CHAPTER - III

DENSITY VARIATION PARAMETER ($\lambda$)
3.1 INTRODUCTION

The central density plays an important role when dense nucleon matter collapses under gravitational force. We know that space time can be used to develop static models describing the gravitational field in the interior of superdense condensation of matter like neutron stars. Einstein's field equation for a spherical distribution of matter in the form of a perfect fluid at rest gives the relationship between the matter density and the geometry of the associated physical 3-space governed by the parameter 'R' and 'K'. An exact solution of the field equations for a particular value K=2 is used to develop the model for a superdense star.

As the gravitational contraction going on, to clear out the picture in radius density around the boundary surface, we subdivide the domain into three regions

\[ \rho < 2 \times 10^{14} \text{g.cm}^{-3} \]
\[ \rho = 2 \times 10^{14} \text{g.cm}^{-3} \]
\[ \rho > 2 \times 10^{14} \text{g.cm}^{-3} \text{ (will be valid if we relax the boundary conditions)} \]

In the domain \( \rho < 2 \times 10^{14} \text{g.cm}^{-3} \), matter consists predominantly of neutrons along with protons, electrons. Their interaction is substantial in the upper part of the region, near \( 10^{14} \).

As \( \rho \rightarrow 2 \times 10^{14} \text{g.cm}^{-3} \) the weak intera-
ction becomes more effective and due to interaction electrons and protons are converted into neutrons and neutrinos. As a result, predominancy of neutrons increases while the number of electrons and protons decreases gradually, which in turn becomes smaller and smaller, finally negligible.

If $\lambda$ denotes the ratio of the matter density at the boundary of a star to the matter density at its center, it is then possible to estimate the size of the configuration for different values of $\lambda$ and $K$, if the order of magnitude of the density on the boundary is known. With the assumption, the physical 3-space $t$-constant in a superdense star is spheroidal and the matter density on the boundary surface of the configuration to be $\rho_0 = 2 \times 10^{14} \text{g.cm}^{-3}$ (i.e. the average matter density in a neutron star) Vaidya and Tikekar (1982) proposed an exact relativistic model for a superdense star and have raised the maximum upper limit of mass of a neutron star from $3.2 M_\odot$, obtained by Rhoades and Ruffini (1974), to $3.575 M_\odot$, when the density variation parameter $\lambda$ attains minimum permissible value $0.4598$.

A static spherically symmetric model, based on an exact solution of Einstein's equation, gives the permissible matter density $\sim 2 \times 10^{14} \text{g.cm}^{-3}$. Using the change in the ratio of central density to the radius $r=a$
(i.e. central density per unit radius \( \rho / a \)) we can call it radius density.) minimum, it has estimated the upper limit of the density variation parameter (\( \lambda \)) and minimum mass limit of a superdense star like neutron star. It is found that minimum radius occurs when \( \lambda \) attains the maximum permissible value 0.68 with the corresponding mass \( m = 1.5707 M_\odot \). This limit gives an idea of the domain where the neutron abundance with negligible number of electrons and protons (may be treated as pure neutrons) and equilibrium in neutrons begins.

3.2 FIELD EQUATIONS AND MATTER DENSITY RELATIONS

Consider a four dimensional Euclidean flat space with metric

\[
d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2 \quad (1)
\]

A 3-spheroid immersed in this four dimensional flat space will have

\[
\frac{\omega^2}{b^2} + \frac{\kappa^2 + \gamma^2 + \tau^2}{R^2} = 1 \quad (2)
\]

The sections \( w = \text{const.} \) of the 3-spheroid are concentric spheres while sections \( x = \text{const.} \), \( y = \text{const.} \), \( z = \text{const.} \) represent systems of confocal ellipsoids. So, the metric on the 3-space can be written as

\[
d\sigma^2 = \left[ 1 - \kappa (\gamma^2/R^2) \right] \left[ 1 - \gamma^2/R^2 \right]^{-1} d\gamma^2 + \gamma^2 (d\kappa^2 + \sin^2 \gamma \, d\beta^2) \quad (3)
\]
where \[ K = 1 - \frac{b^2}{R^2} \quad ; \quad \alpha = R \sin \lambda \sin \alpha \cos \beta \]
\[ w = b \cos \lambda \quad ; \quad \gamma = R \sin \lambda \sin \alpha \sin \beta \quad (4) \]
\[ r = R \sin \lambda \quad ; \quad z = R \sin \lambda \cos \alpha. \]

For \( K < 1 \), the metric (3) is regular and positive definite at all points \( r^2 < R^2 \).

