3.3(ii) are independently and normally distributed random variables. Then function $\sum_{i=1}^{L} \frac{c_i}{x_i}$, will also be normally distributed with mean as

$$
E\left(\sum_{i=1}^{L} \frac{c_i}{x_i}\right) = \sum_{i=1}^{L} \frac{E(c_i)}{x_i} = \sum_{i=1}^{L} \frac{\mu_i}{x_i},
$$

(3.4)

where $\mu_i = E(c_i)$, $i = 1, \ldots, L$,

and variance as

$$
V\left(\sum_{i=1}^{L} \frac{c_i}{x_i}\right) = \sum_{i=1}^{L} \frac{1}{x_i^2} V(c_i) = \sum_{i=1}^{L} \frac{\sigma_i^2}{x_i^2},
$$

(3.5)
where $\sigma_i^2 = V(c_i)$.

Now let $f(c) = \sum_{i=1}^{L} \frac{c_i}{x_i}$, where $c = (c_1, ..., c_n)$, then \{3.3 (ii)\} is given by

$$P\left( f(c) \leq C \right) \geq p_o,$$  \hspace{1cm} (3.6)

$$\frac{C - E\{f(c)\}}{\sqrt{V\{f(c)\}}} \geq K_{\alpha},$$

Following the derivations in chapter 2, the probabilistic constraint (3.6) is equivalent to the deterministic constraint

$$E\left(f(c)\right) + K_{\alpha} \sqrt{V(f(c))} \leq C.$$  \hspace{1cm} (3.7)

Substituting from (3.4) and (3.5) in (3.7), we get

$$\left( \sum_{i=1}^{L} \frac{\mu_i}{x_i} \right) + K_{\alpha} \sqrt{\sum_{i=1}^{L} \frac{\sigma_i^2}{x_i^2}} \leq C.$$  \hspace{1cm} (3.8)

If the constants $\mu_i$ and $\sigma_i$ in (3.8) are unknown then we use their estimators $\hat{\mu}_i$ and $\hat{\sigma}_i^2$.

Thus

$$\hat{E}\left( \sum_{i=1}^{L} \frac{c_i}{x_i} \right) = \sum_{i=1}^{L} \frac{\hat{E}(c_i)}{x_i} = \sum_{i=1}^{L} \frac{\hat{c}_i}{x_i},$$  \hspace{1cm} say,

$$\hat{V}\left( \sum_{i=1}^{L} \frac{c_i}{x_i} \right) = \sum_{i=1}^{L} \frac{\hat{\sigma}_c^2}{x_i^2},$$  \hspace{1cm} say,

where $\hat{c}_i$ and $\hat{\sigma}_c^2$ are the estimated means and variances from the sample.
Thus, an equivalent deterministic constraint to the stochastic constraint \{3.3(ii)\} is given by

\[
\left( \sum_{i=1}^{L} \bar{c}_i + c_0 \right) + K_a \sqrt{\sum_{i=1}^{L} \frac{\sigma_i^2}{x_i}} \leq C .
\]

The equivalent deterministic non-linear programming problem to the chance constrained programming problem (3.3) is obtained as

\[
\begin{align*}
\text{Min. } V_j &= \sum_{i=1}^{L} a_j x_j , \quad j = 1, ..., p \quad (i) \\
\text{Subject to } \left( \sum_{i=1}^{L} \bar{c}_i + K_a \sqrt{\sum_{i=1}^{L} \frac{\sigma_i^2}{x_i}} \right) &\leq C \quad (ii) \\
\frac{1}{N_i} \leq x_i \leq \frac{1}{2}, \quad i = 1, ..., L \quad (iii) .
\end{align*}
\]

### 3.3. Convex Chebyshev Approximation Problem

Consider \( p \) convex smooth functions

\[
g_j(x) = g_j(x_1, ..., x_n), \quad j = 1, ..., p
\]

and a region \( \Omega \) defined by \( q \) inequalities

\[
\Psi_i(x) = \Psi_i(x_1, ..., x_n) \leq 0 , \quad i = 1, ..., q
\]

where \( \Psi_i \) are also convex smooth functions.

