This is the Computer Program for generating the Henon map given in eqn. (2.3.1). This programme is meant to be run on an IBM PC or a compatible microcomputer. The programme was originally given in ref. [10] of chapter II, and has been modified wherever necessary for our use.

program henon1;
const
  MAXREAL = 100000000000.0;
var
  i,j,p1,p2,orbits, points : integer ;
  r,l,t,b,a,xold,yold,xnew,ynew,x0,y0,dx0,dy0,xscale,yscale,
  cosa,sina : real;
  grid : boolean;  
(set this to false to turn off the grid)
BEGIN
  grid := true;
  a:=1.111;
  l:=-1.2;
  r:= 1.2;
  b:=-1.2;
  t:=1.2;
  x0:=0.098;
  y0:=0.061;
  dx0:=0.04;
  dy0:=0.03;
  orbits:=25;
  points:=500;
  begin
    clrscr;
    gotoxy(2,1);
    writeln('INPUT PHASE ANGLE A (IN RADIANS BETWEEN 0 AND PI)');
    readln(a);
    writeln('INPUT STARTING POINT FOR FIRST ORBIT (X0,Y0)');
    readln(x0,y0);
    writeln ('INPUT X AND Y INCREMENTS IN ORBITS: DX0, DY0');
    readln(dx0,dy0);
    writeln('INPUT NUMBER OF ORBITS ');
    readln(orbits);
    writeln('NUMBER OF POINTS PER ORBIT');
    readln(points);
    writeln('INPUT LEFT AND RIGHT WINDOW VALUES');
    readln(l,r);
writeln('INPUT BOTTOM AND TOP WINDOW VALUES');
readln(b,t);

hires;
hirescolor(WHITE);

if grid then (draw reference grid if GRID is TRUE)
begin
  p1:=round((0.0-1)*640/(r-1));  \{find origin\}
p2:=round((t-0.0)*200/(t-b));  \{origin off screen so\}
end;
end;
draw(0,p2,640,p2.1);
draw(p1,0,p1,200.1);
for j:=0 to 20 do
begin
  draw(64*j,p2+3.64*j,p2-3.1);
  draw(p1+5.20*j,p1-5.20*j,1);
end;
gotoxy(1,1);
writeln('A= ',a:6:5);
cosa:=cos(a);
sina:=sin(a);
xold:=x0;
yold:=y0;
xscale:=640/(r-1);
yyscale:=200/(t-b);
for j:=1 to orbitn do
begin
  i:=1;
  while i<=points do
  begin
    if (abs(xold) < MAXREAL) and (abs(yold) < MAXREAL) then
    begin
      xnew:=xold*cosa-(yold-xold*xold)*sina;
ynew:=xold*sina+(yold-xold*xold)*cosa;
    if (abs(xnew-1) < MAXINT/xscale) and (abs(1-ynew) < MA
end;
begin
  p1:=round((xnew-1)*xscale);
p2:=round((1-ynew)*yscale);
plot(p1,p2,1);
end;
xold:=xnew;
yold:=ynew;
end;
    if keypressed then i:=points+1 else i:=i+1;
end; (while i)
xold:=x0+j*dx0;
yold:=y0+j*dy0;
end; {for j}
gotoxy(1,22);
end;
end.
APPENDIX 2.A.2

COMPUTER PROGRAMMES FOR THE MAPPING EQUATION (2.1.7)

This and the next two programmes are written in Microsoft Quickbasic. The first programme generates the bifurcation diagram given in chapter II. The next two programmes generate the fixed points of the map (2.1.7) for the first and the second iterates respectively.

PROGRAM 1

'basic program for modified logistic map
'this program draws bifurcation of the fixed points
'of the map given in eqn (2.1.7) and accepts the
'iteration step size as an input.

input "No. of iterations desired":n
input "initial value of x":xold
input "iteration step size":size
cls: key off: screen 2: window (1.9,-0.1)-(4.1,6.5)
line (2.1)-(4.1)   'x-axis 2-4
line (2.5)-(2.1)   'y-axis 1-6
for j=0 to 4 step .5 'marking x-axis
line(j,.91)-(j,1)
ext j
for k=1 to 6 step .5 'marking y-axis
line(-.80,k)-(.2,k)
ext k
for r=2.0 to 3.5 step size
   for i=1 to n
      xnew=xold * exp(r*(1-xold))
xold=xnew
      if i>=300 then pset(r,xnew+2)
ext i
next r
'def seg=&hb800
'save"expmap4.pic",0,&hb800
end