In order to see the space-time with the metric, we can write
\[
d s^2 = e^\nu dt^2 - d\sigma^2 \\
= e^\nu dt^2 - [1 - K \left( \frac{r^2}{R^2} \right)] \left[ 1 - \frac{R^2}{R^2} \right]^{-1} dr^2 - r^2 \left( d\alpha^2 + \sin^2 \alpha d\beta^2 \right) \quad (5) \]

where \[ \nu = \nu \left( \frac{r}{R} \right), \quad \alpha^1 = r, \quad \alpha^2 = \alpha, \quad \alpha^3 = \beta, \quad \alpha^4 = t \]

It is clear that when \( K = 0 \), the physical 3-space \( t = \text{const.} \) becomes spherical and when \[ e^\nu = \left[ A + B \left( 1 - \frac{r^2}{R^2} \right)^{\frac{1}{2}} \right]^2, \quad K = 0 \]
the metric (5) gives the metric of Schwarzschild's interior solution. For \( K = -2 \), the metric describing the solution of the field equation explicitly is
\[
d s^2 = \left[ B \left( 1 - \frac{2}{3} z^2 \right)^{\frac{3}{2}} + A \left( 1 - \frac{4}{3} z^2 \right) \right]^2 dt^2 \\
- \frac{3 - 2 z^2}{z^2} dr^2 - r^2 \left( d\alpha^2 + \sin^2 \alpha d\beta^2 \right) \quad (6) \]

where \( z^2 = 1 - \frac{r^2}{R^2} \), \( A, B \) are constants of integration.
As we are considering the perfect fluid distribution represented by metric (5) when $K < 1$, $K \neq 0$, the energy momentum tensor for a perfect fluid

$$T_{ik} = \left(\rho + \frac{P}{c^2}\right) U_i U_k - \frac{P}{c^2} g_{ik}$$

and Einstein field equation

$$R_{ik} - \frac{1}{2} g_{ik} R = -\left(\frac{8\pi G}{c^2}\right) T_{ik}$$

gives the density ($\rho$) and pressure ($P$) relation for an equilibrium situation with the metric (5) as

$$\left(\frac{8\pi G}{c^2}\right) \rho = \frac{3(1-k)}{R^L} \frac{(1-k)\tau^2/3R^L}{(1-k\tau^2/3R^L)^2}$$

$$\left(\frac{8\pi G}{c^2}\right) \frac{P}{c^2} = \frac{(1-k\tau^2/R^L)}{(1-k\tau^2/R^L)^2} \left[\frac{\tau'}{\tau} + \frac{1}{\tau} \right] - \frac{1}{R^L}$$

together with the consistency condition ($T^1_1 = T^2_2$)

which reads as

$$\left(1 - \frac{\tau^2}{R^L}\right) \left(1 - \frac{k\tau^2}{R^L}\right) \left(\frac{\tau'}{\tau} + \frac{1}{\tau} \right)^2 \frac{\tau'}{\tau} - \frac{2(1-k)^2}{R^L} \left(\frac{\tau'}{\tau} \right)^2$$

$$+ 2 \frac{(1-k)}{R^L} \left(1 - \frac{k\tau^2}{R^L}\right) = 0$$

where

$$\tau' = \frac{d\tau}{d\tau}$$

Denoting the matter density at $r = 0$ by $\rho_0$ expression (7) becomes

$$\left(\frac{8\pi G}{c^2}\right) \rho_0 = \frac{3(1-k)}{R^L}$$

As $K < 1$, the central density $\rho_0$ is positive
and if $0 < k < 1$, $\rho$ remains positive in the spherical region $r^2 < 3R^2/K$ which imposes a restriction on the size of the configuration that can be waived out by considering $K < 0$.

So, on the boundary $r = a$ the expression (7) gives

$$\left(\frac{\kappa \pi G \kappa}{c^2}\right) \rho_a = 3(1 - k)(1 - ka^2/3R^2)/R^2 \left(1 - ka^2/R^2\right)^2$$  \hspace{1cm} (11)

$$\lambda = \frac{\rho_a}{\rho_0} = \left(1 - ka^2/3R^2\right)/\left(1 - ka^2/R^2\right) < 1$$  \hspace{1cm} (12)

Thus the field in the exterior region $r \geq a$ is described by the Schwarzschild's exterior metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\alpha^2 + \sin^2 \alpha d\beta^2\right)$$  \hspace{1cm} (13)

But the metric (5) should be continuous with the metric (13) as we cross the boundary.

