CHAPTER 2

EOQ Model for Deteriorating Items with Exponential Demand Rate and Partial Backlogging

2.1. Introduction

Inventory modeling is a mathematical approach to determine when to order and how much order to minimize the total inventory cost. The deterioration of items plays a significant role in the inventory system. The deterioration of goods is a usual phenomenon in daily life. It includes a variety of consumable goods such as vegetables, fruits, milk products etc. are a few examples of objects in which deterioration can take place during the ordinary storage of goods and consequently this loss must be taken into account when analyzing the model. Shortages are permitted and so the customers will wait till the next replenishment. All demands are fulfilled instantly. Donaldson (1977) is the first author who developed the classical no shortage inventory strategy for linear, time-dependent demand. An ELSP for non-instantaneous deteriorating items using price discount was established by Jeyaraman and Sugapriya (2008). Sana (2010a) considered that the demand rate is price dependent; the deterioration rate is time-dependent and partial backlogging. Shah and Shah (2000), Goyal and Giri (2001) extended review on inventory model with deteriorating items. Sanni (2012) developed an inventory model for three parameters Weibull deterioration, with shortages and quadratic demand rate.

The aim of the present chapter is to give a new approach to the inventory system. Exponential demand rate is an increasing function of time and shortages are allowed. To explain the inventory model a numerical illustration and sensitivity analysis has been carried out to analyze the result of parameters on decision variables and the entire cost of this inventory model.

2.2 Assumptions

We need the following assumptions for developing mathematical model.

1. The Inventory system consider single item.
2. The demand rate \(D(t)\) at any time \(t\) is given by

\[
D(t) = \begin{cases} 
  e^{\lambda t}, & i(t) > 0 \\
  \alpha, & i(t) \leq 0
\end{cases}
\]

Where \(\lambda > 0\) and \(\alpha > 0\) and \(i(t)\) is the inventory level at time \(t\).

3. The lead time is zero.

4. Shortage are allowed and partially backlogged.

5. The rate of deterioration \(\theta(0 < \theta < 1)\) is constant.

6. Replenishment rate is infinite.

7. The backlogging rate through the shortage phase is inconsistent and depends on the length of the waiting time till the next replenishment. The partial backlogging rate will be \(B(t) = \frac{1}{1 + \delta(T - t)}\), where \(\delta > 0\) denote the backlogging parameter during \(t_1 \leq t \leq T\).

8. \(I_a(t)\) is the inventory level at time \(t (0 \leq t \leq t_1)\) in which the product has no deterioration. \(I_b(t)\) is the inventory level at time \(t (t_1 \leq t \leq t_2)\) in which the product has deterioration. \(I_c(t)\) is the inventory level at time \(t (t_2 \leq t \leq T)\) in which the product has shortage.

### 2.3 Mathematical Model

At the start of the cycle, the inventory level reaches its maximum \(I_{\text{max}}\) units of items at time \(t = 0\). The inventory depletes due to exponential demand rate during \([0,t_1]\). The inventory depletes due to both exponential demand rate and deterioration till it becomes zero during the interval \([t_1,t_2]\). The inventory level reaches to zero at time \(t = t_2\). The inventory system \(I_a(t)\) is defined by the differential equation:

\[
\frac{dI_a(t)}{dt} = -e^{\lambda t}, \quad 0 \leq t \leq t_1
\]

With condition \(I_a(0) = I_{\text{max}}\)

The solution of Eq. (1) is
\[ I_a(t) = I_{\text{max}} + \frac{1}{\lambda} \left[ 1 - e^{\lambda t} \right], \quad 0 \leq t \leq t_1 \]  

(2)

Inventory level decreases due to exponential demand rate and deterioration rate during the interval \([t_1, t_2]\), and reaches zero at time \(t_2\) and then shortages occur. The inventory system \(I_b(t)\) is defined by the differential equation:

\[ \frac{dI_b(t)}{dt} + \theta I_b(t) = -e^{\lambda t}, \quad t_1 \leq t \leq t_2 \]  

With condition \(I_b(t_2) = 0, \quad t_1 \leq t \leq t_2\)

The solution of Eq. (3) is

\[ I_b(t) = \frac{1}{\theta + \lambda} \left( e^{-\theta t} \cdot e^{(\theta + \lambda)t_2} - e^{\lambda t} \right), \quad t_1 \leq t \leq t_2 \]  

(4)

Considering continuity of \(I(t)\) at \(t = t_1\), it follows that

\[ I_a(t_1) = I_b(t_1) \]

\[ I_{\text{max}} + \frac{1}{\lambda} \left[ 1 - e^{\lambda t_1} \right] = \frac{1}{\theta + \lambda} \left[ e^{-\theta t_1} \cdot e^{(\theta + \lambda)t_2} - e^{\lambda t_1} \right] \]

