Chapter 1

Preliminary

In this chapter we collect some basic definitions and theorems on graphs and digraphs which are needed for the subsequent chapters. For graph theoretic notation and terminology, we follow [5, 7, 15, 36].

1.1 Basic Definitions and Theorems on Graphs

Definition 1.1.1. [5] The distance $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the minimum length of a path joining them if any; otherwise $d(u, v) = \infty$. A shortest $u - v$ path is called a $u - v$ geodesic.

Definition 1.1.2. [5] Let $G$ be a connected graph and let $v$ be a vertex of $G$. The eccentricity $e(v)$ of $v$ is the distance to a vertex farthest from $v$. Thus, $e(v) = \max \{d(u, v) : u \in V\}$. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $d(G)$ is the maximum eccentricity of the vertices.
Theorem 1.1.3. [5] For any connected graph \( G \), the radius and diameter satisfy \( r(G) \leq d(G) \leq 2r(G) \).

Definition 1.1.4. [5] A vertex \( v \) is called a central vertex if \( e(v) = r(G) \), and the center of a graph \( G \) is the subgraph induced by the vertices of minimum eccentricity.

Definition 1.1.5. [5] A vertex \( v \) is called a peripheral vertex if \( e(v) = d(G) \), and the periphery of a graph \( G \) is the subgraph induced by the vertices of maximum eccentricity.

Definition 1.1.6. [5] For a vertex \( v \) in a connected graph \( G \), each vertex at distance \( e(v) \) from \( v \) is an eccentric vertex for \( v \).

Definition 1.1.7. [5] A graph \( G \) is called a self-centered graph if all vertices have the same eccentricity.

Definition 1.1.8. [5] A graph \( G \) is called a self-complementary graph if \( G \cong \overline{G} \).

Theorem 1.1.9. [5] Every non-trivial self-complementary graph has diameter 2 or 3.

Theorem 1.1.10. [5] If \( G \) is a simple graph with diameter at least 3, then \( \overline{G} \) has diameter at most 3.

Theorem 1.1.11. [5] If \( G \) is a simple graph with diameter at least 4, then \( \overline{G} \) has diameter at most 2.

Theorem 1.1.12. [5] If \( G \) is a simple graph with diameter at least 3 and if \( \overline{G} \) is connected, then \( \overline{G} \) has radius at most 2.
Theorem 1.1.13. [5] If $G$ is a self-centered graph with $r(G) \geq 3$, then $\overline{G}$ is a self-centered graph of radius 2.

Definition 1.1.14. [5] A graph $G$ is hamiltonian-connected if for each pair of vertices $u, v$ there is a spanning path joining $u$ and $v$.

Definition 1.1.15. [17] The detour distance $D(u, v)$ from a vertex $u$ to a vertex $v$ is defined as the length of a longest $u-v$ path in $G$.

Definition 1.1.16. [17] The detour eccentricity $e_D(v)$ of a vertex $v$ in a connected graph $G$ is $e_D(v) = \max \{D(v, x) : x \in V(G)\}$. The detour radius $rad_D(G)$ of $G$ is $rad_D(G) = \min \{e_D(v) : v \in V(G)\}$ and the detour diameter $diam_D(G)$ of $G$ is $diam_D(G) = \max \{e_D(v) : v \in V(G)\}$. A vertex $v$ in a graph $G$ is called a detour central vertex if $e_D(v) = rad_D(G)$, while the subgraph induced by the detour central vertices of $G$ is the detour center $C_D(G)$ of $G$. A vertex $v$ in a graph $G$ is called a detour peripheral vertex if $e_D(v) = diam_D(G)$, while the subgraph induced by the detour peripheral vertices of $G$ is the detour periphery $P_D(G)$ of $G$.

