Chapter-4

Integrated Production Inventory Model: Multi-Item, Multiple Suppliers and Retailers, Exponential Demand Rate

4.1 Introduction

In recent, there is lot of competition between companies in marketing system. Supply chain plays an important role to deliver the items at right place, in right quantities, at right time and in minimum cost according to customer’s satisfaction. These days, multi-retailers shops are increased because people become professional to achieve their needs in minimum time. Now, multiple suppliers’ supplies different type of raw materials to producer and producer delivers different types of products according to the demand of multiple retailers.

Hill and Pakkala (2007) extended a multi-item inventory model where each customer orders a particular list of items. They select at least one item is being order independently from the list of items. Li et al. (2011) developed an inventory model for multi-item with dynamic lot-size taking both single-level and multi-level cases. They consider limited production resources and each item faces a particular demands. Sana (2011) developed an integrated multi-level production-inventory model of good and defective quality goods where supplier, producer and retailer are the members of the chain. Sana et al. (2012b) introduced a multi-echelon inventory model considering good and defective quality objects, product consistency and reworking of imperfect objects in the position of supply chain running. Freshly, Sana et al. (2012a) developed a three layer multi-item production-inventory model for various suppliers and retailers.

In this chapter, we have developed an integrated multi-item production inventory model where single producer, multiple suppliers and retailers are considered. Producer manufactured different types of finished items by a combination of fixed percentage of different types of raw materials. Each supplier delivers one type of raw material to the producer. Multiple retailers orders different types of products to producer and sell these products to customers in the market according to their
demands. The integrated profit function of the integrated supply chain is maximized with respect to the ordering lot size of raw materials. We also find the optimal solution of the model including a numerical example and sensitivity analysis.

### 4.2 Essential Assumptions

We used the following assumptions in the model:

1. The demand rates are exponential increasing function of time for each member of three layer supply chain.
2. Production rate is demand dependent i.e.
   \[ P(t) = \lambda D(t), \text{ where } D(t) = be^{at} \text{ and } \lambda > 1 \]
3. Holding cost per unit item varies with manufactured items, raw materials and each member of the supply chain.
4. Multiple \(m\) items are considered for joint effect of suppliers, producer and retailers in a three layer supply chain.
5. Single producer, multiple suppliers \(n\) and multiple retailers \(l\) are considered.
6. Producer produces various types of products on the basis of fixed percentage of different types of raw materials.
7. No shortages at each stage of three-layer supply chain.
8. Insignificant lead time.
9. Retailers purchases different types of manufactured items from the producer according to their demands.

![Figure 11 Integrated Logistic System](image-url)

**Figure 11 Integrated Logistic System**
4.3 Mathematical Model

We develop a three layer supply chain production inventory model for multi-item with single producer, multiple suppliers and retailers. The producer orders different types of raw materials to different suppliers but one supplier delivers one type of raw material only. The producer produces multi-items where every object is produced by the amalgamation of a definite proportion of the raw materials. The retailer purchase different types of products from the producer and sells to the customers according to their demands which depends on locations and environment of society. Shortages are not allowed at any stage of the model (Fig. 12).

4.3.1 Inventory model of supplier

We develop this segment with \( n \) suppliers where every supplier supplies only one type of raw material. We assume that multiple supplier supplies multiple raw materials. The producer orders \( R \) quantity of multiple raw materials at the opening of the manufacturing procedure and collects at a rate \( \lambda b e^{at} \) from the multiple suppliers (Fig. 12).

![Figure 12 Supply Chain for Suppliers](image-url)
The leading differential equation for multi-item is
\[
\frac{dQ_s(t)}{dt} = -\lambda be_s^{\alpha}, \text{ with } Q_s(0) = R \text{ and } Q_s(T_s) = 0, \quad 0 \leq t \leq T_s
\]  
(1)

From equation (1) we have, 
\[
Q_s(t) = \frac{\lambda b}{a} (1 - e^{\alpha t}) + R, 0 \leq t \leq T_s
\]  
(2)

Now, we have 
\[
Q_s(T_s) = 0, \Rightarrow T_s = \frac{1}{a} \log \left(1 + \frac{aR}{\lambda b}\right)
\]  
(3)

Supplier’s inventory cost of multi-item is
\[
C_s = h_s \int_0^{T_s} Q_s(t) dt = h_s \int_0^{T_s} \left(\frac{\lambda b}{a} (1 - e^{\alpha t}) + R\right) dt = \frac{h_s}{a} \left[\left(\frac{\lambda b}{a} + R\right) \log \left(1 + \frac{aR}{\lambda b}\right) - R\right]
\]  
(4)

using Eq. (3).