'remove the comment indicator
'if picture is to be saved.
PROGRAM 2

'program for exponential map
'only the parameter is the input for this program
'generating the plot of the second generation map with
'the line x = x.

n=1550
input "parameter":r
cls:screen 2 :window(-1,-1) - (6,6)
line(0.4)-(0.0) 'y-axis
line(0.0)-(4.0) 'x-axis
for i=0 to 4 step .5
  line(i+.05)-(i,0) 'x-axis marking
  line(.05,i)-(0,i) 'y-axis marking
next i
locate 5,18:print "r =":r
for i=1 to n
  xnew=xold*exp(r*(1-xold))
  ynew=xnew*exp(r*(1-xnew))
  pset(xold,ynew):pset(xold,xold)
  xold=xold + .005
  i=i+1
next i
end
PROGRAM 3

' program for exponential map
' this program plots the first iterate of the map (2.1.7)
' with the line x = x.
'
n=1500
input "parameter":r
cls:screen 2:window(-1,-1) - (6.6)
line(0.4)-(0,0)  'y axis
line(0.0)-(4.0)  'x axis
for i=0 to 4 step .5
    line(i,.05)-(i,0)  'x-axis marking
    line(.05,i)-(0,i)  'y-axis marking
next i
locate 5,18:print "r =":r
for i=1 to n
    xnew=xold*exp(r*(1-xold))
    pset(xold,xnew):pset(xold,xold)
    xold=xold + .005
    i=i+1
next i
end
ONE DIMENSIONAL POTENTIALS HAVING EQUI-SPACED SPECTRUM

Here we show that one-dimensional potentials giving equi-spaced energy spectrum are either the Kramers type or the linear harmonic oscillator type and in the former case the Hamiltonian form a representation of the compact generator $\gamma_3$ of SU(1,1) Lie-algebra.

A quantum Hamiltonian $H$ possesses equi-spaced spectrum if there exist raising and lowering operators $Q_\pm(x)$ such that

$$[H, Q_\pm] = \pm Q_\pm$$

(3.A.1)

For a Hamiltonian

$$H = -\frac{i}{2} \frac{d^2}{dx^2} + V(x)$$

(3.A.2)

we investigate for what form of the potential $V(x)$ one can construct a raising operator $Q_+$. As the Hamiltonian contains only the second derivative it is not difficult to see that $Q_+$ cannot involve higher order derivatives than the second. Further, the coefficient of $\frac{d^2}{dx^2}$ in $Q_+$ should be a constant which can be conveniently taken to be unity. Then the most general form of $Q_+$ is
\[ Q_+ = \frac{d^2}{dx^2} + \alpha'_+ (x) \frac{d}{dx} + \alpha_0 (x) \quad (3.A.3) \]

Substituting the operators (3.A.2) and (3.A.3) in the eqn. (3.A.1) we get the following equations, denoting differentiation w.r.t. x by primes

\[ \alpha'_+ (x) = -1 \quad (3.A.4) \]

\[ 2 \nu (x) + \alpha'_0 (x) = - \alpha_+ (x) \quad (3.A.5) \]

\[ \frac{1}{2} \alpha''_0 (x) + \nu'' (x) + \alpha_+ (x) \nu (x) = - \alpha_0 (x) \quad (3.A.6) \]

From eqn. (3.A.4) we get

\[ \alpha_+ (x) = - (x + \kappa), \quad \kappa \text{ being a constant} \quad (3.A.7) \]

Differentiating eqn. (3.A.5) w.r.t. x and substituting in eqn. (3.A.6) we get

\[ \frac{1}{2} + \alpha_0 \nu' = - \alpha_0 \quad (3.A.8) \]

Eliminating \( \nu' \) between eqns. (3.A.8) and (3.A.5) we get

\[ \frac{1}{2} (a_1^2 + a_0 a'_1) = a_0 + \frac{1}{2} \quad (3.A.9) \]

Since \( a_1 \) is known from eqn. (3.A.7) this gives a first order equation for \( \alpha_0 (x) \) which is easily solved to get
\[ a_0(x) = \frac{(x + \alpha)^2}{4} + \frac{2C}{(x + \alpha)^2} - \frac{1}{2} \]  

(3.A.10)

where \( C \) is a constant.

