Chapter 4

Weakly $\gamma$-endowed graphs

The aim of this chapter is to study a concept called Weakly $\gamma$-endowed graphs. In $k$-extendable graphs, the condition is that every independent set of cardinality $k$ is part of a maximum independent set. If there exists a maximal independent set of cardinality $k$ which is not a maximum independent set, then the graph is not $k$-extendable. Hence $k$-extendability excludes the presence of maximal independent sets of cardinality $k$. This concept can be weakened and one may stipulate that a graph is weakly $k$-extendable if every non-maximal independent set of cardinality $k$ is a part of a maximum independent set. In such graphs, maximal independent sets of cardinality $k$ may exist. A similar concept in the case of domination called weakly $\gamma$-endowed sets is introduced in this chapter.
4.1 Introduction

A graph $G$ is $\gamma$-endowed if every dominating set of cardinality $k$ contains a minimum dominating set of the graph. This condition may be relaxed in the sense that minimal dominating set of cardinality $k$ may be exempted. This led to the notion of weakly $\gamma$-endowed sets. This chapter is devoted to the study of weakly $\gamma$-endowed graphs.

4.2 Weakly $\gamma$-endowed graphs

Definition 4.2.1 Let $G$ be a graph. Let $k$ be any positive integer, $k \geq \gamma(G)$. $G$ is said to be weakly $k \gamma$-endowed if every non-minimal dominating set of cardinality $k$ contains a minimum dominating set.

Example 4.2.2 $P_5$ is weakly $k \gamma$-endowed for all $k$, $2 \leq k \leq 5$.

Example 4.2.3 $D_{3,2}$ is not weakly $k \gamma$-endowed when $k = 4, 5, 6$ and weakly $k \gamma$-endowed for $k = 2, 3$ and 7.
\{u_1, u_4, u_6, u_7\} is a non-minimal dominating set of \(G\) which does not contain the unique minimum dominating set of \(G\). \(\{u_1, u_2, u_4, u_6, u_7\}, \{u_1, u_2, u_3, u_4, u_6, u_7\}\) are non-minimal dominating sets of \(G\) which do not contain the unique minimum dominating set of \(G\). 

**Example 4.2.4**

Let \(G = \)

\[ \gamma(G) = 2, \text{that is } \{u_2, u_3\} \text{ is the unique minimum dominating set of } G. \{u_1, u_3, u_5\}, \{u_1, u_2, u_5, u_6\}, \{u_1, u_2, u_4, u_5, u_6\}\text{ are non-minimal dominating set of } G \text{ not containing any minimum dominating set of } G. \text{ Therefore, } G \text{ is weakly trivially } \gamma\text{-endowed. Note that } G \text{ has a unique minimum dominating set and } \{u_4, u_5, u_6\} \text{ is a minimal dominating set of } G \text{ not containing any minimum dominating set of } G. \text{ } G \text{ is trivially } \gamma\text{-endowed.} \]

**Remark 4.2.5** Any weakly trivially \(\gamma\)-endowed graph is is trivially \(\gamma\)-endowed.

**Observation 4.2.6** Let \(G\) be a graph. Then \(G\) is \((\gamma + 1)\) weakly \(\gamma\)-endowed.
Proof:

Let $D$ be a non-minimal dominating set of $G$ of cardinality $\gamma(G)+1$. since $D$ is non-minimal, $D$ contains a minimal dominating set of $G$ of cardinality less than $(\gamma + 1)$. Therefore, $D$ contains a minimum dominating set of $G$. Hence the observation.

Definition 4.2.7 A graph $G$ is said to be trivially weakly $\gamma$-endowed if for any positive integer $k$ such that $(\gamma + 2) \leq k \leq n-1$, any non-minimal dominating set of cardinality $k$ does not contain a minimum dominating set of $G$.

Example 4.2.8

$G$ is trivially $\gamma$-endowed since $\{2, 3, 5, 6\}$ and $\{1, 2, 3, 5, 6\}$ are non minimal dominating sets of $G$ which do not contain any minimum dominating set of $G$. Hence $G$ is also trivially $\gamma$-endowed.
**Remark 4.2.9** Suppose $G$ is trivially weakly $\gamma$-endowed. If $G$ has a minimal dominating set of cardinality $\gamma + 1$, then $G$ is trivially $\gamma$-endowed.

\[ \blacksquare \]

**Remark 4.2.10** A trivially $\gamma$-endowed graph need not be trivially weakly $\gamma$-endowed. For: Let $G = P_3$. $G$ is trivially $\gamma$-endowed but not trivially weakly $\gamma$-endowed.

\[ \blacksquare \]

**Remark 4.2.11** Let $G$ be a graph and $k$ be a positive integer and not $\gamma(G) \leq n - 1$. Suppose $G$ is $k\gamma$-endowed. Then $G$ is weakly $k\gamma$-endowed. But the converse is not true. $P_5$ is weakly $3\gamma$-endowed but not $3\gamma$-endowed.

\[ \blacksquare \]

**Lemma 4.2.12** Let $G$ be a graph without isolates which is weakly $k\gamma$-endowed for all $k$, $\gamma \leq k \leq n - 2$ and $G$ be not weakly $(n - 1)\gamma$-endowed. Then $G$ has at least two minimum dominating sets.

**Proof:**

Suppose $G$ satisfies the hypothesis and $G$ has a unique minimum dominating set say $D$. Suppose there exists a minimal dominating set $S$ of $G$ such that $|S| = t < n - 2$. Then $S \cup \{u\}$ for all $u \in V - S$ is a non-minimal dominating set of $G$ and hence contain a minimum dominating set of $G$. Since $G$
has a unique minimum dominating set of \( G \), \( u \in D \). Therefore, \( V - S \subseteq D \).

Since \( S \) is a minimal dominating set of \( G \) and \( G \) has no isolates, \( V - S \) is a dominating set of \( G \). Therefore, \( V - S = D \). Clearly \( D \subseteq S \cup \{u\} \).

Therefore \( V - S = S \cup \{u\} \). Therefore, \( V - S = \{u\} \). Therefore \( D = \{u\} \).

Therefore, \( |S| = n - 1 \), a contradiction (since \( |S| < n - 2 \)). Therefore any minimal dominating set \( S \) of \( G \) is of cardinality \( \geq n - 2 \). If \( |S| = n - 1 \), then \( G \) is a star and hence \( G \) is weakly \( k \)-\( \gamma \)-endowed for all \( k \), \( \gamma(G) \leq k \leq n \), a contradiction. Therefore, \( |S| = n - 2 \). Since \( S \cup \{u\} \) for every \( u \in V - S \) is a non-minimal dominating set of \( G \), \( S \cup \{u\} \) contain \( D \) for every \( u \) in \( V - S \).

Therefore, \( u \in D \) for \( u \in V - S \). Therefore, \( V - S \subseteq D \). Since \( S \) is a minimal dominating set of \( G \) and \( G \) has no isolates \( V - S \) is a dominating set of \( G \). Therefore, \( V - S = D \). Therefore, \( V - S = \{u, v\} \) (say).

