A Generalisation of Linear and Linear Fractional Programming

In this chapter, a generalisation of linear and piece-wise linear programming problem (GLLP) is considered. The problem can be stated as:

Minimize \( f(x) = \frac{a^T x + (c^T x + \alpha)}{d^T x + \beta} \)

Subject to

(GLLP) \hspace{1cm} Ax \leq b \hspace{1cm} x \geq 0

where \( a, c, d \) and \( x \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{m \times n} \). The problem was first directly studied by Teterev [109] though a parametric study of a generalisation was done in [85]. It has applications in business enterprise. Let \( S = \{ x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0 \} \), \( d^T x + \beta > 0 \) for every \( x \in S \) and \( S \) be regular.

In the Section 3.1 the methods to solve the problem are considered and in the Section 3.2 an algorithm for the case (which arises in systems design [26] when coefficients of a component of \( x \) vary over a bounded polyhedral set, is given. The Section 3.3 contains an algorithm for integer solution of the problem.
3.1 Computational Aspects:

In this section, various approaches to the solution of the problem are given. In the first subsection, simplex-like approach relevant to pinpoint the discrepancy in [109], is discussed. Based on this approach more general conditions of validity of algorithms in [109] than given in [98], are derived. The second subsection contains parametric approaches briefly. A modified simplex-like algorithm to solve the problem in the general case, is suggested in the third subsection. Also a simple numerical example is solved with this algorithm.

3.1.1 Simplex-like Approach:

A simplex like algorithm to solve the (GCP) along-with its applications was suggested in [109]. The algorithm is based on the Theorem 1 page 249 in [109], which can briefly be stated as:

**Theorem 3.1.1**: Let

\[ E = \{ x : a^T x \geq 0, \ b^T x + \beta \geq 0 \} \]

\[ E_2 = \{ x : a^T x \geq 0, \ b^T x + \beta \geq 0 \} \]

\[ E_3 = \{ x : a^T x \leq 0, \ b^T x + \beta \leq 0 \} \]

\[ E_4 = \{ x : a^T x \geq 0, \ b^T x + \beta \leq 0 \} \]
If $S \subseteq S_1(\mathbb{R}_2)$, then the function $f(x)$ on $S$,

(A) has unique minimum (maximum),

(B) may take several local maxima (minima); all of which are reached at corner points of $S$. Further, nothing can be said when $S \subset \mathbb{R}_3$ or $\mathbb{R}_4$.

To prove (B), the crux of the argument in this theorem is based on:

"$P$ is a local maximum of $f(x)$ on any (open) segment $(x^1, x^2)$ in $S$ is impossible". This is proposed to follow from the analysis given earlier in the treatment that $f(x)$ has no more than one extremum in any direction $s$ originating from $d^2x + \beta = 0$ and not parallel to it i.e. $d^2s \neq 0$. In case $s$ is parallel to $d^2x + \beta = 0$ i.e. $d^2s = 0$ and originates from a point $x$ not in $d^2x + \beta = 0$ i.e. $d^2x + \beta \neq 0$, $f(x)$ is either monotonic or constant in the direction $s$.

From the table on page 3 in [109], in cases

\[(3.1.1) \quad [d^2s > 0, a^2s < 0, c^T x^* + \alpha < 0] \]

or

\[(3.1.2) \quad [d^2s < 0, a^2s < 0, c^T x^* + \alpha > 0] \]

where $x^*$ is the point on $d^2x + \beta = 0$ from which $s$ originates, $f(x)$ has its unique directional maximum in the direction $s$ at a point corresponding to
\[ e^* = \left( \frac{\partial^2 f}{\partial x^2} + \alpha \right) \left( \frac{\partial^2 f}{\partial y^2} \right)^{1/2}, \]

where any point \( x \) in the direction \( s \) is given by \( x = x^* + \epsilon s \), \( 0 < \epsilon < \infty \). Since \( d^2 f + \beta \neq 0 \) for all \( x \in \delta_s \), \( s \) can always be chosen to satisfy \( d^2 f > 0 \). If the segment \( (x^1, x^2) \) in \( \delta \) is a part of this ray from \( x^* \) and if \( \epsilon^* \) corresponds to some point on \( (x^1, x^2) \) then, in fact, \( f(x) \) has a directional maximum at a point in the segment \( (x^1, x^2) \). This directional maximum in \( (x^1, x^2) \) in the direction \( s \) may not be a local maximum of \( f(x) \) in \( (x^1, x^2) \). For example, the point \( x = (1, 0, 5, 0) \) in the Example 1, in [47] viz.,

\[ f(x) = 2x_1 - 0.48x_2 + x_3 + (-x_1 + x_2)/(2x_1 + x_2), \]

\[ s = \left\{ x : 1 \leq x_1 \leq 2, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1 \right\}, \]

is a directional maximum in the direction \( s = (0, 1, 0) \) in the segment \( (x^1, x^2) \) where \( x^1 = (1, 0, 0) \) and \( x^2 = (1, 1, 0) \), without being a local maximum. Here in this case \( x^* = (1, -2, 0) \). But in case such a maximum qualifies to be a local maximum in \( (x^1, x^2) \) in \( s \), statement \( P \) stands contradicted. This is proved by the Example 2 in [47] viz.,

\[ f(x) = -0.1x_2 - 0.48x_2 + x_3 + (-x_1 + x_2)/(2x_1 + x_2), \]

\[ s = \left\{ x : 1 \leq x_1 \leq 2, 0 \leq x_2 \leq 1, 1 \leq x_3 \leq 2 \right\}, \]

in which case \( x^* = (1, -2, 0) \) and \( f(x) \) has directional maximum.
at \((1,0,5,3)\) in the segment \((x^1, x^2)\) in \(S\) where \(x^1 = (1,0,2)\), \(x^2 = (1,1,2)\) and \(s = (0,1,0)\). Thus part (3) of the theorem, in general, is not true.

Now consider the statement,

'\(Q \subseteq \) a directional maximum of \(f(x)\) on any segment \((x^1, x^2)\) in \(S\) in a direction \(s\) is impossible'. Obviously, \(Q \Rightarrow P\) but \(P \not\Rightarrow Q\) so that \(P\) holds under the conditions under which \(Q\) holds.