Then eqn. (3.A.8) determines \( V(x) \) to be

\[ V(x) = \frac{(x + \alpha)^2}{8} - \frac{C}{(x + \alpha)^2} + V_0 \]  

(3.A.11)

where \( V \) is a constant. Without any loss of generality we can set \( \alpha = 0 \) and \( V = 0 \). Thus we get

\[ Q_+ = \frac{d^2}{dx^2} - \frac{x}{x} + \frac{2C}{4} - \frac{1}{2} \]  

(3.A.12)

and

\[ V(x) = \frac{x^2}{8} - \frac{C}{x^2} \]  

(3.A.13)

The essence of what we have obtained is that for the Hamiltonian (3.A.2) a raising operator \( Q_+ \) can be found provided as given by eqn. (3.A.12) provided the potential \( V(x) \) is of the form given eqn. (3.A.13).

Putting the form of the potential (3.A.13), the Hamiltonian is

\[ H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{8} - \frac{\lambda}{x^2} \]  

(3.A.14)
Using the realisation of SU(1,1) given in eqn (3.4.15)

with \( \phi(y) = y \) and \( \nu(y) = 0 \),

\[
H = -\left( \frac{5}{4} \Gamma_3 - \frac{3}{4} \Gamma_1 \right)
\]

and

\[
Q_+ = 2 \left\{ \left( \frac{5}{4} \Gamma_1 - \frac{3}{4} \Gamma_3 \right) - i \Gamma_2 \right\}
\]

Applying a rotation about \( \Gamma_2 \) axis through an 'angle' \( \Theta \)

as in eqn (3.4.15) where \( \Theta \) is given by

\[
tanh \Theta = \frac{3}{5}
\]

we have

\[
\Gamma_1 \rightarrow \Gamma'_1 = \frac{5}{4} \Gamma_1 + \frac{3}{4} \Gamma_3
\]

\[
\Gamma_2 \rightarrow \Gamma'_2 = \Gamma_2
\]

\[
\Gamma_3 \rightarrow \Gamma'_3 = \frac{3}{4} \Gamma_1 + \frac{5}{4} \Gamma_3
\]

and hence,

\[
H \rightarrow H' = \left( \frac{5}{4} \Gamma'_3 - \frac{3}{4} \Gamma'_1 \right) = - \Gamma_3
\]

and

\[
Q_+ \rightarrow Q'_+ = 2 \left\{ \left( \frac{5}{4} \Gamma'_1 - \frac{3}{4} \Gamma'_3 \right) - i \Gamma'_2 \right\} = 2 (\Gamma_1 + i \Gamma_2) = 2 \Gamma_2
\]
Thus in this case, the Hamiltonian (3.A.15) is essentially the compact generator $\hat{\gamma}_3$ and the raising operator $Q_+$ is essentially the SU(1,1) lowering operator $\hat{\gamma}_2$ (not $\hat{\gamma}_+ \gamma_2$ because $H = -\hat{\gamma}_3$). Then the eigenvalue spectrum of $H$ is essentially that of $\hat{\gamma}_3$ whose discrete spectrum is equispaced. It is easy to see that the lowering operator $Q_-$ is essentially $\hat{\gamma}_+ \gamma_2$.

In the above discussion we have taken in the construction of $Q$, the highest possible order of derivative (that is second order). One more possibility is that $Q_+$ contains only up to the first order derivative. In this case, $Q_+$ has the general form

$$Q_+ = a_4(x) \frac{d}{dx} + a_0(x)$$  \hspace{1cm} (3.A.17)

Then, imposing the condition $[H, Q_+] = Q_+$ we get

$$a_4' = 0$$

$$a_0' = -a_4$$ \hspace{1cm} (3.A.18)

$$\frac{1}{2} a_0'' + a_4' \nu' = -a_0$$
This set of equations can be easily solved to get

\[ a_1 = 0, \]
\[ a_2 = a(x) \]

and

\[ \nu = \frac{1}{2} (x^2 + \kappa^2) + \nu_0 \]

where \( a, \kappa, \nu_0 \) are all constants.