Suppose two vertices say \( w_1, w_2 \) in \( S \) are adjacent. Then \( w_1 \) and \( w_2 \) have private neighbours in \( V - S \). Therefore, \( u \) is adjacent with \( w_1 \) only and \( v \) is adjacent with \( w_2 \) only in \( S \). But \( \{u, v\} \) is a dominating set of \( G \). Therefore, \( |V - S| = 2 \) and \( G \) is \( P_4 \), a contradiction. Therefore, \( G \) contains more than one dominating set of \( G \). Suppose no two vertices in \( S \) are adjacent. Since \( \gamma(G) = 2 \) and \( |S| \geq \gamma(G) \), we get that \( |S| \geq 2 \). If \( u \) is adjacent with only one vertex of \( S \) then \( G \) contain more than one dominating set. If \( u \) is adjacent with at least two vertices of \( S \) and \( v \) is adjacent with at least two vertices of \( S \) then \( G \) has a non-minimal dominating set of cardinality \( \leq n - 2 \) and not containing \( D \),
a contradiction. Therefore, \( G \) has at least two minimum dominating sets.

Illustration 4.2.13

Let

\[
G =
\]

\[
S = \{1, 2, 3, 4\}, D = \{u, v\}, \{1, 2, 3, v\} \text{ is a non-minimal dominating set}
\]

of cardinality 4 = \( n - 2 \) and this does not contain the unique minimum dominating set \( D \) of \( G \).
**Example 4.2.14**

There exists a graph which is weakly $k$ $\gamma$-endowed for all $k$, $\gamma(G) \leq k \leq n - 2$, and which is not weakly $(n - 1)\gamma$-endowed. For: consider

$$G = \begin{array}{c}
\text{u_1} \\
\text{u_2} \quad \text{u_3} \quad \text{u_n} \\
\text{u_{n+1}}
\end{array}$$

$G$ is not weakly $(n - 1)\gamma$-endowed since $\{u_2, u_3, \ldots, u_n, u_{n+1}\}$ is a non-minimal dominating set of $G$ but it does not contain any minimum dominating set of $G$. Clearly $G$ is $k$ $\gamma$-endowed for all $k$, $\gamma(G) \leq k \leq n - 2$ and hence weakly $k$ $\gamma$-endowed all $k$, $\gamma(G) \leq k \leq n - 2$. $G$ has two minimum dominating sets namely $\{u_1, u_n\}, \{u_1, u_{n+1}\}$. $\blacksquare$

**Theorem 4.2.15** Let $G$ be a graph without isolates which is weakly $k$ $\gamma$-endowed for all $k$, $\gamma(G) \leq k \leq n - 2$ and $G$ be not weakly $(n - 1)\gamma$-endowed. Then $G$ has at least one $\gamma$-fixed vertex.

**Proof:**

Suppose there exist two $\gamma$-fixed vertices say $u$, $v$
Case(i) \(\gamma(G) \leq n - 3\). let \(S = V(G) - \{u,v\}\). If \(u\) and \(v\) are not adjacent with any vertex of \(S\), then \(u\) and \(v\) are either isolates of \(G\) or \(u\) and \(v\) are adjacent and \(\{u,v\}\) is a component of \(G\). Since \(G\) has no isolates and \(u\), \(v\) are \(\gamma\)-fixed vertices we get, a contradiction. Therefore, \(u\) or \(v\) is adjacent with some vertex of \(S\). Suppose \(u\) is adjacent with a vertex of \(S\) and \(v\) is not adjacent with any vertex of \(S\), \(v\) is adjacent with \(u\) (since \(G\) has no isolates). Therefore, for any minimum dominating set \(D\), \(D - \{v\}\) is a dominating set, a contradiction. Therefore, both \(u\) and \(v\) are adjacent with at least one vertex of \(S\). Therefore \(S\) is a dominating set of \(G\). if \(S\) is a non-minimal dominating set of \(G\), then \(S\) contains a minimum dominating set of \(G\) and hence \(u,v \in S\), a contradiction. Therefore, \(S\) is a minimal dominating set of \(G\).

Suppose there exist two vertices say \(x, y\) in \(S\) which are adjacent. Since \(S\) is minimal, \(x\) and \(y\) must have private neighbours in \(V - S\). Since \(V - S = \{u,v\}\), \(u\) is a private neighbour of \(x\) say and \(v\) is a private neighbour of \(y\). Therefore, \(S - \{x,y\}\) is independent and also no vertex of \(S - \{x,y\}\) is adjacent with \(u\) as well as \(v\). Therefore, every vertex in \(S - \{x,y\}\) is an isolate of \(G\), a contradiction. Therefore \(S = \{x,y\}\). Therefore, \(G = P_4\), a contradiction since \(G\) is not weakly \((n - 1)\gamma\)-endowed. Therefore, no two vertices of \(S\) are adjacent. Therefore \(G\) is the union of two stars or a double star which may be \(P_4\). That is either \(G\) has a unique minimum dominating
set which gives a contradiction to lemma or $G$ is weakly $k$ $\gamma$-endowed for all $k$, $\gamma \leq k \leq n$, a contradiction.

**Case(ii):** $\gamma(G) = n - 2$. Let $S$ be a $\gamma$-set of $G$. Then $|V - S| = 2$. Let $V - S = \{x, y\}$

**Subcase(i):** Suppose $\Gamma(G) = n - 1$. Let $D$ be a $\Gamma$-set of $G$. If two vertices of $D$ are adjacent, then they must have private neighbours in $V - D$. Since $V - D$ is a singleton, this is not possible. Therefore, $D$ is independent. Let $V - D = \{x\}$. If $x$ is not adjacent with a vertex say $w$ of $D$, then $w$ is an isolates of $G$, a contradiction. Therefore, $x$ is adjacent with every vertex of $D$. Therefore $G$ is a star, a contradiction, since $G$ is not weakly $(n - 1)$ $\gamma$-endowed. Therefore $\Gamma(G) = n - 2$.

**Subcase(ii):** $\Gamma(G) = n - 2$. In this case $\gamma(G) = \Gamma(G)$. Therefore, $G$ is well dominated. Therefore, $G$ is $k$ $\gamma$-endowed for all $k$, $\gamma(G) \leq k \leq n$. Therefore, $G$ is weakly $k$ $\gamma$-endowed for all $k$, $\gamma(G) \leq k \leq n$, a contradiction.

**Case(iii):** $\gamma(G) = n - 1$. In this case $G$ has isolates or $G = K_2$, a contradiction.

**Case(iv):** $\gamma(G) = n$. In this case $G = \overline{K_n}$. Since $G$ has no isolates, $n = 1$. Therefore $G$ is not weakly $k$ $\gamma$-endowed for all $k$, $\gamma(G) \leq k \leq n$, a contradiction. Hence there exists at most one $\gamma$-fixed vertex in $G$. 

$\blacksquare$
4.3 Weakly $\gamma$-endowed graphs except for specific values of $k$.

**Theorem 4.3.1** Let $G$ be a graph without isolates which is weakly $k$ $\gamma$-endowed for all $k$ , $\gamma(G) \leq k \leq n-2$ and $G$ be not weakly $(n - 1 ) \gamma$-endowed. Then $G$ is one of the following

(i) $G = K_2 \cup K_{1,t} (t \geq 2)$
(ii) $G$ is obtained from $K_3$ by attaching 1 or more pendant vertices with exactly one vertex of $K_3$
(iii) $G$ is obtained from a star $K_{1,n}$ by attaching exactly one pendant vertex at exactly one pendant vertex of $K_{1,n}$.