The supplier’s total profit of multi-item = Total revenue from sales - Total inventory cost
\[
STP = \left[ (w_s - C_s)R - \frac{h_s}{a} \left(\frac{\lambda b}{a} + R\right) \log \left(1 + \frac{aR}{\lambda b}\right) + h_s R \right]
\]  
(5)

4.3.2. **Inventory model of producer**

The producer manufactured multi-item where the production rate of multi-item is \(\lambda be_p^{\alpha t}\). One unit of multi-item is manufactured by the amalgamation of \(\alpha_i\) percentage of multi raw materials for \(i = 1, 2, 3,..., n\). Therefore, the manufacturing of multi-item is
\[
\lambda be_p^{\alpha t} = \sum_{i=1}^{n} \alpha_{sji} \lambda be_s^{\alpha t}, 0 \leq \alpha_s \leq 1 \text{ and } \sum_{j=1}^{m} \alpha_{sji} = 1 \text{ and the manufacturing run-time of multi-item by putting the value of } T_s \text{ from equation (3) is (Fig.13).}
\]

\[
T_{p_{ji}} = \frac{\sum_{j=1}^{n} \alpha_{sji} \lambda be_s^{\alpha t} T_s}{\lambda be_p^{\alpha t}} = \frac{\sum_{j=1}^{n} \alpha_{sji} e_s^{\alpha t} \log \left(1 + \frac{aR}{\lambda b}\right)}{ae_p^{\alpha t}} = \frac{1}{a} \log \left(1 + \frac{aR}{\lambda b}\right)
\]  
(6)
The producer supplies the finished goods to the retailers. Producer delivers multi-item to the retailers according to their demands at a rate $be_r^{at}$. 

Then, the leading differential equation with multi-item is

$$\frac{dQ_p(t)}{dt} = \lambda be_p^{at} - be_r^{at}, \text{ with } Q_p(0) = 0, 0 \leq t \leq T_p$$

(7)

And

$$\frac{dQ_p(t)}{dt} = -be_r^{at}, \text{ with } Q_p(T_p) = 0 \text{ and } Q_p(T_p) = \left(\lambda be_p^{at} - be_r^{at}\right)T_p, T_p \leq t \leq T_p$$

(8)

From equations (7) and (8), we get

$$Q_p(t) = \left[\frac{b}{a} \left(1 - e_r^{at}\right) - \frac{\lambda b}{a} \left(1 - e_p^{at}\right)\right], 0 \leq t \leq T_p$$

(9)

And

$$Q_p(t) = \frac{b}{a} \left(1 - e_r^{at}\right) - \frac{\lambda b}{a} \left(1 - e_p^{aT_p}\right), T_p \leq t \leq T_p$$

(10)

Now, we have

$$Q_p(T_p) = \left(\lambda be_p^{at} - be_r^{at}\right)T_p$$

$$T_p = \frac{1}{a} \log \left(1 + \frac{a}{b} R\right) = \frac{1}{a} \log \left(1 + \frac{aR}{\lambda b}\right)$$

(11)

using Equation (6) for multi-item.

Producer’s inventory cost for multi-item is
The unit manufacturing cost for multi-item product is

\[ C_P = \frac{h_p b}{a} \left[ \left( \frac{a}{b} \right) R + (1 - \lambda) \log \left( 1 + \frac{aR}{\lambda b} \right) \right] \]  

The unit manufacturing cost for multi-item product is

\[ C(P) = \frac{L}{\lambda be_p^{at}} + \gamma \lambda be_p^{at} \]  
Where \( \lambda be_p^{at} = \sum_{i=1}^{n} \alpha_i \lambda be_s^{at} \)  

\( \frac{L}{\lambda be_p^{at}} \) is equally distributed labor/energy cost over manufacturing lot size \( \lambda be_p^{at} \). Therefore, unit manufacturing cost reduces with increase in manufacturing lot-size.