Without any loss of generality we may set \( \kappa = 0, \nu = 0 \) and \( a = 1 \). Then we have

\[ H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} \kappa^2 \quad (3.19) \]

and the raising operator

\[ Q_+ = \frac{d}{dx} - \kappa \quad (3.20) \]

The corresponding lowering operator \( Q_- \) is easily obtained as

\[ Q_- = \frac{d}{dx} + \kappa \quad (3.21) \]

The Hamiltonian \((3.19)\) is the familiar linear harmonic oscillator Hamiltonian.

\[ H = \frac{1}{2} (p^2 + x^2) \]
and the raising and lowering operators are the one found in elementary Quantum Mechanics.

\[ Q_+ = -x + i\hat{p} \]
\[ Q_- = x + i\hat{p} \]
APPENDIX 4.A

Derivation of Equation (4.2.11)

On derivation of eqn. (4.2.8) w.r.t. \( \lambda \) we get

\[
\frac{1}{2} \frac{\partial \varepsilon_i}{\partial \lambda^2} - \frac{1}{2} \frac{\partial}{\partial \lambda} \langle \psi_i | \frac{\partial^2 H}{\partial \lambda^2} | \psi_i \rangle = \sum_j \left( \frac{\partial h_{ij}}{\partial \lambda} \xi_j + h_{ij} \frac{\partial \xi_j}{\partial \lambda} \right) \tag{4.2.1}
\]

Now for \( i \neq j \)

\[
\frac{\partial \xi_{ij}}{\partial \lambda} = \frac{1}{\lambda - (E_j - E_i)} \frac{\partial}{\partial \lambda} \left( \frac{\partial h_{ij}}{\partial \lambda} \xi_j - (E_j - E_i) \xi_{ij} \frac{\partial \xi_j}{\partial \lambda} \right) \tag{4.2.2}
\]

On using eqns. (4.2.10) and (4.2.2), the right hand side of eqn. (4.2.1) becomes

\[
\frac{\partial h_{ij}}{\partial \lambda} \xi_j + h_{ij} \frac{\partial \xi_j}{\partial \lambda} = \frac{\partial h_{ij}}{\partial \lambda} \xi_j + (E_j - E_i) \xi_{ij} \frac{\partial \xi_j}{\partial \lambda}
\]

\[
= \frac{\partial}{\partial \lambda} \xi_{ij} - \xi_{ij} \frac{\partial h_{ij}}{\partial \lambda} + \xi_{ij} \left( \frac{\partial E_i}{\partial \lambda} - \frac{\partial E_j}{\partial \lambda} \right)
\]

\[
= \langle \psi_i | \frac{\partial h_{ij}}{\partial \lambda} | \psi_j \rangle \xi_{ij} - \langle \psi_j | \frac{\partial^2 H}{\partial \lambda^2} | \psi_i \rangle \xi_{ij}
\]

\[
+ \xi_{ij} \left( \frac{\partial E_i}{\partial \lambda} - \frac{\partial E_j}{\partial \lambda} \right) \sum_k \left( h_{ik} \xi_k \xi_{ij} - \xi_{ik} \frac{\partial h_{ij}}{\partial \lambda} \right)
\]

\[
- \left( h_{ij} \xi_i \xi_{ij} - \xi_{ij} \frac{\partial h_{ij}}{\partial \lambda} \right) \tag{4.2.3}
\]
The summation term (last term) on the right hand side of eqn. (4.1.3) is

\[
\sum_{k \neq i} \left\{ h_{ik} \alpha_{kj} \alpha_{ji} - \alpha_{ik} h_{kj} \alpha_{ji} - h_{jk} \alpha_{ki} \alpha_{ij} + \alpha_{jk} h_{ki} \alpha_{ij} \right\} + 2 h_{ii} \alpha_{ij} \alpha_{ji} - \alpha_{ii} (h_{ij} \alpha_{ji} + \alpha_{ij} h_{ji})
\]

Now for \( i \neq j \)

\[ h_{ij} \alpha_{ji} + \alpha_{ij} h_{ji} = 0 \quad (4.1.4) \]

Collecting together these results we then have from eqn. (4.1.1)

\[
\frac{1}{2} \frac{\partial^2 E_i}{\partial \lambda^2} - \frac{1}{2} \frac{\partial}{\partial \lambda} \langle \psi_i | \frac{\partial H}{\partial \lambda} | \psi_i \rangle = \sum_{j} \left( \frac{\partial h_{ij}}{\partial \lambda} \alpha_{ji} + h_{ij} \frac{\partial \alpha_{ji}}{\partial \lambda} \right)
\]