The converse is also true.

**Proof :**

By hypothesis, there exists a non-minimal dominating set $S$ of $G$ of cardinality $(n-1)$ such that any minimal dominating set say $T \subset S$ is not a $\gamma$-set of $G$. Therefore, $\gamma \neq \Gamma$. Suppose $|S - T| \geq 2$. Let $u \in S - T$. $T \cup \{u\}$ is a non-minimal dominating set of $G$ of cardinality less than $(n - 1)$. Therefore, $T \cup \{u\}$ contains a $\gamma$-set of $G$. Therefore, $S$ contains a $\gamma$-set of $G$, a contradiction. Therefore $|S - T| = 1$. Therefore $|T| = n - 2$. Therefore $\Gamma(G) \geq n - 2$. Suppose $\Gamma(G) = n - 1$. Let $D$ be a $\Gamma(G)$-set of $G$. Let $V - D = \{u\}$. Suppose two vertices say $x$, $y$ of $D$ are adjacent. Since $D$ is minimal, $x$, $y$ must have a private neighbour in $V - D$ which is not
possible since $V - D$ is a singleton. Therefore, $D$ is independent. If $u$ is not adjacent to a vertex say $z \in D$, then $z$ is an isolates of $G$, a contradiction. Therefore $u$ is adjacent with any vertex of $D$. Therefore $G$ is a star. Therefore $G$ is weakly $k \gamma$-endowed for any $k$, $\gamma(G) \leq k \leq n$, a contradiction. Therefore $\Gamma(G) = n - 2$. Let $D_1$ be a $\Gamma$-set of $G$. Let $V - D_1 = \{u, v\}$.

Suppose two vertices say $x, y$ of $D_1$ are adjacent. Since $D_1$ is minimal $x$ and $y$ must each have a private neighbour in $V - D_1$. Without loss of generality, let $u$ be a private neighbour of $x$ and $v$ be a private neighbour of $y$. If $z \in D, z \neq x, y$ then $z$ is not adjacent with $u$ and $v$. If $z$ is adjacent with any other vertex of $D_1$, then $z$ must have a private neighbour in $V - D_1$ which is not possible. Therefore $z$ is an isolate of $<D_1>$ and hence of $G$, a contradiction. Therefore, $D_1 = \{x, y\}$ and hence $G = P_4$. But $P_4$ is $k \gamma$-endowed for all $k$, $\gamma(G) \leq k \leq n$, a contradiction. Therefore $D_1$ is independent.

Any vertex of $D_1$ is adjacent with either $u$ or $v$. Let $T_1 = N(u) \cap D_1$ and $T_2 = N(v) \cap D_1$.

**Case(i):** $u$ and $v$ are not adjacent.

**Subcase(i):** $T_1 \cap T_2 = \phi$. If cardinality of $T_1$ or $T_2$ is one, then $G$ is the union of $K_2$ and $K_{1,t}$ where $t \geq 1$. If $t = 1$ then $G = 2K_2$ which does not satisfy the hypothesis. If $t \geq 2$ then $G = K_2 \cup K_{1,t}(t \geq 2)$ which satisfy the hypothesis. If $|T_1|$ and $|T_2|$ are $\geq 2$, than $\{u_1, z_1\} \cup T_2$ where $z_1 \in T_1$ is a non-minimal dominating set of cardinality $\leq n - 2$ and this does not contain
the unique minimum dominating set \(|u, v\) of \(G\), a contradiction.

**Subcase(ii):** \(T_1 \cap T_2 \neq \emptyset\)

**Subsubcase(i):** \(T_1 = T_2\). In this case \(u\) and \(v\) are adjacent with every vertex of \(D_1\) and hence \(G\) is a complete bipartite graph. Therefore, any non-minimal dominating set of cardinality \((n - 1)\) contains a minimum dominating set of \(G\), a contradiction.

**Subsubcase(ii):** \(T_1 \subset T_2\) (similar proof if \(T_2 \subset T_1\)). In this case \(v\) is \(\gamma\)-fixed vertex (if \(|T_2 - T_1| \geq 2\)). If \(|T_2 - T_1| = 1\) then every non-minimal dominating set of cardinality \((n - 1)\) contains a minimum dominating set of \(G\) and in the latter case there exists a non-minimal dominating set of cardinality \((n - 2)\) which does not contain a minimum dominating set of \(G\), a contradiction.

**Subsubcase(iii):** \(T_1 \not\subset T_2\) and \(T_2 \not\subset T_1\). If \(|T_1 \cap T_2| = 1\) and \(|T_1 - T_2| = |T_2 - T_1| = 1\), then \(G = P_5\). In this case, every non-minimal dominating set of cardinality 4 contains a minimum dominating set of \(G\), a contradiction.

If \(|T_1 \cap T_2| = 1\) and \(|T_1 - T_2| = 1\), then there exists a non-minimal dominating set \(\{u\} \cup T_2\) of cardinality \((n - 2)\) which does not contain a minimum dominating set of \(G\), a contradiction.

If \(|T_1 \cap T_2| = 1\) and \(|T_1 - T_2| \geq 2\) then \(V(G) - \{v, w\}\) where \(w \in T_1 - T_2\) is a non-minimal dominating set of cardinality \((n - 2)\) which does not contain a minimum dominating set of \(G\), a contradiction. If \(|T_1 \cap T_2| \geq 2\) and \(|T_1 - T_2| = |T_2 - T_1| = 1\), then every non-minimal dominating set of cardinality
(n - 1) contains a minimum dominating set of $G$, a contradiction. If $|T_1 \cap T_2| \geq 2$ and $|T_1 - T_2| = 1$, $|T_2 - T_1| \geq 2$, then $\{u\} \cup T_2$ is a non-minimal dominating set of cardinality $(n - 2)$ which does not contain a minimum dominating set of $G$, a contradiction.

If $|T_1 \cap T_2|, |T_1 - T_2|, |T_2 - T_1| \geq 2$, then $V(G) - \{v, w\}$ where $w \in T_1 - T_2$ is a non-minimal dominating set of cardinality $(n - 2)$ which does not contain a minimum dominating set of $G$, a contradiction.

Case(ii): $u$ and $v$ are adjacent.

Subcase(i): $T_1 \cap T_2 = \phi$.

If $|T_1| = |T_2| = 1$ then $G = P_4$ which is weakly well dominated, a contradiction. If $|T_1| = 1, |T_2| \geq 2$ or $|T_2| = 1, |T_1| \geq 2$, then every non-minimal dominating set of cardinality $(n - 1)$ contains a minimum dominating set of $G$, a contradiction.

If $|T_1|, |T_2| \geq 2$, then there exists a non-minimal dominating set of cardinality $(n - 2)$ which does not contain a minimum dominating set of $G$, a contradiction.