Next we briefly investigate the conditions under which \(Q\) holds. Since by our choice of \(s\), \(d^2s > 0\) for every segment \((x^1, x^2)\) in \(S\) lying on the ray in the direction \(s\), possibility of (3.1.2) being satisfied, is ruled out. On the basis of (3.1.1), following can be stated:

**Theorem 3.1.2**: Let \(S \subseteq R_4\). \(Q\) holds if

\[(3.1.3) \quad \quad [a = vd_s, \, \, v > 0]\]

or

\[(3.1.4) \quad \quad \begin{bmatrix}
\mu > 0, \, \, a^T x \leq -\mu x - \nu v, \text{ for all } x \in S \text{ or} \\
\mu < 0, \, \, a^T x \geq -\mu x - \nu v, \, \, d^2 x + \rho \leq -(1/\mu) \text{ for all } x \in S \circ \text{ or} \\
\mu < 0, \, \, a^T x \leq -\mu x - \nu v, \, \, d^2 x + \rho \geq -(1/\mu) \text{ for all } x \in S.
\end{bmatrix}\]

**Proof**: The proof of (3.1.3) and the first part of (3.1.4) is trivial, therefore, the proof of remaining two parts of (3.1.4) is briefly given.
Let $\mu < 0$, $a = \mu c + \nu d$ and (3.1.1) be satisfied for a passing through some segment $(x^1, x^2)$ in $S$. The directional maximum of $f(x)$ in the direction $s$ occurs at a point corresponding to $c^*$ given by

$$(c^*)^2 = (c^T x^* + \alpha)/\langle a^T s, d^T s \rangle,$$

where, as before, $x^*$ is the point on $d^T x + \beta = 0$ where $s$ originates from and $s$ is chosen to satisfy $d^T s > 0$. As $\alpha$ holds if no point of $(x^1, x^2)$ corresponds to $c^*$, i.e., if for every $x \in [x^1, x^2]$, $x = x^* + \epsilon s$, $0 < \epsilon < \infty$ either,

$$\text{(3.1.5)} \quad (c)^2 \leq (c^T x^* + \alpha)/\langle a^T s, d^T s \rangle$$

or

$$\text{(3.1.6)} \quad (c)^2 \geq (c^T x^* + \alpha)/\langle a^T s, d^T s \rangle.$$

The proof of the third part of (3.1.4) would follow from (3.1.6) on similar lines as the proof of the second part of (3.1.4) from (3.1.5). So we consider the proof of the second part of (3.1.4) only.

Now $d^T s = (1/\mu)(d^T x + \beta)$, $a^T s = (1/\mu)(a^T x - a^T x^*)$ and $(c^T x^* + \alpha) = (1/\mu)(a^T x^* + \mu x + \nu \beta)$ so that (3.1.5) reduces to

$$\text{(3.1.7)} \quad d^T x + \beta \leq (a^T x^* + \mu x + \nu \beta)/(\mu(a^T x - a^T x^*)).$$

As $a^T s < 0$, we have
\[ a^2 x + \mu x + \nu \leq a^2 x^* + \mu x + \nu \]

therefore,

\[
\frac{(a^2 x^* + \mu x + \nu)/(a^2 x - a^2 x^*)}{(a^2 x^* + \mu x + \nu)/(a^2 x + \mu x + \nu - (a^2 x^* + \mu x + \nu))}
\]

\[ \leq 1 \text{ for } a^2 x + \mu x + \nu \geq 0. \]

Since \( u < 0 \), therefore,

\[
(a^2 x^* + \mu x + \nu)/(\mu^{-1}(a^2 x - a^2 x^*)) \geq -(1/\mu) \text{ for } a^2 x \geq -\mu x - \nu. \]

Thus (3.1.7) and, therefore, (3.1.5) is true if

\[ d^2 x^* + \rho \leq -(1/\mu) \text{ and } a^2 x \geq -\mu x - \nu. \]

Hence if for every \( x \in S \), second part of (3.1.4) is satisfied (naturally then it is satisfied on any closed segment in \( S \) as well), \( Q \) holds.

Remark: The conditions in the second and the third part of (3.1.4) for the case when \( \alpha = \beta = \rho = 0 \) and \( \nu = 0 \) confirm with corresponding conditions in part II of Proposition in [98] where by misprint \( \mu \) appears in place where here we have \( - (1/\mu) \).

In general, (3) holds for a function on \( S \) if and only if it is quasiconvex while (4) holds for functions on \( S \) if
and only if these are explicitly quasiconvex [63] and that there are quasiconvex functions e.g. in [57, p. 236] for which (A) does not hold. But there on the basis of Proposition 6 in [95] it is easy to prove that (A) holds for \( f(x) \) if \( \xi \) is true. Thus under the conditions (3.1.3) or (3.1.4) \( f(x) \), in fact, is explicit quasiconvex function on \( S \) (in [95] \( f(x) \) is shown to be quasiconvex under (3.1.3) or (3.1.4)). Hence simplex-type algorithms to find a local maximum of \( f(x) \) on \( S \), given in [109], is applicable under these conditions only.

So far an attempt was made to pinpoint the discrepancy in the argument in [109] as clearly as possible. Also, under what conditions simplex-like technique can be applied, were discussed which broadly confirm with the conditions obtained in [95] though with a different approach.

### 3.1.2 Parametric Approaches

In general, the problem can be solved by parametric methods suggested in [85] and [48]. In the former, equivalent parametric problem to be solved is:

(PLP) \[ \text{Minimize } [g(x^2 + (1/t)c^2)x + \alpha/t) ; x \in S, c^2x = t - \beta] \]

which is a linear parametric problem in which the parameter \( t \) appears in both the price vector and the requirement vector. Also \( t \) varies over \([t_{\min}, t_{\max}]\) where \( t_{\min} \) (\( t_{\max} \)) is the
optimal value of the objective function in the (LP):

\[
\text{Minimise (Maximise)} \; [(d^T x + \beta) : x \in \mathcal{S}],
\]

As usual, the problem is to find an optimal solution function \(x^*(t)\) to the problem

\[
\text{Minimise } [(a^2 + (1/t)c^T)x : x \in \mathcal{S}, \; d^T x = t - \beta]
\]

over some interval \([\xi_1, \xi_2] \subseteq [t_{\min}, t_{\max}]\) resulting in division of \([t_{\min}, t_{\max}]\) in finite number of, say \(n\), intervals such that

\[t_{\min} = \xi_1, \; t_{\max} = \xi_n, \; \xi_{i-1} = \xi_i \text{ and } x^i(\xi_{i-1}) = x^i(\xi_i)\]

for \(2 \leq i \leq n\). Let \(x(t) = x^i(t), \; t \in [\xi_{i-1}, \xi_i]\) over \([t_{\min}, t_{\max}]\).

Then, if \(t^*\) is a solution to the one-dimensional problem

\[
\text{Minimise } [(a^2 + (1/t)c^T)x(t) + \zeta/t) : t \in [t_{\min}, t_{\max}]],
\]

corresponding \(x(t^*)\) solves the (GLLP).

In [48], a Branch-and-Bound technique to solve the (GLLP), is given. This, in general, is difficult to apply.

A parametric approach which emanates from this work is the algorithm as follows:

Let \(f_1(x) = a^T x\) and \(f_2(x) = (c^T x + \zeta)/(d^T x + \beta)\).

**Step 1**: Solve
Minimize \( \{ f_2(x) : x \in S \} \) for \( i = 1, 2 \).

Let \( h_i = \min \{ f_2(x) : x \in S \} \) and \( X_i^* = \{ x \in S : f_i(x) = h_i \} \)
for \( i = 1, 2 \).

If \( X_1^* \cap X_2^* \neq \emptyset \), any \( x \in X_1^* \cap X_2^* \) solves the (GLLP).

Otherwise,

Step 2: Solve

\[
\min \{ f_2(x) : x \in X_2^* \},
\]

let \( t_1^* = \min \{ f_2(x) : x \in X_2^* \} \).

Step 3: Solve the parametric fractional programming problem [106]:

\[
\min \{ f_2(x) : x \in S, f_2(x) \leq t, t \in [h_2, t_1^*] \}.
\]

Let \( x(t) \) be the optimal solution function over \([h_2, t_1^*]\).