Definition 1.1.17. [21] For a simple connected graph $G$ and for two vertices $u$ and $v$ of $G$, let $D_{u,v} = N[u] \cup N[v]$. We define a $D_{u,v}$-walk as a $u-v$ walk in $G$ that contains every vertex of $D_{u,v}$. The superior distance $d_{SD}(u, v)$ from $u$ to $v$ is the length of a shortest $D_{u,v}$-walk.
Definition 1.1.18. [21] The superior eccentricity $e_{SD}(v)$ of a vertex $v$ in a connected graph $G$ is $e_{SD}(v) = \max\{d_{SD}(u,v) : u \in V(G)\}$. The superior radius $r_{SD}(G) = \min\{e_{SD}(v) : v \in V(G)\}$ and the superior diameter $d_{SD}(G)$ of $G$ is $d_{SD}(G) = \max\{e_{SD}(v) : v \in V(G)\}$. A vertex $v$ in a graph $G$ is called a superior central vertex if $e_{SD}(v) = r_{SD}(G)$, while the subgraph induced by the superior central vertices of $G$ is the superior center $C_{SD}(G)$ of $G$. A vertex $v$ in a graph $G$ is called a superior peripheral vertex if $e_{SD}(v) = d_{SD}(G)$, while the subgraph induced by the superior peripheral vertices of $G$ is the superior periphery $P_{SD}(G)$ of $G$.

Definition 1.1.19. [15] A set $S \subseteq V$ of vertices in a graph $G$ is called a dominating set if for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that $v$ is adjacent to $u$. A dominating set of minimum cardinality is a minimum dominating set and its cardinality is the domination number $\gamma(G)$.

Definition 1.1.20. [9] Let $G$ be a nontrivial connected graph. For each vertex $v \in V(G)$, define $d^-(v) = \min\{d(u,v) : u \in V(G) - v\}$. A vertex $u(\neq v)$ is called a neighbor of $v$ if $d(u,v) = d^-(v)$. A vertex $v$ is said to dominate a vertex $u$ if $u = v$ or $u$ is a neighbor of $v$. Since $d^-(v) = 1$ for all $v \in V(G)$, this is equivalent to the standard definition of neighbor and the standard definition of domination.
Definition 1.1.21. [9] For each vertex $v$ in a nontrivial connected graph $G$, define $D^{-}(v) = \min\{D(u,v) : u \in V(G) - v\}$. A vertex $u(\neq v)$ is called a detour neighbor of $v$ if $D(u,v) = D^{-}(v)$. The detour neighborhood $N_D(v)$ of a vertex $v$ is the set of all detour neighbors of $v$; and its closed detour neighborhood is $N_D[v] = N_D(v) \cup \{v\}$.

Definition 1.1.22. [9] A vertex $v$ is said to detour dominate a vertex $u$ if $u = v$ or $u$ is a detour neighbor of $v$. A set $S$ of vertices of $G$ is called a detour dominating set if every vertex of $G$ is detour dominated by some vertex in $S$. A detour dominating set of minimum cardinality is a minimum detour dominating set and its cardinality is the detour domination number $\gamma_D(G)$.

Definition 1.1.23. [22] For each vertex $v$ in a nontrivial connected graph $G$, define $d^{-}_{SD}(v) = \min\{d_{SD}(u,v) : u \in V(G) - v\}$. A vertex $u(\neq v)$ is called a superior neighbor of $v$ if $d_{SD}(u,v) = d^{-}_{SD}(v)$. The superior neighborhood $N_{SD}(v)$ of a vertex $v$ is the set of all superior neighbors of $v$; and its closed superior neighborhood is $N_{SD}[v] = N_{SD}(v) \cup \{v\}$.

Definition 1.1.24. [22] A vertex $v$ is said to superior dominate a vertex $u$ if $u$ is a superior neighbor of $v$. A set $S$ of vertices of $G$ is called a superior dominating set if every vertex of $V - S$ is superior dominated by some vertex of $S$. A superior dominating set of minimum cardinality is a minimum superior dominating set and its cardinality is the superior domination number $\gamma_{SD}(G)$. 
Definition 1.1.25. [34] Two vertices of a graph are said to be antipodal to each other if the distance between them is equal to the diameter of the graph.

Definition 1.1.26. [34] The antipodal graph of a graph $G$ is the graph with the same vertices as in $G$ and two vertices are adjacent if they are antipodal in $G$. The antipodal graph of a graph $G$ is denoted by $A(G)$. If $G$ is disconnected, then two vertices are adjacent in $A(G)$ if they are in different components of $G$.