The Convex Chebyshev Approximation Problem (CCAP) for minimizing the system (3.10) under Constraints (3.11) consists in finding \( x^* \in \Omega \) for which
\[ \max_j g_j(x) = \min_{\omega \in \Omega} \max_j f_j(x). \] (3.12)

Since \( \max_j g_j(x) \) is convex as can be seen from the figure (3.1) below, the (CCAP) is convex.

Corresponding to the points \( (x_1, \ldots, x_j) \in \Omega \), we have

\[ \max_j f_j(x_j) = \{ f_1(x_1), f_2(x_2), f_3(x_3) = f_1(x_1), f_2(x_2), f_3(x_3) \} \]

Fig 3.1: Convexity of the function \( \max_j f_j(x) \).

In the general (CCAP) (3.10) & (3.11) we introduce an auxiliary variable \( x_{n+1} \) and the auxiliary constraints \( a_j g_j(x) \leq x_{n+1}, j = 1, \ldots, p \), where \( a_j \) are some constants.

The problem (3.12) then is equivalent to

\[
\begin{align*}
\text{Min } Z &= x_{n+1} \\
\text{subject to} & \\
 a_j g_j(x_1, \ldots, x_n) - x_{n+1} & \leq 0, j = 1, \ldots, p \\
\text{and } & \\
\psi_i(x_1, \ldots, x_n) & \leq 0, i = 1, \ldots, q
\end{align*}
\] (3.13)
3.4. Solutions Using Chebyshev Approximation Technique

The \( p \) objective functions in \( \{3.9(i)\} \) are linear. The single constraint \( \{3.9(ii)\} \) is convex [see Kokan and Khan (1967)] and \( \{3.9(iii)\} \) are upper and lower bounds on \( x_i \). So \( (3.9) \) represents \( p \) convex programming problems. Let us denote the feasible region defined by \( 3.9 \) (ii) and (iii) by \( \Omega \). Suppose that the feasible region is not void. Let us introduce an auxiliary variable \( x_{L+1}, j = 1, \ldots, p \). From \( (3.10) \) to \( (3.12) \) it follows that the problem \( (3.9) \) is equivalent to the convex Chabyshev’s approximation problem of finding \( \bar{x}^* \in \Omega \) such that

\[
\max_j a_j V_j(x) = \min_{\bar{x} \in \Omega} \max_j a_j V_j(x),
\]

where \( a_j \) are the weights assigned to the \( p \) variances according to their importance.

The problem \( (3.10) \) is then equivalent to the following problem with a linear objective function:

\[
\begin{align*}
Z &= x_{L+1} \quad \text{(i)} \\
\text{subject to} \quad a_j V_j(x) &\leq x_{L+1} \text{ or } a_j \sum_{i=1}^{L} a_j x_i - x_{L+1} \leq 0, \quad j = 1, \ldots, p \quad \text{(ii)} \\
\sum_{i=1}^{L} \frac{c_i}{x_i} + K \alpha \sqrt{\sum_{i=1}^{L} \frac{\sigma^2}{x_i^2}} &\leq C \quad \text{(iii)} \\
\text{and } \frac{1}{2} \leq x_i \leq \frac{1}{2}, \quad i = 1, \ldots, L \quad \text{(iv)}
\end{align*}
\]

The non-linear programming problem in \( (3.15) \) is convex as the objective function \( \{3.15(i)\} \) is linear and the constraints \( \{3.15(ii)\} \) are linear. Further, the left hand side in \( \{3.15(ii)\} \) is convex. So it is possible to solve the convex programming
problem (3.15) by using any standard convex programming algorithm. The optimal sample numbers thus obtained may turn out to be fractional. However, it is known that the variance functions are flat at the optimum solution. So for large or even moderate sample size it is enough to round the fractional values to the nearest integers. However, for small $n_i = \frac{1}{x_i}$ the branch and bound method should be applied for finding the optimal integer solution. The approach of this chapter has been accepted for publication in *American Journal of Computational Mathematics*. See Khan, M. Faisal et al.

3.5. Numerical Illustration

Let us consider the data of numerical illustration considered in section 2.8.