The maximum inventory level per cycle is

\[ I_{\text{max}} = \frac{1}{\theta + \lambda} \left( e^{-\theta t_1} \cdot e^{(\theta + \lambda)t_2} - e^{\lambda t_1} \right) - \frac{1}{\lambda} \left( 1 - e^{\lambda t_1} \right) \]  

(5)

Substitute the value of \(I_{\text{max}}\) from Equation (5) into Equation (2), we have

\[ I_a(t) = \frac{1}{\theta + \lambda} \left( e^{-\theta t_1} \cdot e^{(\theta + \lambda)t_2} - e^{\lambda t_1} \right) + \frac{1}{\lambda} \left( e^{\lambda t_1} - e^{\lambda t} \right), \quad 0 \leq t \leq t_1 \]  

(6)

The inventory level reaches to zero at time \(t_2\) and then shortages take place. The exponential demand is partially backlogged. The amount of demand backlogged is given by the following differential equation:

\[ \frac{dI_c(t)}{dt} = -\alpha, \quad t_2 \leq t \leq T \]  

(7)

With condition \(I_c(t_2) = 0, \quad t_2 \leq t \leq T\)

The solution of Equation (7) is
\[ I_c(t) = -\frac{\alpha}{\delta} \left[ \ln \left( 1 + \delta \left( T - t \right) \right) - \ln \left( 1 + \delta \left( T - t \right) \right) \right], t_2 \leq t \leq T \tag{8} \]

The maximum amount of demand backlogged per cycle by putting \( t = T \) in Equation (8), we have

\[ S = -I_c(T) = \frac{\alpha}{\delta} \left[ \ln \left( 1 + \delta \left( T - t \right) \right) \right], t_2 \leq t \leq T \tag{9} \]

From Equation (5) and Equation (9), we obtained the order quantity \( Q \) per cycle as

\[ Q = I_{\text{max}} + S \]

\[ Q = \frac{1}{\theta - \lambda} \left[ e^{-\theta t} e^{(\theta + \lambda) t} - e^{\delta t} \right] - \frac{1}{\lambda} \left[ 1 - e^{\delta t} \right] + \frac{\alpha}{\delta} \left[ \ln \left( 1 + \delta (T - t) \right) \right], t_2 \leq t \leq T \tag{10} \]

The holding cost per cycle is

\[ H_c = \int_0^{t_2} I_a(t)dt + \int_{t_1}^{t_2} I_b(t)dt \]

\[ = \frac{e^{\delta t_2}}{\theta + \lambda} \left\{ e^{-\theta t} e^{\theta t} \left( t_1 + \frac{1}{\theta} \right) - \left( \frac{1}{\theta} + \frac{1}{\lambda} \right) \right\} + e^{\delta t} \left( \frac{t_1 - t_1}{\lambda} - \frac{1}{\theta + \lambda} - \frac{1}{\lambda^2} + \frac{1}{\lambda (\theta + \lambda)} \right) + \frac{1}{\lambda^2} \tag{11} \]

The deterioration cost per cycle is

\[ D_c = I_b(t_1) - \int_{t_1}^{t_2} D(t)dt = e^{\delta t_2} \left( \frac{e^{-\theta t} e^{\theta t} \left( t_1 + \frac{1}{\theta} \right)}{\theta + \lambda} - \frac{1}{\lambda} \right) + e^{\delta t} \left( \frac{1}{\lambda} - \frac{1}{\theta + \lambda} \right) \tag{12} \]

The shortage cost per cycle is

\[ S_c = \int_{t_2}^{T} \left[ -I_c(t) \right] dt = \frac{\alpha}{\delta} \left[ (T - t_2) - \frac{\ln \left( 1 + \delta (T - t_2) \right)}{\delta} \right] \tag{13} \]

The opportunity cost per cycle is

\[ O_c = \int_{t_2}^{T} \alpha \left[ 1 - \beta (T - t) \right] dt = \alpha \left[ (T - t_2) - \frac{\ln \left( 1 + \delta (T - t_2) \right)}{\delta} \right] \tag{14} \]

Total relevant cost is

\[ TC(t_2) = \frac{1}{T} \left[ A + H_c + D_c + S_c + O_c \right] \]

\[ TC(t_2) = B + Ce^{\delta t_2} - Fe^{\delta t_2} - G \frac{\ln \left( 1 + \delta t_2 \right)}{\delta} \]
Where,
\[ B = \frac{A}{T} + \frac{1}{\lambda^2 T} + \frac{e^{\lambda t}}{T} \left( \frac{t_1 + 1}{\lambda} - \frac{1}{\lambda^2} - \frac{t_1 + 1}{\theta + \lambda} + \frac{1}{\lambda(\theta + \lambda)} \right) + \alpha \left( \frac{1}{\delta} + 1 \right), \]
\[ C = \frac{e^{-\theta t}}{(\theta + \lambda)T} \left( t_1 + 1 + \frac{1}{\theta} \right), D = \theta + \lambda, E = \frac{1 + \theta}{\theta T}, F = \frac{\alpha}{T} \left( \frac{1}{\delta} + 1 \right), G = 1 + \delta T. \]