Satisfying the boundary condition that the fluid pressure must vanish at $r = a$ and the physical conditions

$$\rho_0 > 0 \quad ; \quad p_a > 0 \quad ; \quad \left(\frac{\rho_0}{c^2} - \frac{3\rho_a}{c^2}\right) > 0$$

we obtain the following relations:

$$\frac{a^2}{R^2} = \left(6\lambda - 1 - \sqrt{1 + 24\lambda}\right)/6\kappa\lambda$$  \hspace{1cm} (14)

$$\frac{m}{a} = \frac{3}{2} \left(\frac{a^2}{R^2}\right)/\left(1 + 2a^2/R^2\right)$$  \hspace{1cm} (15)

$$\Lambda = \frac{3}{2} \left(1 - 2a^2/R^2\right) \left(1 - a^2/R^2\right)^{-\frac{1}{2}} \left(1 - \frac{2m}{a}\right)^{\frac{1}{2}}$$  \hspace{1cm} (16)
\[ B = \frac{\sqrt{3}}{2} \left( 1 + \frac{4a^2}{R^2} \right) \left( 1 + \frac{2a^2}{R^2} \right)^{-\frac{1}{2}} \left( 1 - \frac{2m}{a} \right)^{\frac{1}{2}} \quad (17) \]

\[ R^L = \frac{3 \left( 1 - \kappa \right)}{\left( \frac{8\pi G M}{c^2} \right) \rho_0} \quad (18) \]

Given \( \rho_a \), \( \lambda \), and \( K \), the parameter \( R \), the radius \( a \) and the total mass \( m \) of the configuration can subsequently be calculated from the above equations.

3.3 RESULTS AND DISCUSSIONS

The matter density on the boundary \( r = a \) has been taken as \( \rho_a = 2 \times 10^{14} \text{ g cm}^{-3} \). For each chosen value of \( \lambda \cdot \rho_0 \) has been calculated with the assumed value of \( \rho_a \), using the relations mentioned above. \( R \), \( a \) and \( m \) have been calculated using the relations (14), (15) and (18).

Applying the above for neutron star we have

1) in the pre-neutron star stage, when the star undergoes gravitational contraction, its mass density goes on increasing and the last stage leads to the forma-
tion of superdense condensation of matter. A pre-neutron star stage i.e. \( \rho < 10^{14} \text{g.cm}^{-3} \), consists of abundant neutrons along with electrons and protons. As contraction going on, weak interaction takes place among electrons and protons more effectively producing neutrons and neutrinos.

iii) As \( \rho \rightarrow 2 \times 10^{14} \text{g.cm}^{-3} \), constituents are then only neutrons along with negligible numbers of electrons and protons. The number of neutrons increases as the gravitational contraction going on. So, it estimates the minimum in central density per unit equilibrium radius i.e. radius density is minimum, stability established among non-interacting neutrons. From the tabulated data (Table 3.1) it is seen that stability begins in a neutron star at \( \lambda = 0.68 \) with the corresponding mass \( m = 1.5707 \text{ M}_\odot \).

iii) From the fig 3.1 it is seen that at \( \rho = 2 \times 10^{14} \text{g.cm}^{-3} \) neutrons are not in equilibrium because radius density is not minimum as the number of neutrons produced in weak interactions increasing. At \( \lambda = 0.68 \) the radius density is minimum. In this domain, the number of remaining electrons and protons are negligible. In other words, it can say in this domain the constituents are only neutrons, they are non-interacting and equilibrium in neutrons begins.
Fig 3.1 Neutron Star Central Density per unit radius \( \left( \frac{f_0}{a} \right) \)
as a function of radius \( (r = a) \) for \( \rho = 2 \times 10^{14} \text{g cm}^{-3} \).

A → B = Pre-stable neutron star stage

B = Stable neutron star minimum mass \( m = 1.5707 \text{M}_\odot \)

B → E = Neutron Star stage

C = Maximum mass of Neutron star 3.2 \text{M}_\odot \) (Rhoades et al)

D = Maximum mass of Neutron star 3.5 \text{M}_\odot \) (Vaidya et al)

E = Maximum mass of Neutron star 5.03 \text{M}_\odot \) (Theoretical)
Table 3.1  Masses, equilibrium radii and radius density of a neutron star model corresponding to \( K = -2 \), \( \rho_0 = 2 \times 10^{14} \text{ gm. cm}^{-3} \)