Step 4: Solve the one-dimensional minimization problem

\[
\min \{ [t + f_2(x(t))] : t \in [h_2, t_1^*] \}.
\]

Let \( t^* \) be the optimal solution then \( x(t^*) \) solves the (GLLP).

In the next subsection a direct approach to find a global solution to the (GLLP) is given [76].

3.1.3 Modified Simplex-like Technique:

Hereafter, the constraint set \( S \) would be considered in
the ready for simplex method form i.e.,
\[ \mathcal{S} = \{ x \in \mathbb{R}^n : Ax = b , \ x \geq 0 \} , \ b \geq 0. \] Other conditions remain the same. Absence of degeneracy is also assumed.

In case conditions of Theorem 3.1.2 for the problem

\[ \text{Maximize } [-f(x) : x \in \mathcal{S}] \]

are satisfied; a local minimum of \( f(x) \) is attained at an extreme point of \( \mathcal{S} \). In this case the simplex-like algorithm given in [109] solves the (GLLP) for a local minimum provided, so obtained solution does not satisfy

\[ (3.1.8) \quad \begin{align*}
\mu &= \nu c + \rho d , \\
\mu &= -1/(d^T x + \rho) \quad \text{and} \quad v = (c^T x + \alpha)\mu \beta,
\end{align*} \]

because in this case such a search may lead to a stationary point [44, Sec. 9.2]. In fact, if we search over all the extreme points of \( \mathcal{S} \) (which are finite in number) we get the global minimum of \( f(x) \) over \( \mathcal{S} \). This justifies the assumption that the conditions of Theorem 3.1.2 are not satisfied for the problem

\[ \text{Maximize } [-f(x) : x \in \mathcal{S}], \]

in what follows.

From the (PLP) in the subsection 3.1.2 or Step 3 in the parametric algorithm in subsection 3.1.2, it is evident that a global minimum of \( f(x) \) in the (GLLP) lies
on an edge of $S$. A modified simplex-type technique is given here.

Let $x = (x_B, 0)$ be an initial basic feasible solution where $B$ is a basis matrix. It is assumed that $A$ is of full rank and $m \leq n$. For any $x \in S$, $x = (x_B, x_N)$ where $N$ denotes the matrix corresponding to nonbasic variables,

\[(3.1.9) \quad x_B = B^{-1}b - B^{-1}Nx_N\]

and

\[(3.1.10) \quad x_N = B^{-1}d.\]

The direction joining $\tilde{x}$ with the adjacent extreme point $\bar{x}$ obtained when $k$th column $A_k$ of $A$ is entered in the basis, is given by

\[a_k = (-B^{-1}A_k, e_k)\]

where $e_k$, as usual, is a vector in $\mathbb{R}^n$ with 1 at $k$th position and zero elsewhere. Now for sufficiently small $\varepsilon > 0$,

\[f(x) = f(\bar{x}) + \varepsilon a_k^T \nabla f(\bar{x}) + \varepsilon \sigma^-(x, \varepsilon)\]

where $\sigma^-(x, \varepsilon) = 0$ as $\varepsilon \to 0$. From (3.1.9) and (3.1.10)
\[
f(x) = e^x - (e^{\theta} - e^{\theta} x_N) + (e^{\theta} x + \alpha - (e^{\theta} - e^{\theta} x_N)/(e^{\theta} + \beta - (e^{\theta} - e^{\theta} x_N)
\]

so that
\[
e^{\theta} / \partial x_N \Delta \partial x N
\]

\[
= \Delta_k \text{ where ,}
\]

\[\begin{align*}
\Delta_k &= \Delta_k^{(1)} - \Delta_k^{(2)} + (f(3)(\Delta)(\Delta^{(2)} - \Delta_k) - f(2)(\Delta)(\Delta^{(3)} - \Delta_k))/f(3)(\Delta)^2
\end{align*}
\]

\[
\text{and } \Delta_k^{(1)} = \Delta_k^{(2)} = \Delta_k^{(3)} \text{ etc.}
\]

Therefore, \( f(x) \) decreases (increases) in the direction \( s_k \)
for sufficiently small \( \varepsilon > 0 \), if \( \Delta_k > 0 \) (< 0). Since \( f(x) \)
is either linear or strictly convex or else strictly concave on the edge \([\varepsilon, \overline{\varepsilon}]\), in fact, it is so on any closed line segment in \( S \) \([46]\), following cases arise,
\[
\begin{array}{c}
\begin{cases}
(1) \quad \Delta_k > 0 & \text{or} \\
(11) \quad \Delta_k \leq 0
\end{cases}
\end{array}
\]
and
\[
(3.1.12) \quad \begin{cases}
(iii) \quad f(x) \text{ is linear on } \left[ x, x' \right] \text{ or} \\
(iv) \quad f(x) \text{ is strictly convex on } \left[ x, x' \right] \text{ or} \\
(v) \quad f(x) \text{ is strictly concave on } \left[ x, x' \right]
\end{cases}
\]

In either of the combinations \([1], (iii)] \text{ or } \[(iii), (iii)]\), either \(x\) or \(x'\) respectively, is a minimum solution of \(f(x)\) over \([x, x']\) in the direction \(s_k\) and therefore, investigation for the minimum on the open segment \([x, x']\) is not required. Similarly, this holds in cases \([(iii), (iv)], [(i), (v)], \text{ and } [(i), (v)]\). But in case \([(i), (iv)]\) holds, \(f(x)\) may have its minimum on \([x, x']\) and therefore, investigation on the edge \([x, x']\) is required. 

Thus apart from \(\Delta_k\) which is known, indicators for \(f(x)\) being linear or strictly convex or else strictly concave and \(x'\), the minimum in the direction \(s_k\) if it lies in \([x, x']\) in case \([(i), (iv)]\) in \((3.1.12)\) holds, are required in terms of known functions at \(x\). This is done one by one in what follows.

To derive the first, we know
\[ f(\xi + \epsilon s_k) = f(\xi - \epsilon s_k) \]
\[ = f(\xi) - \epsilon(s_k' - a_k) \]
\[ + ((f^{(2)}(\xi) - \epsilon(s_k' - a_k))/f^{(3)}(\xi) - \epsilon(s_k' - a_k)) \]
so that
\[ f''(\xi) = -\epsilon(s_k' - a_k)(f^{(3)}(\xi)(s_k' - a_k) \]
\[ - f^{(2)}(\xi)(s_k' - a_k))/f^{(3)}(\xi) - \epsilon(s_k' - a_k))^3. \]

Therefore, as
\[ f^{(3)}(\xi) - \epsilon(s_k' - a_k) = f^{(3)}(\xi + \epsilon s_k) > 0, \text{ for all } \epsilon > 0 \]
such that \[ \xi + \epsilon s_k \in [\xi_k, \xi] \subset \delta, \]
\[ \text{sign}(f''(\xi)) = \text{sign}( - \delta_k) \] where,
\[ (3.1.13) \delta_k = (s_k' - a_k)(f^{(3)}(\xi)(s_k' - a_k) - \epsilon(s_k' - a_k) - f^{(2)}(\xi)(s_k' - a_k)). \]

Thus, \( f(x) \) is either linear or strictly convex or else strictly concave according as \( \delta_k \) is either zero or negative or else positive, respectively.