Definition 1.1.27. [27] Two vertices of a graph are said to be radial to each other if the distance between them is equal to the radius of the graph.

Definition 1.1.28. [27] The radial graph of a graph $G$, denoted by $R(G)$, has the vertex set as in $G$ and two vertices are adjacent in $R(G)$ if and only if they are radial in $G$. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of $G$.

Definition 1.1.29. [27] A graph $G$ is called a radial graph if $R(H) \cong G$, for some graph $H$.

Proposition 1.1.30. [27] If $\text{rad}(G) > 1$, then $R(G) \subseteq \overline{G}$.

Theorem 1.1.31. [27] Let $G$ be a graph of order $n$, then $R(G) = G$ if and only if $\text{rad}(G) = 1$.

Let $S_i(G)$ be the subset of the vertex set of $G$ consisting of vertices with eccentricity $i$. 
Lemma 1.1.32. [27] Let $G$ be a graph of order $n$, then $R(G) = \overline{G}$ if and only if $S_2(G) = V(G)$ or $G$ is disconnected in which each component is complete.

Theorem 1.1.33. [27] A graph $G$ is a radial graph if and only if it is the radial graph of itself or the radial graph of its complement.

Definition 1.1.34. [5] Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ respectively. Their union $G = G_1 \cup G_2$ has, as expected, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Their join denoted by $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining $V_1$ with $V_2$.

Definition 1.1.35. [5] For three or more disjoint graphs $G_1, G_2, \ldots, G_n$, the sequential join $G_1 + G_2 + \ldots + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \ldots \cup (G_{n-1} + G_n)$.

Definition 1.1.36. [27] For a pair of integers $m \geq 3$ and $n \geq 3$, let $L(n, m)$ be the lollipop graph of order $m + n$ obtained from a cycle $C_n$ by attaching a path of length $m - 1$ to a vertex of the cycle.

Definition 1.1.37. For a pair of integers $m \geq 3$ and $n \geq 4$, let $U(n, m)$ be the umbrella graph of order $m + n + 1$ obtained from a fan $F_n$ by attaching a path of length $m - 1$ to a vertex $K_1$ of the fan.
Definition 1.1.38. [7] If each edge $e = uv$ of a graph $G$ is replaced by a new vertex $w$ and the new edges $uw$ and $vw$, then the resulting graph is called the subdivision graph of $G$ and is denoted by $S(G)$.

Definition 1.1.39. [32] Let $G$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be an arbitrary supersubdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for arbitrary $m_i$, $1 \leq i \leq q$ in such a way that the ends of $e_i$ are merged with the two vertices of the 2-vertices part of $K_{2,m_i}$ after removing the edge $e_i$ from $G$.

Definition 1.1.40. [29] A graph $G$ with $p$ vertices and $q$ edges is numbered if each vertex $v$ is assigned a non negative integer $f(v)$ and each edge $uv$ is assigned the value $|f(u) - f(v)|$. A numbering is called graceful, if, further, the vertices are labeled with distinct integers from $\{0, 1, ..., q\}$ and the edges with integers from $\{1, 2, ..., q\}$ where $q$ is the number of edges of $G$. A graph which admits a graceful numbering is said to be graceful.

Theorem 1.1.41. [29] If every vertex has even degree and the number of edges is congruent to 1 or 2($mod$ 4) then the graph is not graceful.

Theorem 1.1.42. [29] Cycle $C_p$ is graceful if and only if $q \equiv 0$ or $3($mod $4)$, where $q$ is the number of edges in $C_p$. 
Definition 1.1.43. [4] Let \( v_0, v_1, ..., v_{n-1} \) be the consecutive vertices of the path \( P_n \). The \( y \)-tree is the tree of order \( n + 1 \) whose vertex set is \( V = \{v_0, v_1, ..., v_{n-1}, v_n\} \) and its edge set is \( E = \{e_i = v_{i-1}v_i : 1 \leq i \leq n - 1\} \cup \{e_n = v_{n-2}v_n\} \). In other terms, \( y \)-tree is obtained by attaching the vertex \( v_n \) to the vertex \( v_{n-2} \) of \( P_n \).