2.4 Solution Procedure

\[ \frac{\partial TC}{\partial t_2} = -F + CDe^{Dt_2} - \lambda E e^{Lt_2} + \frac{F}{G - \delta t_2}, \]
\[ \frac{\partial^2 TC}{\partial t_2^2} = CD^2 e^{Dt_2} - \lambda^2 E e^{Lt_2} + \frac{\delta F}{(G - \delta t_2)^2}. \] (16)

Main objective is to minimize the total relevant cost of the inventory model. The necessary condition to minimize the total relevant cost is

\[ \frac{\partial TC}{\partial t_2} = 0, \quad \text{We have} \]
\[ \delta \lambda E t_2 e^{Lt_2} - \delta CD t_2 e^{Dt_2} + CD Ge^{Dt_2} - \lambda E Ge^{Lt_2} + \delta Ft_2 + F - FG = 0 \] (18)

Using the software Mathematica-9, we can calculate the optimal value of \( t_2 \) by equation (18) and the optimal value \( TC(t_2) \) of the total relevant cost is determined by equation (15). The optimal value of \( t_2 \) satisfy the sufficient condition for minimizing total relevant cost \( TC(t_2) \) is

\[ \frac{\partial^2 TC}{\partial t_2^2} > 0 \] (19)

The sufficient condition is satisfied.
Graphical representation of total relevant cost $TC$ and time $t_2$ is shown by figure 5.

Figure 5. Graphical representation of total relevant cost $TC$ and time $t_2$

Convexity on the cost

Figure 6. Effects of changes of the parameters on total relevant cost
2.5 Numerical Example

Consider the values;

\[ A = 100, H_c = 30, D_c = 15, S_c = 25, O_c = 10, \alpha = 20, \delta = 5, t_i = 0.5, \lambda = 0.1, \theta = 0.5, T = 1 \text{ year}. \]

\[ B = 146.1, C = 4.5, D = 0.6, E = 30, F = 24, G = 6 \]

Thus, the optimal value of \( t_2 \) is \( t_2^* \to 0.98 \). The optimal ordering quantity is \( OQ = 1.7 \). The minimum relevant cost is \( TC = 97.14 \).

2.6 Sensitivity Analysis

To know, how the optimal solution is affected by the values of parameters, we derive the sensitivity analysis for some parameters. The particular values of some parameters are increases or decreases by \(+10\%, -10\%\) and \(+20\%, -20\%\). After that, we derive the value of \( t_2 \) and \( TC \) with the help of increased or decreased values of \( H_c, D_c, S_c \) and \( O_c \). The result of the minimum relevant cost is existing in the following table 1.

**Table 1 Sensitivity Analysis**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Actual Values</th>
<th>+10%</th>
<th>+20%</th>
<th>-10%</th>
<th>-20%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>100</td>
<td>110</td>
<td>120</td>
<td>90</td>
<td>80</td>
</tr>
<tr>
<td>( H_c )</td>
<td>30</td>
<td>33</td>
<td>36</td>
<td>27</td>
<td>24</td>
</tr>
<tr>
<td>( D_c )</td>
<td>15</td>
<td>16.5</td>
<td>18</td>
<td>13.5</td>
<td>12</td>
</tr>
<tr>
<td>( S_c )</td>
<td>25</td>
<td>27.5</td>
<td>30</td>
<td>22.5</td>
<td>20</td>
</tr>
<tr>
<td>( O_c )</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>( TC )</td>
<td>112.89</td>
<td>124.5</td>
<td>135.8</td>
<td>101.85</td>
<td>90.5</td>
</tr>
</tbody>
</table>

From the result of above table, we observe that total relevant cost and ordering quantity is affected by holding cost, deterioration cost, shortage cost and opportunity cost.
2.7 Conclusion

In this chapter, an optimal inventory model developed for deteriorating item with exponential demand rate. This type of demand has an improved version of time unstable marketplace. This type of demand is fairly suitable for seasonal goods like festival goods, ice cream in summer etc. Shortages are permitted in the inventory system and are partially backlogged. The backlogging rate is depends on the lead time for the next replenishment. We have derived the most favorable order quantity model by minimizing the total inventory cost. To explain the model a numerical illustration and sensitivity analysis has been carried out to study the outcome of parameters on variables and the total inventory cost of this model.

In the third chapter, we developed an inventory model where rate of deterioration is time dependent, which is two parameters weibull distribution deterioration.