For the second part, as \([(i),(iv)] \) in \((3.1.13)\) is true, we have
\[ \delta_k > 0 \quad \text{and} \quad \delta_k < 0. \]

Also
\[ d^2s_k = - d_0^2 \delta - 1 \Delta_k + \delta_k = -(s_k^{(3)} - d_k) \]

and \((s_k^{(3)} - d_k)\) is a factor of \(\delta_k\) therefore,

\[ d^2s_k = -(s_k^{(3)} - d_k) \neq 0. \]

Let \(x^*\) be the point on \(d^2x + \beta = 0\) i.e., \(f^{(3)}(x^*) = 0\), where
from \(s_k\) originates. Let

\[ x = x^* + \delta s_k \]

where \(\delta > 0\) or \(< 0\) according as \(d^2s_k > 0\) or \(< 0\), so that

\[ x^* = (x_0 + \delta \bar{\delta}^{-1} A_{k \theta} - \delta s_k) \cdot \]

As \(d^2x^* + \beta = 0\) we have

\[ \delta = - f^{(3)}(x)/\delta s_k^{(3)} - d_k \cdot \]

Thus

\[ \begin{cases} \delta = - f^{(3)}(x)/\delta s_k^{(3)} - d_k \\ x^* = (x_0 + \delta \bar{\delta}^{-1} A_{k \theta} - \delta s_k) \end{cases} \]  
\((3.1.14)\)

and

\[ x^* = (x_0 + \delta \bar{\delta}^{-1} A_{k \theta} - \delta s_k) \cdot \]

Since \(d^2s_k = -(s_k^{(3)} - d_k)\) and \(a^2s_k = -(s_k^{(1)} - a_k)\), in cases [100],
\[
\begin{align*}
(3.1.16) \quad &
\begin{cases}
(1) & s_k^{(3)} - d_k < 0, \quad s_k^{(1)} - a_k < 0, \quad c^T x' + \alpha > 0 \\
\text{or} & \\
(11) & s_k^{(3)} - d_k > 0, \quad s_k^{(1)} - a_k < 0, \quad c^T x' + \alpha < 0,
\end{cases}
\end{align*}
\]

\(f(x)\) has a minimum in the direction \(s_k\) at a point corresponding to

\[
(3.1.16) \quad \delta^* = (f'(x')(x')/(s_k^{(3)} - d_k)(s_k^{(1)} - a_k))^{1/2}
\]

where positive or negative square root is taken according as 
(1) or (11) in (3.1.15) holds. Also \(\delta^*\) corresponding to \(\bar{x}\) is given by

\[
(3.1.17) \quad \delta^* = \delta + \min_i \{b_i/y_{1k} : y_{1k} > 0\},
\]

where \(y_{1k}\) is the \(i^{th}\) component of \(A^{-1}x_k\). Thus in cases

\[
\begin{align*}
(3.1.18) \quad &
\begin{cases}
(1) & \delta < \delta^* < \bar{\delta} \quad \text{if} \quad s_k^{(3)} - d_k < 0 \\
\text{or} & \\
(11) & \delta < \delta^* < \bar{\delta} \quad \text{if} \quad s_k^{(3)} - d_k > 0,
\end{cases}
\end{align*}
\]

\(f(x)\) has the minimum in the direction \(s_k\) in \((\bar{x}, \bar{x})\) at \(x'\) which is given by
\[ (3.1.19) \quad x^* = x^* + \epsilon s_k = \bar{x} + (\epsilon^* - \delta) s_k \]

\[ = (x_B - (\epsilon^* - \delta) B^{-1} A_k) (\epsilon^* - \delta) e_k \]

and

\[ (3.1.20) \quad f(x^*_i) = f^{(1)}(\bar{x}) - (\epsilon^* - \delta)(x_k^{(1)} - a_k) + (f^{(2)}(\bar{x}) - (\epsilon^* - \delta)(x_k^{(2)} - a_k)) \]

Thus by searching on all the extreme points (which are finite in number) and on appropriate edges of \( S \), the global minimum of \( f(x) \) over \( S \) can be found.

**Algorithm:**

**Step 1:** Check whether the conditions of the Theorem 3.1.2 are satisfied for

\[ \text{Maximize } [-f(x) : x \in S] \]

If no, go to Step 3.

**Step 2:** Solve the problem by simplex-like technique given in [109]. (If conditions (3.1.8) are satisfied at the solution obtained, a stationary point of \( f(x) \) is reached; otherwise a local minimum is obtained). The global minimum may be obtained by searching over all the extreme points. Go to END.

**Step 3:** Initialize \( S \) in the form,
\[ Ax = b \]
\[ x \geq 0 \]

with \( b \geq 0 \).

\textbf{Step 4:} Find a basic feasible solution

\[ \tilde{x} = (x_{B'}, 0). \]

\textbf{Step 5:} Calculate

\[ z_k^{(1)} - a_k = s_B^{T}b - a_k, \quad z_k^{(2)} - c_k = s_B^{T}a - c_k, \]
\[ z_k^{(3)} - d_k = d_B^{T}a - d_k, \]
\[ f^{(1)}(\tilde{x}) = s_B^{T}x, \quad f^{(2)}(\tilde{x}) = c_B^{T}x + c, \]
\[ f^{(3)}(\tilde{x}) = d_B^{T}x + \beta \]

and

\[ \Delta_k = z_k^{(1)} - a_k \]
\[ + (f^{(3)}(\tilde{x})(z_k^{(3)} - c_k) - f^{(3)}(\tilde{x})(z_k^{(3)} - d_k))/(f^{(3)}(\tilde{x})^2) \]

for every nonbasic column \( A_k \) of \( A \).

\textbf{Step 6:} Calculate

\[ s_k = (z_k^{(3)} - d_k)(f^{(3)}(\tilde{x})(z_k^{(3)} - c_k) - f^{(3)}(\tilde{x})(z_k^{(3)} - d_k)), \]
\[ ((f^{(3)}(\tilde{x})(z_k^{(3)} - c_k) - f^{(3)}(\tilde{x})(z_k^{(3)} - d_k)) \text{ is already known in the calculation of } \Delta_k \text{ in Step 5).} \]
Step 7: If \( \Delta_j \leq 0 \), for all \( j = 1, \ldots, n \), a local solution to the problem is found.

If there is no \( k \) left for which \( \Delta_k > 0 \) and \( \delta_k < 0 \), go to Step 16.

Step 8: Calculate

\[ \delta = - (z^{(3)}(x)/s_k^{(3)} - d_k) \]

and

\[ x^* = (x_0 + s_k^{-1}a_k - \delta_k) \cdot \]

Step 9: Check whether one of

(i) \[ s_k^{(3)} - d_k < 0, \quad s_k^{(3)} - d_k < 0, \quad c^T x^* + \alpha > 0 \]

or

(ii) \[ s_k^{(3)} - d_k > 0, \quad s_k^{(3)} - d_k < 0, \quad c^T x^* + \alpha < 0 \]

is satisfied.

If no, go to Step 7.

Step 10: Calculate

\[ \delta = \delta + \min \left[ \frac{b_k}{y_{1k}} : y_{1k} > 0 \right] \]

where \( y_{1k} \) is the \( i \)th component of \( \delta^{-1}a_k \).