Subcase(ii): $T_1 \cap T_2 \neq \phi$

Subsubcase(i): $T_1 = T_2$ If $T_1 = T_2$ then $G = K_3$ which is well dominating, a contradiction. $T_1 = T_2 \geq 2$ then $\gamma(G) = 1$ and any non-minimal dominating set of cardinality $(n - 1)$ contains a minimum dominating set of $G$, a contradiction.
Subsubcase (ii): $T_1 \subseteq T_2$ (similar proof if $T_2 \subseteq T_1$). In this case $\{v\}$ is a dominating set of $G$. If $T_1 = 1$ and $|T_2 - T_1| \geq 1$ then $G$ is obtained from $K_3$ by attaching a pendant vertex to exactly one vertex of $K_3$. In this case $G$ satisfies the hypothesis of the theorem. If $|T_1| \geq 2$, $|T_2 - T_1| \geq 1$, then $V(G) - \{v,w\}$ where $w \in T_1$ is a non-minimal dominating set of cardinality $(n - 2)$ which does not contain a minimum dominating set of $G$, a contradiction.

Subsubcase (iii): Proof is same as in subsubcase (iii) of subcase (ii) of case (i) conversely, if $G$ is any one of the graph mentioned in the statement of the theorem then clearly $G$ satisfies the hypothesis of the theorem. Hence the converse.

Illustration 4.3.2

$\gamma(G) = 2$. Any non-minimal dominating set of cardinality 3 and 4 contain a minimum dominating set of $G$. Therefore $G$ is weakly $k$-endowed for all $k$, $\gamma \leq k \leq 4$ and $\{u_1, u_2, v_1, v_2, v_3\}$ is a non-minimal dominating set which
does not contain any $\gamma$-set of $G$. Therefore, $G$ is not weakly 5 $\gamma$- endowed

\[\square\]

Illustration 4.3.3

\[
\begin{array}{c}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5
\end{array}
\]

$\gamma(G) = 1$. Any non-minimal dominating set of cardinality 2 contains a minimum dominating set of $G$. Therefore $G$ is weakly $k \gamma$- endowed for all $k$, $k = 1, 2$ and \{v_2, v_3, v_4\} is a non-minimal dominating set does not contain any $\gamma$-set of $G$. Therefore $G$ is not weakly 3 $\gamma$- endowed.

\[\square\]

Illustration 4.3.4

\[
\begin{array}{c}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5
\end{array}
\]

$\gamma(G) = 1$. Any non-minimal dominating set of cardinality 2 and 3 contain a minimum dominating set of $G$. Therefore, $G$ is weakly $k \gamma$- endowed for all $k$, $k = 1, 2, 3$ and \{v_2, v_3, v_4, v_5\} is a non-minimal dominating set does not contain any $\gamma$-set of $G$. Therefore $G$ is not weakly 4 $\gamma$- endowed.

\[\square\]
Illustration 4.3.5

\[ \gamma \)-sets are \( \{v_1, v_5\}, \{v_1, v_6\} \). Any non-minimal dominating set of cardinality 2, 3 and 4 contain a minimum dominating set of \( G \). Therefore, \( G \) is weakly \( k \) \( \gamma \)-endowed for all \( k \), \( k = 2, 3, 4 \) and \( \{v_2, v_3, v_4, v_5, v_6\} \) is a non-minimal dominating set which does not contain any \( \gamma \)-set of \( G \). Therefore, \( G \) is not weakly 5 \( \gamma \)-endowed.

Remark 4.3.6 Let \( G \) be a connected graph. \( G \) is weakly \( k \) \( \gamma \)-endowed for all \( k \), \( \gamma(G) \leq k \leq n - 2 \) and \( G \) is weakly \( k \) \( \gamma \)-endowed for \( k = n - 1 \) if and only if \( G = K_3 \) by attaching one pendant vertex with exactly one vertex of \( K_3 \) and \( G = K_{1,t} \) by attaching one pendant vertex with exactly one vertex of \( K_{1,t} \).

Theorem 4.3.7 There exists no graph \( G \) such that \( G \) is not weakly \( k \) \( \gamma \)-endowed exactly for \( k = n - 2 \).
Proof:

By hypothesis, there exists a non-minimal dominating set $S$ of $G$ of cardinality $(n-2)$ such that any minimal dominating set say $T \subset S$ is not a $\gamma$-set of $G$. Therefore, $\gamma(G) \neq \Gamma(G)$. Suppose $|S - T| \geq 2$. Let $u \in S - T$. $T \cup \{u\}$ is a non-minimal dominating set of $G$ of cardinality less than $(n-2)$. Therefore, $T \cup \{u\}$ contains a minimum dominating set of $G$ and hence $S$ contains a $\gamma$-set of $G$, a contradiction. Therefore $|S - T| = 1$. Therefore $|T| = n - 3$. Therefore $\Gamma(G) \geq n - 3$. Suppose $\Gamma(G) = n - 1$. Let $D$ be a $\Gamma(G)$-set of $G$. Let $V - D = \{v\}$. Suppose two vertices say $x$, $y$ of $D$ are adjacent. Since $D$ is minimal $x$ and $y$ must have a private neighbour in $V - D$ which is not possible since $V - D$ is a singleton. Therefore, $D$ is independent. If $v$ is not adjacent to a vertex say $z \in D$, then $z$ is an isolates of $G$, a contradiction. Therefore $v$ is adjacent with every vertex of $D$ and hence $G$ is a star.

Therefore $G$ is weakly $k$ $\gamma$-endowed for every $k$, $\gamma(G) \leq k \leq n$, a contradiction. Therefore, $\Gamma(G) \leq n - 2$. Suppose $\Gamma(G) = n - 2$. Let $D_1$ be a $\Gamma$-set of $G$. Let $V - D_1 = \{u, v\}$. Suppose two vertices say $x$, $y$ of $D$ are adjacent. Since $D_1$ is minimal $x$ and $y$ must each have a private neighbour in $V - D_1$. Without loss of generality, let $u$ be a private neighbour of $x$ and $v$ be a private neighbour of $y$ with respect to $D_1$. If $z \in D_1$, then $z$ is not adjacent with any vertex of $D_1$ as well as $V - D_1$. That is $z$ is an isolates of $G$, a contradiction.
Therefore, \( D_1 = \{x, y\} \) and hence \( G = P_4 \). But \( P_4 \) is \( k \) \( \gamma \)-endowed for all \( k \), \( \gamma(G) \leq k \leq n \), a contradiction. Therefore, \( D_1 \) is independent. Any vertex of \( D_1 \) is adjacent with either \( u \) or \( v \). Let \( T_1 = N(u) \cap D_1 \) and \( T_2 = N(v) \cap D_1 \).

**Case(i):** \( u \) and \( v \) are not adjacent.

**Subcase(i):** \( T_1 \cap T_2 = \emptyset \). If cardinality of \( T_1 \) or \( T_2 \) is one, then \( G \) is the union of \( K_2 \) and \( K_{1,t} \) where \( t \geq 1 \). If \( t = 1 \) then \( G = 2K_2 \) which does not satisfy the hypothesis. Therefore, \( t \geq 2 \). That is, \( G = K_2 \cup K_{1,t} (t \geq 2) \) which does not satisfy the hypothesis. (That is any non-minimal dominating set of \( G \) of cardinality \( n - 2 \) contains a minimum dominating set of \( G \).)