\( \gamma \lambda be_p^{at} \) is per unit instrument cost of finished goods which is proportional to the manufacturing lot-size and the cost for raw-material per unit item is predetermined which is taken independently in the particular profit function.

Producer’s total profit for multi-item

\[ PT_P = \sum_{j=1}^{m} \left( w_p \lambda be_p^{at} - C(P) \lambda be_p^{at} T_p \right) - \sum_{i=1}^{n} Rw_s - \sum_{j=1}^{m} C_p \]

\[ = \left( \frac{mh_p}{a} - \frac{mh_p}{a} - mw_s \right) R + \left( \frac{mw_p \lambda be_p^{at}}{a} + \frac{mh_p b (1 - \lambda)}{a} - mL + m \gamma \lambda b^2 e_p^{2at} \right) \log \left( 1 + \frac{aR}{\lambda b} \right) \]

\[ = \left( \frac{mh_p}{a} - \frac{mh_p}{a} - mw_s \right) R + \left( \frac{mw_p \lambda be_p^{at}}{a} + \frac{mh_p b (1 - \lambda)}{a} - mL + m \gamma \lambda b^2 e_p^{2at} \right) \log \left( 1 + \frac{aR}{\lambda b} \right) \]

\[ 4.3.3. \text{ Inventory model of retailer} \]

According to the location of the retailers, the demand rates of the finished goods for the retailers are dissimilar. The multiple retailers collect the multi-item from the producer at a rate \( \beta_p be_r^{at} \) where \( \beta_p \) is the quantity of multi-item product for multiple retailers, \( 0 \leq \beta_p \leq 1 \) and \( \sum_{k=1}^{l} \beta_k = 1 \) for \( k = 1, 2, 3, \ldots, l \). Multiple retailers facing the customer’s demand \( be_r^{at} \) for the multi-item. The producer supplies the multi-item to the multiple retailers up to time \( T_p \) (Fig.14).
Then, the leading differential equations for multi-item is

\[
\frac{dQ_r(t)}{dt} = b(\beta_p e_r^{a_r} - e_c^{a_c}), \text{ with } Q_r(0) = 0, \quad 0 \leq t \leq T_k
\]

(15)

\[
\frac{dQ_r(t)}{dt} = -be_c^{a_r}, \text{ with } Q_r(T_k) = b(\beta_p e_r^{a_r} - e_c^{a_c})T_k \text{ and } Q_r(T_r) = 0, T_k \leq t \leq T_r
\]

(16)

From Eq. (15) and Eq. (16), we have

\[
Q_r(t) = \frac{b}{a} \left(1 - e_c^{a_c}\right) - \frac{b}{a} \beta_p \left(1 - e_r^{a_r}\right), \quad 0 \leq t \leq T_k
\]

(17)

And

\[
Q_r(t) = \frac{b}{a} \left(e_c^{a_r} - e_c^{a_c}\right), \quad T_k \leq t \leq T_r
\]

(18)

Now, we have

\[
Q_r(T_k) = b(\beta_p e_r^{a_r} - e_c^{a_c})T_k
\]

\[
T_r = \frac{\beta_p be_c^{a_r}T_k}{be_c^{a_c}} = \frac{\beta_p\lambda e_r^{a_r}}{be_c^{a_c}}\frac{1}{a}\log\left(1 + \frac{aR}{\lambda b}\right) = \frac{\beta_p}{a}\log\left(1 + \frac{aR}{\lambda b}\right)
\]

(19)

Using Equation (11) according to model assumptions.

Retailer’s inventory cost for multi-item is
\[ C_i = h_i \left[ \int_0^t Q_i(t) \, dt + \int_t^T Q_i(t) \, dt \right] = \left[ \left( \beta_p h_i - h_i \right) R + \left( \beta_p h_i - h_i \right) R \log \left( 1 + \frac{aR}{\lambda b} \right) \right] \]  

(20)

using Equation (11).