Step 11: If (i) in Step 9 is satisfied then take
\[ \theta^* = \frac{f'(z)}{(s_k^{(3)} - d_k)(s_k^{(1)} - a_k)^{1/2}} , \]
otherwise go to Step 13.

**Step 13**: If \( \bar{\theta} < \theta^* < \bar{\theta} \) is not satisfied, go to Step 7.

Go to Step 15.

**Step 15**: If (11) in Step 9 is satisfied then take
\[ \theta^* = \frac{-f'(z)}{(s_k^{(3)} - d_k)(s_k^{(1)} - a_k)^{1/2}} , \]
otherwise go to Step 7.

**Step 16**: If \( \bar{\theta} < \theta^* < \bar{\theta} \) is not satisfied, go to Step 7.

**Step 16**: Calculate
\[ x^* = (x_0 - (\theta^* - \bar{\theta})s_k^{(3)} - (\theta^* - \bar{\theta})s_k^{(1)} - a_k) \]
and
\[ f(x) = f'(z)(x - (\theta^* - \bar{\theta})(s_k^{(3)} - a_k) \]
\[ + (f'(z)(x - (\theta^* - \bar{\theta})(s_k^{(3)} - a_k) + (f'(z)(x - (\theta^* - \bar{\theta})(s_k^{(3)} - a_k)) \]/(f'(z)(x - (\theta^* - \bar{\theta})(s_k^{(3)} - a_k))). \]
Go to Step 7.

**Step 16**: Calculate
\[ f(x^*) = \min(f(x^*), f(x^*) \text{ over all } k \text{ for which it exists}) \].

**Step 17**: If all the extreme points are not exhausted, then consider next extreme point not considered before and go to Step 4.
Step 18: Minimum of \( f(x) \) for different extreme points gives global minimum of the problem.

The algorithm as such does not contain the optimum choice strategy for the extreme points, objective of which would be - under the condition that we can move to an adjacent extreme point, an extreme point should be visited once only. This may be provided later on. Next, an example is solved to illustrate the algorithm.

Example:

Minimize \( f(x) = (0.1x_1 + 0.45x_2 - x_3 + (x_1 - x_2)/(2x_1 + x_2) \)

Subject to

\[ x \in S = \{x \in \mathbb{R}^3 : 1 \leq x_1 \leq 2, 0 \leq x_2 \leq 4, 1 \leq x_3 \leq 2 \}. \]

After obtaining initial basic feasible solution, the first iteration in the algorithm is as follows:
\[ a_j \rightarrow \begin{array}{cccccccccc} 0.1 & 0.4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{array} \]

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Because $a$, $b$ and $c$ are linear independent, conditions of Theorem 3.12 are not satisfied. There is only one edge for which $\Delta_k > 0$ and $\delta_k < 0$. After calculating $e^k$ we see that $\delta < e^k < \delta^*$ so that

$$x^* = (1, 0, 0, 1, 1, 0, 0, 1, 0, 0)$$

and $f(x^*) = -0.46$. Since no such edge exists further we next choose another extreme point, say, when $s_5$ is entered the basis. Second iteration is:
\[
\begin{array}{cccccccccc}
\text{j} & 0.1 & 0.48 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{k} & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{l} & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

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\[ a_j = \begin{pmatrix} 0.1 & 0.48 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_j = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_j = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \]

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\[ f(1)(x) = -0.8 \quad \begin{array}{c} a_{k}^{(1)} = a_k \end{array} \]
\[ a_k = -0.48 \quad \begin{array}{c} 0.1 \end{array} \]

\[ f(2)(x) = 2 \quad \begin{array}{c} a_{k}^{(2)} = c_k \end{array} \]
\[ c_k = 1 \quad \begin{array}{c} -1 \end{array} \]

\[ f(3)(x) = 4 \quad \begin{array}{c} a_{k}^{(3)} = d_k \end{array} \]
\[ d_k = -1 \quad \begin{array}{c} 2 \end{array} \]

\[ f(x) = -0.3 \quad \begin{array}{c} a_k \end{array} \]
\[ a_k = -0.105 \quad \begin{array}{c} 0.1 \end{array} \]

\[ \Delta_k = -1 \]

\[ \uparrow \]
\[
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\beta_j & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_j & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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</table>
\( f^{(2)}(\tilde{x}) = -0.32 \) | \( s_k^{(1)} = s_k \) | - | - | - | 0.1 | - | 0.48 | - | 1 |
\( f^{(3)}(\tilde{x}) = 1 \) | \( s_k^{(2)} = c_k \) | - | - | - | 1 | - | -1 | - | 0 |
\( f^{(3)}(\tilde{x}) = 5 \) | \( s_k^{(3)} = d_k \) | - | - | - | 2 | - | 1 | - | 0 |
\( f(\tilde{x}) = -0.12 \) | \( c_k \) | - | - | - | 6 | - | 6 | - | 0 |
\( \delta = -5 \) | \( \Delta_k \) | - | - | - | 0.28 | - | 0.24 | - | 1 |
\( f^{(2)}(\tilde{x}) = 6 \) | \( \tilde{x}^* \) = 2 | -4 | 1 | 0 | 1 | 5 | 1 | 0 |

(No\ of\ the\ conditions\ in
Step 9\ is satisfied).
\[
\begin{array}{cccccccccccc}
\alpha_j &=& 0.1 & 0.48 & -1 & 0 & 0 & 0 & 0 & 0 \\
\beta_j &=& 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_j &=& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta_j &=& 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

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<th>(e_j)</th>
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\(f(2)(\hat{x}^*) = 6\)  

(All of the conditions in Step 9 is satisfied).
\[
a_j = \begin{pmatrix}
0.1 & 0.48 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

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</table>

(None of the conditions in step 9 is satisfied).
\[
\begin{align*}
a_j & \rightarrow 0.1 & 0.48 & -1 & 0 & 0 & 0 & 0 & 0 \\
d_j & \rightarrow 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_j & \rightarrow 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{align*}
\]

<table>
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\( f(1)(x) = -0.42 \)

\( s_k = 1 - a_k \)

\( r(2)(x) = 0 \)

\( s_k = 1 - c_k \)

\( r(3)(x) = 3 \)

\( s_k = 1 - d_k \)

\( f(3)(x) = -0.42 \)

\( 6 = -3 \)

\( f(2)(x^*) = 3 \)

(Not all the conditions in Step 0 are satisfied.)
Introduction of any of the nonbasic variables \( z_2, z_3 \) or \( z_5 \) gives the extreme points of iteration 5, 1 or 7 respectively. Hence all the extreme points have been exhausted.

By comparison the optimal solution is on an edge in iteration 2, i.e.,

\[
x^* = (1.0, 5.0, 2)
\]

and optimal value of the objective function is

\[
f(x^*) = -1.46.
\]

3.2 Linear Piecewise Linear Programming with Variable Coefficients:

In what follows in this chapter, it is assumed that the conditions in Theorem 3.1.2 are satisfied so that (GLPLP) can be solved by simplex-like technique [109] for a local minimum.

In this section, based on [30], we consider a (GLPLP) in which the coefficients of an activity can be chosen with some freedom. These coefficients are assumed to be variables that satisfy a system of linear constraints.