If \( |T_1| \) and \( |T_2| \) are \( \geq 2 \), then \( T_2 \cup \{u\} \) is a non-minimal dominating set of \( G \) of cardinality \( \leq n - 3 \). Therefore \( T_2 \cup \{u\} \) contains a unique minimum dominating set of \( G \), a contradiction, since \( \{u, v\} \) is the a unique minimum dominating set of \( G \) and \( v \notin T_2 \cup \{u\} \).

**Subcase(ii):** \( T_1 \cap T_2 \neq \emptyset \)

**Subsubcase(i):** \( T_1 = T_2 \). In this case \( u \) and \( v \) are adjacent with every vertex of \( D_1 \) and hence \( G \) is a complete bipartite graph which does not satisfies the hypothesis, a contradiction.

**Subsubcase(ii):** \( T_1 \subset T_2 \) (similar proof if \( T_2 \subset T_1 \)). In this case \( v \) is a \( \gamma \)-fixed vertex (if \( |T_2 - T_1| \geq 2 \)). If \( |T_2 - T_1| = 1 \), then \( G \) does not satisfy the hypothesis (since \( D_1 \cup \{u\} \) is a non-minimal dominating set of \( G \) of cardinality \( n - 1 \) which does not contain any minimum dominating set of
If $|T_2 - T_1| \geq 2$, then $V(G) - \{v\}$ is a non-minimal dominating set of $G$ of cardinality $(n - 1)$ which does not contain any minimum dominating set of $G$, a contradiction.

**Subsubcase (iii):** $T_1 \nsubseteq T_2$ and $T_2 \nsubseteq T_1$. If $|T_1 \cap T_2| = 1$ and $|T_1 - T_2| = |T_2 - T_1| = 1$, then $G = P_5$. In this case, every non-minimal dominating set of cardinality 3 contains a minimum dominating set of $P_5$, a contradiction to the hypothesis. If $|T_1 \cap T_2| = 1$ and $|T_1 - T_2| = 1, |T_2 - T_1| \geq 2$, then $V(G) - \{v\}$ is a non-minimal dominating set of $G$ of cardinality $(n - 1)$ which does not contain a minimum dominating set of $G$, a contradiction to the hypothesis. If $|T_1 \cap T_2| = 1$ and $|T_1 - T_2| \geq 2$ and $|T_2 - T_1| \geq 2$ then $V(G) - \{v\}$ is a non-minimal dominating set of cardinality $(n - 1)$ which does not contain a minimum dominating set of $G$, a contradiction to the hypothesis.

**Subsubcase (iv):**

If $|T_1 \cap T_2| \geq 2$ and $|T_1 - T_2| = |T_2 - T_1| = 1$, then any non-minimal dominating set of $G$ of cardinality $(n - 2)$ contains a minimum dominating set of $G$, a contradiction. Let $|T_1 \cap T_2| \geq 2$ and $|T_1 - T_2| = 1$ and $|T_2 - T_1| \geq 2$, then $V(G) - \{v\}$ is a non-minimal dominating set of $G$ of cardinality $(n - 1)$ which does not contain a minimum dominating set of $G$, a contradiction. Let $T_1 \cap T_2 \geq 2, |T_1 - T_2| \geq 2$ and $|T_2 - T_1| \geq 2$. Then also we get, a contradiction.
Case(ii): \( u \) and \( v \) are adjacent.

Subcase(i): \( T_1 \cap T_2 = \emptyset \).

If \( |T_1| = |T_2| = 1 \) then \( G = P_4 \) which is weakly well dominated, a contradiction.

If \( |T_1| = 1 \), \( |T_2| \geq 2 \) or \( |T_2| = 1 \), \( |T_1| \geq 2 \), then every non-minimal dominating set of cardinality \( (n - 2) \) contains a minimum dominating set of \( G \), a contradiction.

If \( |T_1|, |T_2| \geq 2 \), then \( V(G) - \{v\} \) is a non-minimal dominating set of cardinality \( (n - 1) \) which does not contain a minimum dominating set of \( G \), a contradiction.

Subcase(ii): \( T_1 \cap T_2 = \emptyset \)

Subsubcase(i): \( T_1 = T_2 \). If \( T_1 = T_2 \) then \( G = K_3 \) which is well dominated, a contradiction. If \( T_1 = T_2 \geq 2 \) then \( \gamma(G) = 1 \) and any non-minimal dominating set of cardinality \( (n - 2) \) contains a minimum dominating set of \( G \), a contradiction.

Subsubcase(ii): \( T_1 \subsetneq T_2 \) (similar proof if \( T_2 \subsetneq T_1 \)). In this case \( \{v\} \) is a dominating set of \( G \). If \( |T_1| = 1 \), \( |T_2 - T_1| \geq 1 \) then \( G \) is obtained from \( K_3 \) by attaching a pendant vertex to exactly one vertex of \( K_3 \). any non-minimal dominating set of cardinality \( (n - 2) \) contains a minimum dominating set of \( G \), a contradiction. If \( |T_1| \geq 2 \), and \( |T_2 - T_1| \geq 1 \), then \( V(G) - \{v\} \) is a non-minimal dominating set of cardinality \( (n - 1) \) which does not contain the
minimum dominating set namely \{v\} of G, a contradiction.

Subsubcase (iii): \(T_1 \not\subset T_2\) and \(T_2 \not\subset T_1\). If \(|T_1 \cap T_2| = 1 \) and \(|T_1 - T_2| = |T_2 - T_1| = 1\), then every non-minimal dominating set of cardinality \(n - 2\) contains a minimum dominating set of G, a contradiction.

If \(|T_1 \cap T_2| = 1 = |T_1 - T_2| = 1\) and \(|T_2 - T_1| \geq 2\), then \(V(G) - \{v\}\) is a non-minimal dominating set of G of cardinality \((n - 1)\) which does not contain a minimum dominating set of G, a contradiction.

If \(|T_1 \cap T_2| = 1\) and \(|T_1 - T_2| \geq 2\) and \(|T_2 - T_1| \geq 2\), then \(V(G) - \{v\}\) is a non-minimal dominating set of G of cardinality \((n - 1)\) which does not contain any minimum dominating set of G, a contradiction.

If \(|T_1 \cap T_2| \geq 2\) and \(|T_1 - T_2| = |T_2 - T_1| = 1\), then any non-minimal dominating set of cardinality \((n - 2)\) contains a minimum dominating set of G, a contradiction.

If \(|T_1 \cap T_2| \geq 2\) and \(|T_1 - T_2| = |T_2 - T_1| \geq 2\), then \(V(G) - \{v\}\) is a non-minimal dominating set of G of cardinality \((n - 1)\) which does not contain a minimum dominating set of G, a contradiction.

If \(|T_1 \cap T_2|, |T_1 - T_2|, |T_2 - T_1| \geq 2\), then \(V(G) - \{v\}\) is a non-minimal dominating set of G of cardinality \((n - 1)\) which does not contain a minimum dominating set of G, a contradiction. Suppose \(\Gamma(G) = n - 3\). Let \(D\) be a \(\Gamma(G)\)-set of G. Let \(V - D = \{u, v, w\}\). Suppose \(D\) is independent. Let \(T_1 = N(u) \cap D, T_2 = N(v) \cap D\), and \(T_3 = N(w) \cap D\).
Subcase (ii) $T_1 \cap T_2 = \phi$

Subsubcase (i) $|T_1| = |T_2| = |T_3| = 1$. Then $G$ is

Here $\gamma(G) = \Gamma(G) = 3$. Therefore $G$ is well dominated, a contradiction.