\[
= \sum_{j} \left\{ \langle \psi_i | \frac{\partial H}{\partial \lambda} | \psi_j \rangle \alpha_{ji} - \langle \psi_j | \frac{\partial H}{\partial \lambda} | \psi_i \rangle \alpha_{ij} + \alpha_{ij} \partial_{\lambda} \left( \frac{\partial E_i}{\partial \lambda} - \frac{\partial E_j}{\partial \lambda} \right) \right\} + \sum_{j \neq i} \left\{ h_{ik} \alpha_{kj} \alpha_{ji} - \alpha_{ik} h_{kj} \alpha_{ji} - h_{jk} \alpha_{ki} \alpha_{ij} + \alpha_{ij} \alpha_{jk} h_{ki} \right\} + \sum_{j \neq i} 2 h_{ii} \alpha_{ij} \alpha_{ji} \quad (4.1.5)
\]

We shall now simplify the double summation term on the right hand side of eqn. (4.1.5). It equals

\[
\sum_{j \neq i} \sum_{k \neq i} \left[ h_{ik} \alpha_{kj} \alpha_{ji} + \alpha_{ij} \alpha_{jk} h_{ki} \right] - (\alpha_{ik} h_{kj} \alpha_{ji} + \alpha_{ij} h_{jk} \alpha_{ki})
\]

\[
= \sum_{j \neq i} \sum_{k \neq i} \left[ h_{ik} \alpha_{kj} \alpha_{ji} + \alpha_{jk} h_{ki} \alpha_{ji} \right] - (\alpha_{ik} h_{kj} \alpha_{ji} + \alpha_{ij} h_{jk} \alpha_{ki})
\]
(on interchanging the dummy summation indices \( j \) and \( k \) in the second and last terms.)

\[
\sum_{j \neq i} \sum_{k \neq i} \alpha_{kj} (h_{ik} \alpha_{ji} + \alpha_{ik} h_{ji}) - \sum_{j \neq i} \sum_{k \neq i} 2 \alpha_{ik} h_{kj} \alpha_{ji}
\]

(in the first double summation \( k = j \) terms do not contribute because of eqn. (4.A.4))

\[
\sum_{j \neq i} \sum_{k} \alpha_{kj} \alpha_{ij} \alpha_{ji} (\xi_{k} - \xi_{i} + \xi_{i} - \xi_{j}) - \sum_{j} \sum_{k} 2 h_{kj} \alpha_{ij} \alpha_{ji} - \sum_{j} \sum_{i \neq j} 2 h_{jj} \alpha_{ij} \alpha_{ji}
\]

(where the prime over the summation over \( j \) indicates the exclusion of \( j = i \) term and the double prime over the summation over \( k \) indicates exclusion of the \( k = i \) and \( k = j \) terms.)

\[
= \sum_{j \neq i} \sum_{k} \alpha_{kj} \alpha_{ij} (\xi_{k} - \xi_{i}) - \sum_{j} \sum_{k} 2 \alpha_{ik} h_{kj} \alpha_{ji} - \sum_{j} 2 h_{jj} \alpha_{ij} \alpha_{ji}
\]

\[
= 3 \sum_{j \neq i} \sum_{k} \alpha_{ik} h_{kj} \alpha_{ji} - 2 \sum_{j} h_{jj} \alpha_{ij} \alpha_{ji}
\]

(on using the relation \( h_{kj} = (\xi_{j} - \xi_{k}) \alpha_{kj} \).)

Putting back in eqn. (4.A.5) we get
\[
\frac{1}{2} \frac{\partial^3 E_i}{\partial \lambda^3} - \frac{d}{d\lambda} \left( \frac{\partial^2 H}{\partial \lambda^2} \right) \psi_i + \sum_j \left[ \left( \frac{\partial^2 H}{\partial \lambda^2} \right) \psi_j \right] \alpha_{ij} - \left( \frac{\partial^2 H}{\partial \lambda^2} \right) \psi_i \alpha_{ji}
\]

\[
= \sum_j \alpha_{ij} \alpha_{ji} \left( \frac{\partial E_i}{\partial \lambda} - \frac{\partial E_j}{\partial \lambda} \right) + 2 \sum_j \alpha_{ij} \alpha_{ji} \left( k_{ji} - k_{jj} \right) - 3 \sum_j \sum_k \alpha_{ij} k_{kj} \alpha_{ji}
\]

\[
= 3 \frac{\partial E_i}{\partial \lambda} \sum_j \alpha_{ij} \alpha_{ji} - 3 \sum_j \sum_k \alpha_{ij} k_{kj} \alpha_{ji} - 3 \sum_j \sum_k \alpha_{ij} k_{kj} \alpha_{ji}
\]