Retailer’s total profit for multi-item is

\[ RTP = \sum_{j=1}^m \left[ \left( \log \left( 1 + \frac{ar}{\lambda b} \right) - \beta_p \omega_p \log \left( 1 + \frac{ar}{\lambda b} \right) \right) \left( \frac{h_j}{a} \right) \right] \]

\[ R TP = \sum_{j=1}^{m} \left[ \left( \frac{m \beta_p h_j}{a} - \frac{mh_j}{a} \right) R \log \left( 1 + \frac{ar}{\lambda b} \right) \right] \]

\[ = \left( \frac{mh_j}{a} - \frac{m \beta_p h_j}{a} \right) R + \left( \frac{mw_p \omega_p \beta_p h_j}{a} - \frac{m \beta_p h_j}{a} \log \left( 1 + \frac{ar}{\lambda b} \right) \right) \]  

(21)

4.4 Integrated total profit

The integrated total profit for multi-suppliers, single producer and multi-retailers, with the help of equations (5), (6), (14), (19) and (21) is

\[ ITP (R) = \sum_{i=1}^n STP + PTP + \sum_{k=1}^l RTP = \left( \frac{mh_j h_i}{a} - \frac{mh_j}{a} + \frac{mh_i}{a} - \frac{ml \beta_p h_i}{a} - nC \right) R \]

\[ + \left( \frac{m \beta_p h_j}{a} - \frac{mh_j}{a} - \frac{m \beta_p h_j}{a} \log \left( 1 + \frac{ar}{\lambda b} \right) \right) \]  

\[ ITP (R) = AR + \left( B - CR \log (1+DR) \right) \]  

(22)

Where

\[ A = \frac{mh_j h_i}{a^2} - \frac{mh_j}{a} + \frac{mh_i}{a} - \frac{ml \beta_p h_i}{a} - nC, \quad C = \frac{nh_j}{a} + \frac{ml \beta_p h_i}{a} - \frac{mlh_i}{a}, \quad D = \frac{a}{\lambda b}, \]

\[ B = \frac{m \beta_p h_j}{a} - \frac{mh_j}{a} - \frac{m \beta_p h_j}{a} - \frac{ml \beta_p h_j}{a} + \frac{mlh_i}{a} - \frac{nh_j \lambda b}{a^2} \]

Solution Procedure:

Differentiating equation (22) partially with respect to \( R \) for \( i = 1, 2, 3, \ldots, n \), we have

\[ \frac{\partial (ITP)}{\partial R} = A + \frac{D (B - CR)}{1 + DR} - C \log (1 + DR) \]  

(23)
\[
\frac{\partial (ITP)}{\partial R} = 0, \quad A + \frac{D(B - CR)}{1 + DR} - C \log(1 + DR) = 0
\]

Now, we get,

\[
R = \frac{AD - CD - 2C \log D + D \sqrt{A^2 - 2AC + C^2 + 4(\log D)^2 + 4BC \log D}}{2C(\log D)^2}
\] (24)

Again differentiate equation (23) partially with respect to \( R \) for \( i = 1, 2, 3, \ldots, n \), we have

\[
\frac{\partial^2 (ITP)}{\partial R^2} = -\frac{2CD}{1 + DR} \frac{D^2(B - CR)}{(1 + DR)^2}
\] (25)

Thus \( \frac{\partial^2 (ITP)}{\partial R^2} < 0 \) holds at \( R \) satisfying necessary and sufficient condition, and then \( ITP(R) \) is maximum.