$|T_1| = |T_2| = 1$

Case shown in figure 1 any non-minimal dominating set of cardinality 3 (that is cardinality $n-2$) contains a minimum dominating set of $G$, a contradiction. In both cases shown in figure 2 and figure 3, $\gamma(G) = \Gamma(G) = 2$. Therefore $G$ is well dominated, a contradiction.
In figure 1 \( \{u, v, w, 2\} \) is a non-minimal dominating set of cardinality 4 (that is \( n - 1 \)) which does not contain a \( \gamma \)-set of \( G \), a contradiction. In figure 2 the graph is weakly well dominated. Therefore it does not satisfies the hypothesis. The graph shown in figure 3 is well dominated, a contradiction.

The graph in figure 1 is weakly well dominated and in figure 2 is well dominated. Therefore a contradiction. The graph \( G \) in figure 3 \( \{w\} \) is the unique minimum dominating set of \( G \) and \( \{1, 2, u, v\} \) is a non-minimal dominating set of \( G \) which does not contain the \( \gamma \)-set of \( G \), a contradiction.
Subsubcase (ii) : One of $T_1, T_2$ and $T_3$ has cardinality $\geq 2$. In this case, one of the vertices say $u$, $v$, $w$ is $\gamma$-fixed and hence there exists a non-minimal dominating set which does not contain a minimum dominating set, a contradiction.

Subcase (ii) : $T_1 \cap T_2 \neq \emptyset$

Subsubcase (i) : $T_1 = T_2$ In this case also, one of the condition in the hypothesis is violated.

Subsubcase (ii) : $T_1 \subset T_2$ If $|T_2 - T_1| = 1$, then $G$ is one of the following

Suppose $u, v$ and $w$ are independent and $w$ is not adjacent with any vertex in $T_2$. Let $n \geq 7$. Then $V(G) - \{w\}$ is a non-minimal dominating set of $G$ which does not contain any minimum dominating set, since $w$ is $\gamma$-fixed, a contradiction.

Suppose $n = 6$. Then $G$ is well dominated, a contradiction.

If $u$ and $v$ are adjacent, then $w$ is $\gamma$-fixed if $n \geq 7$ and $v$ is $\gamma$-fixed if $n = 6$. Therefore, there exists a non-minimal dominating set of $G$ of cardinality

178
If $v$ and $w$ are adjacent, then $w$ is $\gamma$-fixed if $n \geq 7$ and if $n = 6$, every non-minimal dominating set of $G$ of cardinality $(n - 2)$ contains a minimum dominating set, a contradiction.

If both $u$ and $v$ and $v$ and $w$ are adjacent, then $w$ is $\gamma$-fixed if $n \geq 7$ and $v$ is $\gamma$-fixed if $n = 6$, a contradiction.

If $u$, $v$ and $w$ are mutually adjacent, then $w$ is $\gamma$-fixed if $n \geq 7$ and $v$ is $\gamma$-fixed if $n = 6$, a contradiction.

Suppose $u$, $v$ and $w$ independent. If $w$ is adjacent with $y$ in $T_2 - T_1$, then $G$ is $P_5$ in which any non-minimal dominating set of cardinality $(n - 2)$ contains a minimum dominating set of $G$, a contradiction.

If $w$ is adjacent with $x$ in $T_1$, then there exists a non-minimal dominating set of cardinality $(n - 1)$ which does not contain any minimum dominating set of $G$, a contradiction. If $w$ is adjacent with every vertex in $T_2$, then any non-minimal dominating of cardinality $(n - 2)$ contains a minimum dom-
inating set of $G$, a contradiction. If $w$ is adjacent with $z \notin T_2$, then $w$ is not adjacent with any vertex in $T_2$. That is $G$ is $P_6$ which contains a non-minimal dominating set of $G$ of cardinality $(n-1)$ which does not contain any minimum dominating set of $G$, a contradiction. Suppose $|T_2 - T_1| \geq 2$. Then $V(G) - \{v\}$ is a non-minimal dominating set of $G$ of cardinality $(n-1)$ which does not contain any minimum dominating set of $G$, if $w$ is not adjacent with at least one vertex in $T_2 - T_1$. If $w$ is adjacent with every vertex of $T_2 - T_1$, then $\{w, x\}$ is a unique minimum dominating set of $G$, if $T_1 = \{x\}$ and hence there exists a non-minimal dominating set of $G$ of cardinality $(n-1)$ which does not contain any $\gamma$-set of $G$, a contradiction.

If $|T_1| \geq 2$, then any non-minimal dominating set of $G$ of cardinality $(n-2)$ contains a $\gamma$-set of $G$, a contradiction.

Suppose $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$. Let $|T_1 \cap T_2| = 1$ and $|T_1 - T_2| = |T_2 - T_1| = 1$. If $w$ is adjacent with every vertex in $T_1 \cup T_2$ then there exists a unique minimum dominating set containing $w$. Hence $V(G) - \{w\}$ is a non-minimal dominating set of $G$ of cardinality $(n-1)$ which does not contain any $\gamma$-set of $G$, a contradiction. Similar arguments holds if $w$ is adjacent with every vertex in $T_2$ or every vertex in $T_1$ and hence we get a contradiction. If $w$ is adjacent only with the vertex in $T_2 - T_1$ or $T_1 - T_2$ then $G$ is $P_6$, a contradiction. If $w$ is adjacent only with the vertex in $T_1 \cap T_2$, then we get a subdivision of a star $K_{1,3}$ in which exactly two edges are subdivided.
Then any non-minimal dominating set of $G$ of cardinality $(n - 2)$ contains a minimum dominating set of $G$, a contradiction.

Suppose $w$ is adjacent with a vertex not in $T_1 \cup T_2$. Then $w$ is a $\gamma$-fixed vertex of $G$ and hence $V(G) - \{w\}$ is a non-minimal dominating set of $G$ of cardinality $(n - 1)$ which does not contain a minimum dominating set of $G$, a contradiction.

Let $|T_1 \cap T_2| = 1$, $|T_1 - T_2| = 1$ and $|T_2 - T_1| \geq 2$. If $w$ is adjacent with every vertex in $T_1 \cup T_2$, then there exists a unique minimum dominating set of $G$, a contradiction. If $w$ is not adjacent with one of the vertices in $T_2 - T_1$ or $T_1 - T_2$, then $G$ has a $\gamma$-fixed vertex, a contradiction. If $w$ is adjacent with a vertex not in $T_1 \cup T_2$ then $w$ is a $\gamma$-fixed vertex, a contradiction. Similar argument shows that

(a) If $|T_1 \cap T_2| = 1$, $|T_1 - T_2| \geq 2$, $|T_2 - T_1| \geq 2$ (or)

(b) If $|T_1 \cap T_2| \geq 2$, $|T_1 - T_2| = 1$, $|T_2 - T_1| \geq 2$ (or)

(c) If $|T_1 \cap T_2| \geq 2$, $|T_1 - T_2| \geq 2$ and $|T_2 - T_1| \geq 2$ we get, a contradiction.