The method consists of solving the (GLPLP) by the simplex-type algorithm. At each iteration, however, the necessary adjustment of variable coefficients is obtained by solving linear subprograms in these variable coefficients [33]. First the algorithm is developed and then a numerical example is given to illustrate the algorithm.
Consider the problem

\[
\begin{align*}
\text{Minimize } & \quad f(x) = \sum_{i=1}^{n-1} a_i x_i + \frac{\sum_{i=1}^{n-1} d_i x_i + y y x_n}{\sum_{i=1}^{n-1} d_i x_i + y y x_n} \\
\text{Subject to } & \quad A_1 x_1 + A_2 x_2 + \ldots + A_{n-1} x_{n-1} + y x_n = b \\
& \quad x \geq 0
\end{align*}
\]

where \( x \in \mathbb{R}^n \), \( a_i, c_i \) and \( d_i \) are \( i \)-th components of \( a, c \) and \( d \in \mathbb{R}^{n-1} \); \( b \) and \( A_i \)'s are \( n \)-component column vectors.

\( Y = (y_1, y_2, \ldots, y_m)^T \) is \( m \)-component vector of variables.

Let \( P = (y_1, y_2, \ldots, y_m, 3) \) be \((m+3)\)-component vector of variables, that are required to satisfy a system of linear inequalities, which forms a bounded convex polyhedron \( T \).

The constraint set of problem (3.2.1) is assumed to be regular. The problem is to find the vectors \((x_1, x_2, \ldots, x_n)\) and \( P = (y_1, y_2, \ldots, y_m, 3) \in T \), which give the optimum value of the objective function.

From [108], it is known that \( x_0 = B^{-1} b \) (not a stationary point) is a local optimal solution to the (GLLP),

\[
\begin{align*}
\text{Minimize } & \quad \sum_{i=1}^{n} a_i x_i + \frac{\sum_{i=1}^{n} d_i x_i + y y x_n}{\sum_{i=1}^{n} d_i x_i + y y x_n} \\
\text{Subject to } & \quad A_1 x_1 + \ldots + A_n x_n = b \\
& \quad x \geq 0
\end{align*}
\]
if and only if
\[ \Delta_j = (s_j^{(1)} - a_j) \]
\[ + \frac{f^{(2)}(x_B)(s_j^{(2)} - c_j) - f^{(2)}(x_B)(s_j^{(3)} - c_j))}{f^{(2)}(x_B)^2} \leq 0 \]

where \( s_j^{(1)} = a_B^{-1} A_j \) etc.,
\[ f^{(2)}(x_B) = c_B x_B + \lambda \] etc.,

holds for all \( A_j \), the columns of matrix \( A \).

As for the problem (3.2.1) it may be noted that if \( x_n \) is not in the basis, then in the tableau concerned, the variables \( y_i \)'s appear only in the column associated with \( x_n \) and the corresponding \( \Delta_n \) is a linear function of variable vector \( P \).

Algorithm:

**Step 1**: Obtain by simplex-type method, a local minimum \( x_B \) to the problem (3.2.1), without the variable \( x_n \) which has variable coefficients, so that \( \Delta_j \leq 0, j = 1, 2, \ldots, n-1 \). The corresponding \( \Delta_n = f_1(P) \) is a linear function in \( P \).

**Step 2**: Solve the (LP),
\[
(3.2.2) \quad \text{Maximize} \quad \Delta_n = f_1(P) \quad \text{subject to} \quad P \in T
\]
Let \( p^{(1)} \) be an optimal solution to \((3,2,2)\).

**Step 3:** In case \( f_1(p^{(1)}) \leq 0 \), then the basic feasible solution \( x_{B_1} \) is an optimal solution to \((3,2,1)\) with \( P \) being any vector in \( T \). So the procedure terminates.

**Step 4:** In case, \( f_1(p^{(1)}) > 0 \), let \( p^{(1)} = (y_1^{(1)}, \ldots, y_{m+3}^{(1)})^T \). Clearly \( p^{(1)} \) is an extreme point of \( T \). Here we have to adjust these values of the variable coefficients in the problem \((3,2,1)\) and still allow for the possibility of later revising the values of \( y_i^{(1)} \). To do this, a column specified by first \( n \) components of \( p^{(1)} \) is attached in the table and corresponding prices in \( f^{(1)}, f^{(2)} \) and \( f^{(3)} \) are \( y_{m+1}^{(1)}, y_{m+2}^{(1)} \) and \( y_{m+3}^{(1)} \) respectively with \( x_n^{(1)} \) as corresponding variable.

The problem to be considered now in place of \((3,2,1)\) is:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{n-1} a_i x_1 + y_{m+1}^{(1)} x_n^{(1)} + y_{m+2}^{(1)} x_n^{(2)} \\
& \quad + \sum_{i=1}^{n-1} c_i x_1 + y_{m+2}^{(1)} x_n^{(1)} + y_{m+3}^{(1)} x_n^{(2)} \\
& \quad + \sum_{i=1}^{n-1} d_i x_1 + y_{m+3}^{(1)} x_n^{(1)} + y_{m+3}^{(1)} x_n^{(2)} \\
\end{align*}
\]

\((3,2,3)\) Subject to

\[
A_1 x_1 + \cdots + A_{n-1} x_{n-1} + y_1^{(1)} x_n^{(1)} + y_2^{(1)} x_n^{(2)} = b \\
x \geq 0
\]

where \( y^{(1)} = (y_1^{(1)}, \ldots, y_{m+3}^{(1)}) \) with other conditions and
assumptions remaining similar to that of (3.2.1).

**Step 5:** Solve (3.2.3) by introducing \( x^{(1)}_n \) into the basis treating \( x^{(2)}_n \) the same way as \( x_n \) in case of (3.1.1) in Step 1 with \( A_j \geq 0 \), for all \( j = 1, \ldots, n \).

If \( x^{(1)}_n \) is not in the basis, then the solution so obtained is another local solution and replace \( x_{B_1} \) in Step 1 by this solution. Go to Step 2.

**Step 6:** If \( x^{(1)}_n \) is in the basis, then a solution \( x^* \), \( p^* \) to (3.2.3) also gives a solution to (3.2.1) as \( \bar{x}_n \), \( \bar{p} \) given by

\[
\bar{x}_1 = x_1^* \quad 1 \leq 1 \leq n-1
\]

\[
\bar{x}_n = x_n^{(1)} + x_n^{(2)} \quad \text{and} \quad \bar{p} = P(1) x_n^*(1) / (x_n^*(1) + x_n^*(2)) + P^* x_n^*(2) / (x_n^*(1) + x_n^*(2))
\]

\( (\because \bar{p} \in T_+ \text{ as } x_n^*(1) + x_n^*(2) > 0) \). In this sense problems (3.2.1) and (3.2.3) are equivalent.

Take problem (3.2.3) in place of problem (3.2.1) and go to Step 2.

In case, in Step 5, \( x^{(1)}_n \) always remains in the basis, the solution of the corresponding linear subprograms at various stages could be a different extreme point of \( T_+ \). Since \( T \) has finite number of extreme points, two cases arise:
(i) the problem can not be solved at this sequence of local solutions. Go to END.