<table>
<thead>
<tr>
<th>Sl. No</th>
<th>( \lambda )</th>
<th>( \rho_0/\alpha )</th>
<th>( \alpha/R )</th>
<th>( m/\alpha )</th>
<th>( R ) in km</th>
<th>( \alpha ) in km</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0.9</td>
<td>0.2633</td>
<td>0.1810</td>
<td>0.0461</td>
<td>46.5984</td>
<td>8.4343</td>
</tr>
<tr>
<td>2.</td>
<td>0.8</td>
<td>0.2115</td>
<td>0.2690</td>
<td>0.0948</td>
<td>43.9332</td>
<td>11.8180</td>
</tr>
<tr>
<td>3.</td>
<td>0.7</td>
<td>0.1998</td>
<td>0.3483</td>
<td>0.1464</td>
<td>41.0958</td>
<td>14.3136</td>
</tr>
<tr>
<td>4.</td>
<td>0.69</td>
<td>0.19945</td>
<td>0.3562</td>
<td>0.1518</td>
<td>40.8012</td>
<td>14.5334</td>
</tr>
<tr>
<td>5.</td>
<td>0.68</td>
<td>0.19940</td>
<td>0.3641</td>
<td>0.1571</td>
<td>40.5044</td>
<td>14.7476</td>
</tr>
<tr>
<td>6.</td>
<td>0.67</td>
<td>0.1995</td>
<td>0.3720</td>
<td>0.1626</td>
<td>40.2055</td>
<td>14.9564</td>
</tr>
<tr>
<td>7.</td>
<td>0.6</td>
<td>0.2042</td>
<td>0.4290</td>
<td>0.2017</td>
<td>38.0473</td>
<td>16.3223</td>
</tr>
<tr>
<td>8.</td>
<td>0.5</td>
<td>0.2227</td>
<td>0.5173</td>
<td>0.2614</td>
<td>34.7322</td>
<td>17.9669</td>
</tr>
<tr>
<td>9.</td>
<td>0.4598</td>
<td>0.2346</td>
<td>0.5567</td>
<td>0.2870</td>
<td>33.3068</td>
<td>18.5418</td>
</tr>
<tr>
<td>10.</td>
<td>0.4</td>
<td>0.2587</td>
<td>0.6218</td>
<td>0.3270</td>
<td>31.0655</td>
<td>19.3165</td>
</tr>
<tr>
<td>11.</td>
<td>0.3384</td>
<td>0.2955</td>
<td>0.7000</td>
<td>0.3712</td>
<td>28.5734</td>
<td>20.0014</td>
</tr>
</tbody>
</table>

Note: The mass \( m \) recorded in the table is measured in km.

The corresponding value in \( \text{gm} \) is \( m = m \times \alpha^2 \) / G.

\( 1 M_{\odot} = 1.475 \text{ km} \)
<table>
<thead>
<tr>
<th>m</th>
<th>m/m_0</th>
<th>A</th>
<th>B</th>
<th>Value of $\left( \frac{8\pi G}{c^2} \right) \frac{m}{c^2} \times 10^{-8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3889</td>
<td>0.2537</td>
<td>1.3580</td>
<td>0.9041</td>
<td>-2.3545</td>
</tr>
<tr>
<td>1.1203</td>
<td>0.7595</td>
<td>1.1991</td>
<td>0.9393</td>
<td>-2.7005</td>
</tr>
<tr>
<td>2.0955</td>
<td>1.4207</td>
<td>1.0202</td>
<td>0.9705</td>
<td>-0.5264</td>
</tr>
<tr>
<td>2.2061</td>
<td>1.4957</td>
<td>0.9997</td>
<td>0.9730</td>
<td>7.7177</td>
</tr>
<tr>
<td>2.3168</td>
<td>1.5707</td>
<td>0.9800</td>
<td>0.9756</td>
<td>10.1090</td>
</tr>
<tr>
<td>2.4319</td>
<td>1.6487</td>
<td>0.9600</td>
<td>0.9781</td>
<td>12.8436</td>
</tr>
<tr>
<td>3.2922</td>
<td>2.923</td>
<td>0.8101</td>
<td>0.9924</td>
<td>42.9271</td>
</tr>
<tr>
<td>4.6965</td>
<td>3.1684</td>
<td>0.5639</td>
<td>1.0001</td>
<td></td>
</tr>
<tr>
<td>5.3215</td>
<td>3.6078</td>
<td>0.4480</td>
<td>0.9946</td>
<td></td>
</tr>
<tr>
<td>6.3165</td>
<td>4.2823</td>
<td>0.2549</td>
<td>0.9744</td>
<td></td>
</tr>
<tr>
<td>7.4245</td>
<td>5.0335</td>
<td>0.0213</td>
<td>0.9246</td>
<td></td>
</tr>
</tbody>
</table>
The important features in this domain for a neutron star are:

a) The constituents of the superdense star are almost neutrons with negligible number of elementary particles like muons, neutrino, etc. This can be treated as pure neutrons.

b) \( \lambda \) attains maximum permissible value 0.68

c) The corresponding minimum mass \( m = 1.5707 \, M_\odot \)

(Parui et al. 1991).
REFERENCES

5. Oppenheimer, J.R. and Snyder, H. 1939 Phys. Rev. 56, 455