Suppose $\Gamma(G) = n - 3$. Let $D$ be a $\Gamma(G)$-set of $G$. Let $V - D = \{u, v, w\}$.

Suppose $D$ is not independent.

Case 1: Let $x, y, z \in D$ be such that $<x, y, z>$ is connected. As $D$ is a minimal dominating set of $G$, $x$, $y$ and $z$ have private neighbour in $V - D$.

Since $V - D = \{u, v, w\}$, each of $u$, $v$ and $w$ must be a private neighbour of exactly one of $x$, $y$ and $z$. Let without loss of generality private neighbour of
$x$ with respect to $D$ be $u$, that of $y$ be $v$ and that of $z$ be $w$. Then any vertex in $D - \{x, y, z\}$ is an isolate of $G$, a contradiction. Therefore, $D = \{x, y, z\}$.

Therefore $G$ is $<\{x, y, z\}^+$ >

$G$ is one of

\[ (i) \quad (ii) \quad (iii) \quad (iv) \quad (v) \quad (vi) \]

\begin{center}
\begin{tabular}{ccc}
\hspace{0.2cm} x & y & z \\
\hspace{0.2cm} u & v & w \\
\end{tabular}
\begin{tabular}{ccc}
\hspace{0.2cm} x & y & z \\
\hspace{0.2cm} u & v & w \\
\end{tabular}
\begin{tabular}{ccc}
\hspace{0.2cm} x & y & z \\
\hspace{0.2cm} u & v & w \\
\end{tabular}
\begin{tabular}{ccc}
\hspace{0.2cm} x & y & z \\
\hspace{0.2cm} u & v & w \\
\end{tabular}
\end{center}
None of the above satisfies the hypothesis. [(i), (ii) and (iii) are well dominated, (iv) $\gamma(G) = 2, \Gamma(G) = 3$, There exists a non-minimal dominating set namely $V(G) - \{z\}$ which does not contain a minimum dominating set. (v) is similar to (iv). (vi) $\gamma(G) = 2, \Gamma(G) = 3$, every non-minimal dominating set of cardinality $(n - 2)$ contains a minimum dominating set (vii) and (viii) similar to (vi)].

Case (ii): Only two vertices of $D$ are adjacent. Let $x$ and $y$ be adjacent. Let $x$ have private neighbour $u$ with respect to $D$. Let $y$ have private neighbour $v$ with respect to $D$. Any point $z$ in $D - \{x, y\}$ is neither adjacent with $x$ or $y$ nor with any vertex in $D - \{x, y\}$. That is $D - \{x, y\}$ is independent and any vertex in $D - \{x, y\}$ is adjacent with $w$. Then $G$ is one of
Suppose \( w \) has at least two neighbours in \( D - \{ x, y \} \). Then \( w \) is \( \gamma \)-fixed and hence \( v(G) - \{ w \} \) is a non-minimal dominating set of cardinality \( (n - 1) \).
not containing any minimum dominating set of $G$, a contradiction. Suppose $w$ has exactly one neighbour in $D - \{x, y\}$ and $w$ is not adjacent with both $x$ and $y$. In this case $G = P_4 \cup K_2$ or $C_4 \cup K_2$ or $P_6$

$$
\begin{array}{c}
  x & y & z \\
  u & v & w \\
\end{array}
$$

(or)

$$
\begin{array}{c}
  x & y & z \\
  u & v & w \\
\end{array}
$$

where $G = P_4 \cup K_2$ or $C_4 \cup K_2$ then $G$ is well dominated. When $G = P_6, G - \{\text{supportvertex}\}$ is a non-minimal dominating set of cardinality $(n - 1)$ which does not contain a minimum dominating set of $G$, a contradiction.

when $G =

$$
\begin{array}{c}
  x & y & z \\
  u & v & w \\
\end{array}
$$

$\gamma(G) = 2$ and $\{x, w\}$ is the unique minimum dominating set of $G$. Therefore, there exists a dominating set of cardinality $(n - 1)$ which does not contain any minimum dominating set of $G$, a contradiction. Same holds for the next graph. Suppose $w$ has exactly one neighbour in $D - \{x, y\}$ and $w$ is
adjacent with at least one of \( x , y \). Then \( G \) is

\[(i)\] 
\[(ii)\] 

\[(iii)\] 
\[(iv)\] 

\[(v)\] 

\[(vi)\] 
\[(vii)\] 
\[(viii)\] 
\[(ix)\]

\((iii),(iv),(v),(vii)\) and \((viii)\) are well dominated. In \((i),(ii)\) and \((ix)\) there exists a non-minimal dominating set of cardinality \((n - 1)\) which does not contain minimum dominating set of \( G \) and in \((vi)\) every non-minimal dominating set of cardinality \((n - 2)\) contains a minimum dominating set of \( G \), a contradiction.

\[\square\]

**Theorem 4.3.8**

A graph \( G \) is not weakly \( k\gamma\)-endowed exactly for
\[ k = (n - 2) \text{and} (n - 1) \text{ iff } G \text{ is one of the following} \]

(i) \( K_{1,t_1} \cup K_{1,t_2}, \ t_1, t_2 \geq 2 \)

(ii) \( G \) is obtained from \( C_4 \) by attaching at least two pendant vertices with exactly one vertex of \( C_4 \)

(iii) \( G \) is obtained from a star \( K_{1,t}, \ t \geq 3 \) by attaching exactly one \( P_3 \) at exactly one pendant vertex of \( K_{1,t} \)

(iv) \( G \) is double star \( D_{r,s} \), with \( r,s = 2 \).

(v)

\( G \) is

\[ \begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} \]

\[ \begin{array}{c}
x_1 \ x_k \ x_{k+1} \ x_l \\
\end{array} \]
Proof: Arguing as in the theorem, $\Gamma(G) = n - 2$. Let $D_1$ be a $\Gamma$-set of $G$. Let $V - D_1 = \{u, v\}$. Suppose two vertices say $x$, $y$ of $D_1$ are adjacent. Since $D_1$ is minimal, say $x$ and $y$ must each have a private neighbour in $V - D_1$ without loss of generality, let $u$ be a private neighbour of $x$ and $v$ be a private neighbour of $y$. If $z \in D - \{x, y\}$, then $z$ is not adjacent with $u$ and $v$ since otherwise $z$ must have a private neighbour in $V - D_1$ which is not possible. Therefore, $z$ is an isolate of $< D_1 >$ and hence of $G$, a contradiction. Therefore, $D_1 = \{x, y\}$ and hence $G = P_4$ but $P_4$ is $k \gamma$-endowed for all $k$, $\gamma(G) \leq k \leq 4$, a contradiction. Therefore, $D_1$ is independent.

Case (i): $u$ and $v$ are not adjacent.
Subcase (i): \( T_1 \cap T_2 = \emptyset \)

If \(|T_1| \) or \(|T_2| \) is 1 then \( G \) is the union of \( K_2 \) and \( K_{1,t} \) which does not satisfies the hypothesis. If \(|T_1|, |T_2| \geq 2 \) then \( G \) is the union of two stars \( K_{1, t_1} \) and \( K_{1, t_2} \) where \( t_1, t_2 \geq 2 \) clearly \( G \) satisfies the hypothesis.