(on using \( k_{ii} = \frac{\partial E_i}{\partial \lambda} \))

\[
= 3 \frac{\partial E_i}{\partial \lambda} \sum_j \alpha_{ij} \alpha_{ji} - 3 \sum_j \sum_k \alpha_{ij} k_{kj} \alpha_{ji}
\]

(on combining the last two summations)

\[
= 3 \sum_j \sum_k \alpha_{ij} k_{kj} \left( \frac{\partial E_i}{\partial \lambda} \delta_{kj} - h_{kj} \right) \alpha_{ji}
\]

\[
= -3 \sum_j \sum_k \alpha_{ij} h_{kj} \alpha_{ji} = -3 \sum_j \sum_k \alpha_{ij} h_{kj} \alpha_{ki}
\]

(4.1.6)

where

\[
\bar{h}_{kj} = h_{kj} - \frac{\partial E_i}{\partial \lambda} \delta_{kj}
\]

(4.1.7)
Now the second term on the left hand side is

\[
\frac{1}{2} \frac{d}{d\lambda} \langle \psi_i \big| \frac{\partial^2 V}{\partial \lambda^2} \big| \psi_i \rangle
\]

\[
= \frac{1}{2} \langle \psi_i \big| \frac{\partial^3 H}{\partial \lambda^3} \big| \psi_i \rangle + \frac{1}{2} \sum_j \left[ \langle \psi_i \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_j \rangle \langle \psi_j \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_i \rangle + \langle \psi_i \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_j \rangle \langle \psi_j \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_i \rangle \right]
\]

\[
= \frac{1}{2} \langle \psi_i \big| \frac{\partial^3 H}{\partial \lambda^3} \big| \psi_i \rangle + \frac{1}{2} \sum_j \left[ \alpha_{ji} \langle \psi_i \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_j \rangle + \langle \psi_i \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_j \rangle \alpha_{ji} \right]
\]

\[
= \frac{1}{2} \langle \psi_i \big| \frac{\partial^3 H}{\partial \lambda^3} \big| \psi_i \rangle + \frac{1}{2} \sum_j \left[ \langle \psi_i \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_j \rangle \alpha_{ji} - \alpha_{ij} \langle \psi_j \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_i \rangle \right]
\]

\[
= \frac{1}{2} \langle \psi_i \big| \frac{\partial^3 H}{\partial \lambda^3} \big| \psi_i \rangle + \frac{1}{2} \sum_{j \neq i} \left[ \langle \psi_i \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_j \rangle \alpha_{ji} - \alpha_{ij} \langle \psi_j \big| \frac{\partial^2 H}{\partial \lambda^2} \big| \psi_i \rangle \right]
\]
Substituting in eqn. (4.3.6) we get

\[ \frac{1}{3!} \frac{\partial^3 \mathcal{E}_i}{\partial \lambda^3} - \frac{1}{3!} \left( \psi_i | \frac{\partial^2 \mathcal{H}}{\partial \lambda^2} | \psi_i \right) - \frac{1}{2} \sum_j \left\{ \psi_i | \frac{\partial^2 \mathcal{H}}{\partial \lambda^2} | \psi_j \right\} \alpha_{ji} - \alpha_{ij} \left( \psi_j | \frac{\partial^2 \mathcal{H}}{\partial \lambda^2} | \psi_i \right) \]

\[ = - \sum_{ij} \sum_{k} \alpha_{ik} \overline{h}_{kj} \alpha_{ji} \]

\[ = \sum_{ij} \sum_{k} \frac{h_{ik} \overline{h}_{kj} h_{ji}}{(\xi_k - \xi_i)(\xi_j - \xi_i)} \]

(4.3.9)

which is eqn. (4.2.11)
Publications / papers by the author related to the thesis:

Chapter I  Quantum Normal Form and the Harmonic Oscillator with $\chi^6$ Perturbation.

Chapter II A One Dimensional Mapping with Infinite Growth Paper in preparation for submission to the Journal of Physics A.

Chapter III On the Single-Variable Realisation of SU(1,1) Lie Algebra and Discrete Spectra of a Class of One Dimensional Potentials.
            Paper submitted to the International Journal of Theoretical Physics.

Chapter IV Extensions of the Feynman-Hellmann Theorem and Applications.
            Accepted for publication in American Journal of Physics (August, 1989).