(ii) the procedure terminates at the \( k \)th stage with

\[ p(1), \ldots, p(k) \]

being solutions of \( k \) linear subprograms and

\[ (x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*) \]

being the solution of the \( k \)th problem. Then for the optimal solution of the problem \((3.2.1)\) we have

\[ x^* = (x_1^*, \ldots, x_{n-1}^*, x_n^*) \]

where

\[ x_n^* = \sum_{j=1}^{k} \frac{\xi(j)}{x_n} \quad \text{and} \quad \xi_n = \sum_{j=1}^{k} \frac{\xi(j)}{x_n^*} \quad p(j) \]

Obviously, \( p^* \in T \), since \( p(j) \in T, j = 1, \ldots, k \).

END.

Remarks:

(i) It was assumed, for convenience, that the convex polyhedron \( T \) is bounded. As in [24], the method can be used to solve the problem \((3.2.1)\) even if \( T \) is unbounded.

(ii) The method can be generalized for problems in which different sets of variable coefficients say
$P_j (j = 1, 2, \ldots, r, r \leq n)$ are associated with different
activities $x_j (j = 1, 2, \ldots, r)$, and each $P_j$ is to be chosen
from a set $T_j$, where $T_j$ is a convex polyhedron.

To illustrate the method, the following numerical problem
is solved,

\[ \text{Minimize } -2x_1 - x_1^2 - 4x_1 - x_2^2 + \frac{3x_1 + 4x_2}{y_4} + 1 \]

Subject to

\[
\begin{align*}
  x_1 + y_1 x_2 & \leq 2 \\
  -x_1 + x_2 x_3 & \leq 1 \\
  x_1, x_2 & \geq 0
\end{align*}
\]

where $y_i$'s satisfy,

\[
\begin{align*}
  y_1 + y_2 & = 3 \\
  y_3 + y_4 - y_2 & = 4 \\
  y_1 y_2 y_3 y_4 & \geq 0
\end{align*}
\]

so that

\[ T = \left\{ p = (y_1, y_2, y_3, y_4) : y_1 + y_2 = 3, y_3 + y_4 - y_2 = 4, \\
                   y_1 y_2 y_3 y_4 \geq 0 \right\} \]
\[ \begin{align*}
  a_j & = -2 & 0 & 0 & -y_1 \\
  c_j & = -4 & 0 & 0 & -y_3 \\
  d_j & = 5 & 0 & 0 & y_4 \\
\end{align*} \]

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</tr>
<tr>
<td>$f(3) = 1$</td>
<td>$s_j^{(3)} - a_j = 6$</td>
<td>$s_j^{(3)} - c_j = 6$</td>
<td>$s_j^{(3)} - d_j = -5$</td>
<td>$y_1 + y_3$</td>
<td>$y_1$</td>
<td>$y_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f = 0$</td>
<td>$\Delta_j$</td>
<td>$\Delta_j$</td>
<td>$\Delta_j$</td>
<td>$\Delta_j$</td>
<td>$\Delta_j$</td>
<td>$\Delta_j$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
  & -2 & -4 & 5 & x_1 = 2 & 1 & 1 & -y_1 \\
  & 0 & 0 & 0 & x_4 = 3 & -1 & 1 & 1 & y_2 + y_1 \\
  & f(1) = -4 & s_j^{(1)} - a_j = -2 & -1 & 1 & -y_1 \\
  & f(2) = -8 & s_j^{(2)} - a_j = -4 & -1 & -4 & y_1 + y_3 \\
  & f(3) = 11 & s_j^{(3)} - a_j = 6 & -1 & 5y_1 - y_4 \\
  & f = -52/11 & \Delta_j = -24/121 & -1 & -y_1 & +(1/121)(-4y_1 + 11y_3) & -3y_2) |
\]
The solution to the (LP)

\[
\max_{P \in T} \Delta_2(P) = \frac{-123y_1 + 11y_3 - 8y_4}{121}
\]

is \( p(1) = (0,3,7,0) \) and \( \Delta_2(p(1)) = 7/11 > 0 \).

If we now solve the problem equivalent to \((3,2,3)\) right from the beginning, it can be seen that a local solution is

\[
x_1 = 0, x_3 = 2, x_4 = 0, x_2^{(1)} = 1/3
\]

and the solution of the related (LP)

\[
\max_{P \in T} \Delta_2(P) = \frac{3y_1 - 7y_2 + 3y_3 - 7y_4}{3}
\]

is \( p(2) = (3,0,4,0) \) and \( \Delta_2(p(2)) = 7 > 0 \).

In the same way, at the next stage the problem equivalent to \((3,2,3)\) has a solution as

\[
x_1 = 0, x_3 = 0, x_4 = 0, x_2^{(1)} = 1/3, x_2^{(2)} = 2/3
\]

and a solution of related (LP)

\[
\max_{P \in T} \Delta_2(P) = \frac{-4y_1 - 7y_2 + 3y_3 - 15y_4}{3}
\]

is \( p(3) = (3,0,4,0) \) and \( \Delta_2(p(3)) = 0 \). The algorithm terminates at this stage and a solution to the original problem is

\[
x_1 = 0, x_2 = 1/3 + 2/3 = 1, P = 1/3(0,3,7,0) + 2/3(3,0,4,0) = (2,1,5,0)
\]

with an optimum value of the objective function as \((-7)\). But if we go by the algorithm as stated, the next iteration is ;
\[
\begin{array}{ccccccc}
& a_1 & a_2 & a_3 & -y_1 \\
& a_4 & a_5 & a_6 & -y_2 \\
& d_j & 0 & 0 & 0 & -y_3 \\
& & & & & -y_4 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>( a_B )</th>
<th>( c_B )</th>
<th>( d_B )</th>
<th>( x_B )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_2^{(1)} )</th>
<th>( x_2^{(2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(-4)</td>
<td>(6)</td>
<td>(x_2^{(1)})</td>
<td>(1)</td>
<td>(1)</td>
<td>(-3)</td>
<td>(-4)</td>
<td>(-7)</td>
<td>(-7)</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(x_2^{(2)})</td>
<td>(1)</td>
<td>(1)</td>
<td>(3)</td>
<td>(-7)</td>
<td>(4)</td>
<td>(4)</td>
</tr>
<tr>
<td>(f(1))</td>
<td>(-4)</td>
<td>(-4)</td>
<td>(x_3^{(1)})</td>
<td>(1)</td>
<td>(1)</td>
<td>(-2)</td>
<td>(0)</td>
<td>(-y_1)</td>
<td>(-y_1)</td>
</tr>
<tr>
<td>(f(2))</td>
<td>(-8)</td>
<td>(-8)</td>
<td>(x_3^{(2)})</td>
<td>(1)</td>
<td>(1)</td>
<td>(-y_1)</td>
<td>(-y_1)</td>
<td>(-y_1)</td>
<td>(-y_1)</td>
</tr>
<tr>
<td>(f(3))</td>
<td>(11)</td>
<td>(11)</td>
<td>(x_4^{(3)})</td>
<td>(1)</td>
<td>(1)</td>
<td>(-y_1)</td>
<td>(-y_1)</td>
<td>(-y_1)</td>
<td>(-y_1)</td>
</tr>
<tr>
<td>(f)</td>
<td>(-82/11)</td>
<td>(-82/11)</td>
<td>(\Delta_j)</td>
<td>(-246/121)</td>
<td>(-7/11)</td>
<td>(-128y_1)</td>
<td>(-11y_3)</td>
<td>(-5y_4)</td>
<td>(-5y_4)</td>
</tr>
</tbody>
</table>