Subcase (i): \( T_1 \cap T_2 \neq \emptyset \)

Subsubcase (i): \( T_1 = T_2 \).

In this case, \( u \) and \( v \) are adjacent with every vertex of \( D_1 \) and hence \( G \) is a complete bipartite graph. Therefore, any non-minimal dominating set of \( G \), a contradiction.

Subsubcase (ii): \( T_1 \subseteq T_2 \) (similar proof if \( T_2 \subseteq T_1 \)). In this case \( v \) is a \( \gamma \)-fixed vertex (when \(|T_2 - T_1| \geq 2 \)). When \(|T_2 - T_1| = 1 \) every non-minimal dominating set of \( G \) of cardinality \((n - 1)\) contains a minimum dominating set of \( G \). If \(|T_1| \geq 3 \) and \(|T_2 - T_1| \geq 2 \), then there exists a non-minimal dominating set of cardinality \((n - 3)\) which does not contain any minimum dominating set, a contradiction. If \(|T_1| = 1 \) and \(|T_2 - T_1| \geq 2 \), then any non-minimal dominating set of cardinality \((n - 2)\) contains a minimum dominating set of \( G \), a contradiction. Therefore, \(|T_1| = 2 \) and \(|T_2 - T_1| \geq 2 \). Therefore \( G \) is \( C_4 \) with at least two pendant vertices attached with exactly one vertex of \( C_4 \).
Subsubcase (iii): $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$. If $|T_1 \cap T_2| = 1$ and $|T_1 - T_2| = |T_2 - T_1| = 1$, then $G = P_5$. In this case, every non-minimal dominating set of cardinality 4 contains a minimum dominating set of $G$, a contradiction. Suppose $|T_1 \cap T_2| = 1$ and $|T_1 - T_2| = 1$, $|T_2 - T_1| \geq 2$. In this case, $v$ is a $\gamma$-fixed vertex \{u,v\} and $T_2 \cup \{v\}$ are minimum dominating set of $G$. $V(G) - \{v\}$ is a non-minimal dominating set of $G$ of cardinality $(n - 1)$ which does not contain a minimum dominating set of $G$. $T_2 \cup \{u\}$ is a non-minimal dominating set of $G$ of cardinality $(n - 2)$ which does not contain a minimum dominating set of $G$. Any other non-minimal dominating set of $G$ of cardinality $\geq \gamma(G)$ and $\leq (n - 3)$ contains a minimum dominating set of $G$.

If $|T_1 \cap T_2| = 1$ and $|T_1 - T_2| \geq 2$ and $|T_2 - T_1| \geq 2$ then there exists a non-minimal dominating set of cardinality $(n - 3)$ which does not contain any minimum dominating set of $G$, a contradiction. If $|T_1 \cap T_2| \geq 2$ and $|T_1 - T_2| = |T_2 - T_1| = 1$, then every non-minimal dominating set of cardinality $(n - 1)$ contains a minimum dominating set of $G$, a contradiction. If $|T_1 \cap T_2| \geq 2$ and $|T_1 - T_2| = 1$, $|T_2 - T_1| \geq 2$, then there exists a non-
minimal dominating set of cardinality \((n - 3)\) which does not contain any minimum dominating set of \(G\), a contradiction.

If \(|T_1 \cap T_2|, |T_1 - T_2|, |T_2 - T_1| \geq 2\), then there exists a non-minimal dominating set of cardinality \((n - 3)\) which does not contain any minimum dominating set of \(G\), a contradiction.

**Case (ii):** \(u\) and \(v\) are adjacent.

**Subcase (i):** \(T_1 \cap T_2 = \emptyset\). If \(|T_1| = |T_2| = 1\) then \(G = P_4\) which is weakly well dominated, a contradiction.

If \(|T_1| = 1\), \(|T_2| \geq 2\) or \(|T_2| = 1\), \(|T_1| \geq 2\), then every non-minimal dominating set of cardinality \((n - 1)\) contains a minimum dominating set of \(G\), a contradiction.

Let \(|T_1|, |T_2| \geq 2\). If \(|T_1| = |T_2| = 2\), then \(G\) satisfies the hypothesis.

If \(|T_1| \geq 3\) or \(|T_2| \geq 3\) then there exists a non-minimal dominating set of cardinality \((n - 3)\) which does not contain any minimum dominating set of \(G\), a contradiction.

**Subcase (ii):** \(T_1 \cap T_2 \neq \emptyset\)

**Subsubcase (i):** \(T_1 = T_2\).
If \( T_1 = T_2 \) then \( G = K_3 \) which is well dominating, a contradiction.

If \( T_1 = T_2 \geq 2 \) then \( \gamma(G) = 1 \) and any non-minimal dominating set of cardinality \((n - 1)\) contains a minimum dominating set of \( G \), a contradiction.

**Subsubcase (ii):** \( T_1 \subsetneq T_2 \) (similar proof if \( T_2 \subsetneq T_1 \)). In this case \( \{v\} \) is a dominating set of \( G \). If \( T_1 = 1 \), \(|T_2 - T_1| \geq 1\) then \( G \) is obtained from \( K_3 \) by attaching at least one pendant vertex to exactly one pendant vertex of \( K_3 \). In this case any non-minimal dominating set of cardinality \((n - 2)\) contains a minimum dominating set of \( G \), a contradiction. Let \(|T_1| \geq 2 \) and \(|T_2 - T_1| \geq 1 \). Then \( G \) satisfies the hypothesis of the theorem.

![Diagram](image)

**Subsubcase (iii):** \( T_1 \not\subset T_2 \) and \( T_2 \not\subset T_1 \).

If \(|T_1 \cap T_2| = 1 = |T_1 - T_2| = |T_2 - T_1|\), then any non-minimal dominating set of cardinality \( 4 \) (=\( n - 1 \)) contains a minimum dominating set of \( G \), a contradiction.

If \(|T_1 \cap T_2| = 1, |T_1 - T_2| = 1 \) and \(|T_2 - T_1| \geq 2\), then \( G \) satisfies the hypothesis.
If $|T_1 \cap T_2| = 1, |T_1 - T_2| \geq 2$ and $|T_2 - T_1| \geq 2$ then there exists a non-minimal dominating set of cardinality $(n - 3)$ which does not contain any minimum dominating set of $G$, a contradiction.

If $|T_1 \cap T_2| \geq 2, |T_1 - T_2| = |T_2 - T_1| = 1$ then every non-minimal dominating set of cardinality $(n - 1)$ contains a minimum dominating set of $G$, a contradiction.

If $|T_1 \cap T_2| \geq 2, |T_1 - T_2| = 1$ and $|T_2 - T_1| \geq 2$ then there exists a non-minimal dominating set of cardinality $(n - 3)$ which does not contain any minimum dominating set of $G$, a contradiction.

If $|T_1 \cap T_2|, |T_1 - T_2|$ and $|T_2 - T_1| \geq 2$ then there exists a non-minimal dominating set of cardinality $(n - 3)$ which does not contain any minimum dominating set of $G$, a contradiction.