4.5 Numerical experiment:

We consider three suppliers, single producer and three retailers in this integrated production inventory model. Each supplier delivers one type of raw material to the producer. There are three types of raw materials. Producer manufactured three types of products by the combination of three raw materials. Producer supplies three types of products to three retailers according to their demands. The following parameters and numerical data are considered as:

\( m = 3, n = 3, l = 3, \lambda = 2, a = 0.7, b = 0.1, \lambda be_{p_1}^{art} = 200, \lambda be_{p_2}^{art} = 220, \lambda be_{p_3}^{art} = 250 \)

**Table 4  Supplier’s numerical data:**

<table>
<thead>
<tr>
<th>Suppliers ( i )</th>
<th>Raw Materials ( i )</th>
<th>( h_s )</th>
<th>( \lambda be_s^{art} )</th>
<th>( w_s )</th>
<th>( C_s )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
<td>550</td>
<td>18</td>
<td>8</td>
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<td>2</td>
<td>2</td>
<td>3</td>
<td>680</td>
<td>16</td>
<td>7</td>
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<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>450</td>
<td>22</td>
<td>11</td>
</tr>
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</table>
Table 5 Producer’s numerical data

<table>
<thead>
<tr>
<th>Products $j$</th>
<th>$\alpha_s$</th>
<th>$L$</th>
<th>$\gamma$</th>
<th>$h_p$</th>
<th>$b_{e_{r,at}}$</th>
<th>$w_p$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>0.3</td>
<td>4000</td>
<td>0.01</td>
<td>9</td>
<td>200</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>4800</td>
<td>0.02</td>
<td>10</td>
<td>250</td>
<td>70</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>4500</td>
<td>0.01</td>
<td>8</td>
<td>280</td>
<td>65</td>
</tr>
</tbody>
</table>

Table 6 Retailer’s numerical data

<table>
<thead>
<tr>
<th>Retailers $k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_p$</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$b_{e_{r,at}}$</td>
<td>250</td>
<td>300</td>
<td>350</td>
</tr>
<tr>
<td>$w_r$</td>
<td>85</td>
<td>85.5</td>
<td>85</td>
</tr>
<tr>
<td>$h_r$</td>
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<td>1.05</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Applying these numerical values in the Eq. (22) and we have the solutions $R_1 = 1305.5, R_2 = 2077.3$ and $R_3 = 1423.2$. The integrated total profits on the basis of these lot sizes are $ITP_1 = 1280848, ITP_2 = 1567821.5$ and $ITP_3 = 1761590.4$.

4.6 Sensitivity Analysis

To know, how the optimal solution is affected by the parameters, we derive the sensitivity analysis for holding costs of all members of integrated supply chain, purchasing cost of suppliers and production cost of producer. The best values of the parameters of optimal solution are increases or decreases by 25%, -25% and 50%, -50%.

The results of integrated total profits are existing in the following table 4.
Table 7 Sensitivity Analysis

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>+25% Increased</th>
<th>−25% Decreased</th>
<th>+50% Increased</th>
<th>−50% Decreased</th>
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<tbody>
<tr>
<td>$h_{s1}$</td>
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<td>6.25</td>
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<tr>
<td>$h_{s2}$</td>
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<td>3.75</td>
<td>2.25</td>
<td>4.5</td>
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<tr>
<td>$h_{s3}$</td>
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<td>7.50</td>
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<td>5</td>
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<td>0.75</td>
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<td>0.5</td>
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<tr>
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<td>1.56</td>
<td>0.53</td>
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<td>0.83</td>
<td>1.65</td>
<td>0.55</td>
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<td>10</td>
<td>6</td>
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<td>4</td>
</tr>
<tr>
<td>$C_{s2}$</td>
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<td>8.75</td>
<td>5.25</td>
<td>10.5</td>
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<td>8.25</td>
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<td>5.5</td>
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<td>1415153.7</td>
<td>1098591.8</td>
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<td>991455.9</td>
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</table>
4.7 Conclusion
We have developed an integrated multi-item production inventory model where multiple suppliers, single producer and multiple retailers are considered. Producer manufactured different types of finished items by a combination of fixed percentage of different types of raw materials. Each supplier delivers one type of raw material to the producer. Multiple retailers orders different types of products to producer and sell these products to customers in the market according to their demands. We assume that demand rates of each members of integrated supply chain are exponential increasing function of time. The holding costs of raw materials and finished goods are different according to members of integrated supply chain. The integrated profit function of the integrated supply chain is maximized with respect to the ordering lot size of raw materials. We also find the optimal solution of the model including a numerical example and sensitivity analysis.
In the fifth chapter we developed a production inventory model integrated for reworkable items with exponential demand rate in an imperfect production process.