| \(-2\) | \(-4\) | \(5\) | \(x_1^{(1)}\) | \(1\)  | \(1\)  | \(1/3\) | \(1/3\) | \(1\)      | \(1\)      |
| \(0\)  | \(-7\) | \(0\) | \(x_2^{(1)}\) | \(1\)  | \(1/3\) | \(-3\)  | \(0\)   | \(-y_1\)   | \(-y_1\)   |
| \(f(1)\)| \(-4\)  | \(-4\) | \(x_2^{(1)}\) | \(1\)  | \(1/3\) | \(-3\)  | \(0\)   | \(-y_1\)   | \(-y_1\)   |
| \(f(2)\)| \(-15\) | \(-15\)| \(x_2^{(1)}\) | \(1\)  | \(1/3\) | \(-3\)  | \(0\)   | \(-y_1\)   | \(-y_1\)   |
| \(f(3)\)| \(11\)  | \(11\) | \(x_3^{(1)}\) | \(1\)  | \(1/3\) | \(-3\)  | \(0\)   | \(-y_1\)   | \(-y_1\)   |
| \(f\)  | \(-52/11\)| \(-52/11\)| \(\Delta_j\) | \(-710/383\) | \(-7/33\) | \(-128y_1\) | \(-11y_3\) | \(-5y_4\)  | \(-5y_4\)  |

\[ (-527/3y_1-27/3y_2-27/3y_3) \]
\[ (-11y_3-5y_4)/121 \]
The solution to the (LP)
\[
\max_{P \in T} \left( -347y_1 - 77y_2 + 33y_3 - 45y_4 \right) / 363
\]
is
\[
p^{(2)} = (0, 3, 7, 0) \quad \text{and} \quad \Delta_2(p^{(2)}) = 0.
\]
Therefore a solution to the original problem is
\[
x_1 = 2, \quad x_2^{(1)} = 1
\]
and
\[
P = (0, 3, 7, 0)
\]
with an optimum value of the objective function as \((-59/11)\).

3.3 An Algorithm for All Integer Solution to a Generalization of Linear and Piecewise Linear Programs:

There are some real world problems in which some or all of the variables have to be integers. Since rounding or truncation of non-integer solution is not the answer to the problem, for, sometimes such a solution may not remain feasible even, a systematic approach was needed to solve such
problems, Dantzig, Fulkerson and Johnson initiated work in this direction for (LP) but it was Gomory [41] who gave a systematic approach to solve the problem. Recently Glover [38], [39], Young [112], [114] etc. have contributed to this effect.

This section, based on [39] is devoted to develop a primal, all integer algorithm for solving a bounded and solvable (by simplex like technique i.e. appropriate conditions of Theorem 3.1.8 are satisfied) (GLPLP). The method is based on the work of Gomory [41] and borrows some ideas from the works of Glover [39] and Young [114].

To provide a notational context for describing the algorithm, the following form of integer (GLPLP) is considered.

Minimize \( f(x) = a^T x + (c^T x + \alpha)/(d^T x + \beta) \)

Subject to

\[
\begin{align*}
& (3.3.1) \quad x_1 + \sum_{j=1}^{n} a_{1j} x_j = b_{10}, \quad i = 1, 2, \ldots, m \\
& (3.3.2) \quad x_j \geq 0, \quad j = 1, \ldots, n \\
& (3.3.3) \quad x_j \text{'s integers.}
\end{align*}
\]

It is assumed that the constraint set 'S' of feasible solutions is nonempty and bounded. The coefficients \( a_{1j}, b_{10} \) are all integers and \( b_{10} \geq 0 \). This provides initial feasible solution to the problem.
The objective function \( f(x) \) when represented in terms of nonbasic variables is as follows

\[
\begin{align*}
f(x) &= \sum_{i=1}^{m} a_{i1}b_{i0} - \sum_{j=1}^{n} (s_{j}^{(1)} - s_{j})x_{j} + \left( \sum_{i=1}^{m} c_{i1}b_{i0} + \alpha \right) - \sum_{j=1}^{n} (s_{j}^{(2)} - s_{j})\gamma_{j} / \sum_{i=1}^{m} d_{i1}b_{i0} + \beta \\
&\quad - \sum_{j=1}^{n} (s_{j}^{(3)} - d_{j})\gamma_{j}
\end{align*}
\]

where

\[
s_{j}^{(1)} = \sum_{i=1}^{m} a_{i1}x_{i} \text{ etc.}
\]

The symbol \( \lfloor Y \rfloor \) is used to denote the integer part of \( Y \) i.e. \( \lfloor Y \rfloor \) is the largest integer less than \( Y \). Let \( J \) symbolize the index set of nonbasic variables in (3.3.1).

Suppose \( a_{s} > 0 \), so that \( a_{s} \) the column corresponding to \( x_{s} \), is chosen to enter the basis. Let \( s \) be the index of the row to leave the basis i.e.

\[
\theta_{s} = \frac{b_{00}}{a_{0s}} \leq \frac{b_{10}}{a_{1s}} \text{ for all rows } i \text{ with } a_{is} > 0.
\]

Before we go over to next tableau of simplex-like technique, the Gomory cut [41]
(3.3.6) \[ e_k + \sum_{j=1}^{n} \frac{a_{kj}}{a_{kj}} x_j = \frac{b_{kj}}{a_{kj}} \]

is adjoined to the tableau so that the pivot ratio for the cut

\[ \frac{b_{kj}}{a_{kj}} \leq \frac{b_{kj}}{a_{kj}} \]

so that primal feasibility is maintained if the cut (3.3.5) is used as the pivot row. The new tableau obtained by entering \( A_g \) and departing the row corresponding to this cut remains in the same (i.e., integer) form as the previous one. The algorithm is stated as follows:

**Step 1:** Start with the tableau in the form (3.3.1)-(3.3.3) with \( b_{kj} \geq 0 \) and \( a_{kj} \) all integers.

**Step 2:** The current basis is optimal if \( d_j \leq 0 \) for \( j = 1, \ldots, n \). Otherwise

**Step 3:** Select some column \( A_g \) for which \( d_g > 0 \) as pivot column.

**Step 4:** Select as source row for the Gomory cut for which

\[ 0 \leq \frac{b_{kj}}{a_{kj}} \leq \theta_g \]

where
\[ c_s = \min_{a_{1s} > 0} \left( \frac{b_{10}}{a_{1s}} \right) \]

If \( a_{1s} = 1 \) go to Step 6. Otherwise go to Step 5.

**Step 5:** Derive a Gomory cut from the source row given by \((3,3,5)\) with Gomory slack \( s \) and adjacent it at the bottom of the tableau. The value of basic variable \( s \) is \( \left| \frac{b_{00}}{a_{3s}} \right| \) in the new basic feasible solution.

**Step 6:** Use simplex-like procedure with \( A_s \) as entering column and appropriate departing row.

**Step 7:** If the incoming variable is a Gomory slack generated by previous cycle of the algorithm, eliminate the pivot row and the (new basic) variable from the tableau. Go to Step 2.

**END.**

**Remark:**

To establish fitness we shall have to place further restrictions on the rules for pivot column and source row selection as in the case of the simplified primal integer programming algorithm [114].