In order to demonstrate the procedure the following are also assumed. The per unit travel costs $c_i, \ (i = 1, ..., 4)$ of measurement in various strata are independently normally distributed with the following means and variances

$$E(c_1) = 3, \ E(c_2) = 4, \ E(c_3) = 5, \ E(c_4) = 7 \ \text{and} \ V(c_1) = 0.6, \ V(c_2) = 0.5, \ V(c_3) = 0.7, \ V(c_4) = 0.8$$

Let us assign the weights to the variances of the two characters in proportion to the inverse of the sums $\sum_{i=1}^{4} S_{i1}$ and $\sum_{i=1}^{4} S_{i2}$ which turn out to be $a_1 = 0.75 \ \text{and} \ a_2 = 0.25$.

The total amount available for the survey $C$ is assumed as 600 units including an expected overhead cost $t_0 = 100$ units.
Let the chance constraint \( \{3.3(ii)\} \) be required to be satisfied with 99% probability. Then \( k_\alpha \) is such that \( \phi(k_\alpha) = 0.99 \). The value of standard normal variable \( K_\alpha \) corresponding to 99% confidence limits is 2.33. Thus, the problem (3.15) is obtained as:

\[
\begin{align*}
\text{Min.} &= X_5 \\
\text{Subject to} & \\
0.75(552 .640 X_1 + 136 .277 X_2 + 274 .114 X_3 + 2588 .343 X_4) - X_5 & \leq 0 \\
0.25(14926 .197 X_1 + 165 .9747 X_2 + 130 .202 X_3 + 3084 .324 X_4) - X_5 & \leq 0 \\
\left( 3 + \frac{4}{X_1} + \frac{5}{X_2} + \frac{7}{X_4} \right) + 2.33 \sqrt{\frac{0.6}{X_1^2} + \frac{0.5}{X_2^2} + \frac{0.7}{X_3^2} + \frac{0.8}{X_4^2}} & \leq 500 \\
\frac{1}{1419} \leq X_1 \leq \frac{1}{2}, \quad \frac{1}{619} \leq X_2 \leq \frac{1}{2}, \quad \frac{1}{1253} \leq X_3 \leq \frac{1}{2}, \quad \frac{1}{899} \leq X_4 \leq \frac{1}{2}
\end{align*}
\]

The Chebyshev point by solving the convex programming problem (3.16) is

\[
X^{*}_{ch} = (0.02159, 0.12169, 0.10866, 0.03882) \quad \text{with} \; X_5 = 119.0883
\]

The values of sample sizes rounded to nearest integers are \( n_1 = 46, \ n_2 = 8, \ n_3 = 9 \) and \( n_4 = 26 \), with a total of 89. Corresponding to this allocation the values of the variances for the two characters are obtained as \( V_1 = 159.05, \ V_2 = 478.32 \)

Remark: We may compare these results with the compromise solution of chapter 2 which is obtained by solving the following NLP problem:
The integer solution is obtained as $n_1 = 44$, $n_2 = 9$, $n_3 = 10$ and $n_4 = 29$ with a total of 92. The values of the individual variances, corresponding to this allocation are obtained as $V_1 = 144.36$ and $V_2 = 476.90$. 

\[ \text{Min. } V = 0.75 \left( \frac{552 \cdot 640}{n_1} + \frac{136 \cdot 277}{n_2} + \frac{274 \cdot 114}{n_3} + \frac{2588 \cdot 343}{n_4} \right) + 0.25 \left( \frac{14926 \cdot 197}{n_1} + \frac{165 \cdot 39747}{n_2} + \frac{130 \cdot 202}{n_3} + \frac{3084 \cdot 324}{n_4} \right) \]

Subject to

\[ (3n_1 + 4n_2 + 5n_3 + 7n_4) + 2.33 \sqrt{0.6n_1^2 + 0.5n_2^2 + 0.7n_3^2 + 0.8n_4^2} \leq 500 \]

$2 \leq n_1 \leq 1419$, $2 \leq n_2 \leq 619$, $2 \leq n_3 \leq 1253$, $2 \leq n_4 \leq 899$