Theorem 1.1.44. [32] Arbitrary supersubdivisions of any path are graceful.

Theorem 1.1.45. [20] Arbitrary supersubdivisions of any star are graceful.

Theorem 1.1.46. [4] Arbitrary supersubdivisions of any \( y \)-tree are graceful.

Definition 1.1.47. [14] A graph \( G \) with \( p \) vertices and \( q \) edges is odd graceful if there is an injection \( f \) from \( V(G) \) to \( \{0, 1, ..., 2q - 1\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are \( \{1, 3, ..., 2q - 1\} \).

Theorem 1.1.48. [23] If \( G \) is eulerian and odd graceful, then \( q \equiv 0 \) or \( 2(\text{mod} \ 4) \).

Theorem 1.1.49. [14] Cycle \( C_p \) is odd graceful if and only if \( p \) is even.
1.2 Basic Definitions and Examples on Digraphs

Definition 1.2.1. [7] A directed graph or digraph $D$ is a finite, nonempty set $V$ together with a (possibly empty) set $E$ (disjoint from $V$) of ordered pairs of distinct elements of $V$. Each element of $V$ is a vertex and each element of $E$ is called an arc (or directed edge).

Note 1.2.2. In the following figures of digraph, the undirected edge between two vertices indicates the presence of both directed arcs between the vertices.

Example 1.2.3.

![Diagrah](image)

Figure 1.1: A digraph $D$

Definition 1.2.4. [7] If $a = (u, v)$ is an arc of $D$, then we say that $u$ is adjacent to $v$ and $v$ is adjacent from $u$

Example 1.2.5. In the digraph $D$ of Figure 1.1, the vertex $u$ is adjacent to $v$, but $v$ is not adjacent to $u$. Also, the vertex $w$ is adjacent to $u$ and $u$ is adjacent to $w$. 
Definition 1.2.6. [7] The outdegree, $od v$, of a vertex $v$ of a digraph $D$ is the number of vertices of $D$ that are adjacent from $v$. The indegree, $id v$, of a vertex $v$ of a digraph $D$ is the number of vertices of $D$ that are adjacent to $v$.

Example 1.2.7. In the digraph $D$ of Figure 1.1, $od u = 2$, $id u = id v = id w = od w = 1$ and $od v = 0$.

Definition 1.2.8. [7] A digraph $D$ is called symmetric if whenever $(u, v)$ is an arc of $D$, then so too is $(v, u)$. A digraph $D$ is called asymmetric if whenever $(u, v)$ is an arc of $D$, then $(v, u)$ is not an arc of $D$.

Example 1.2.9. The digraph $D_1$ of Figure 1.2 is symmetric while $D_2$ is asymmetric.

![Figure 1.2: Symmetric and asymmetric digraphs](image)

Definition 1.2.10. [7] A digraph $D$ is called complete if for every two distinct vertices $u$ and $v$ of $D$, atleast one of the arcs $(u, v)$ and $(v, u)$ is present in $D$. The complete symmetric digraph of order $p$ has both arcs $(u, v)$ and $(v, u)$ for every two distinct vertices $u$ and $v$ and is denoted by $K_p^*$. 
Definition 1.2.11. [7] The complement $\overline{D}$ of a digraph $D$ is that digraph having the same vertex set as $D$ and such that for $u, v \in V(D)$, $u \neq v$, the arc $(u, v)$ belongs to $\overline{D}$ if and only if $(u, v)$ is not an arc of $D$.

Definition 1.2.12. [7] A walk in a digraph $D = (V, E)$ is a sequence $v_0a_1v_1a_2...a_kv_k$, whose elements $v_i \in V$ and $a_i \in E$ are such that $a_i = v_{i-1}v_i$ for $1 \leq i \leq k$. If $v_0 \neq v_k$ the walk is open; otherwise it is closed. If no arc is repeated, the walk is a trail; a closed trail is a tour. If no vertex is repeated in an open walk, the walk is a path. If no vertex is repeated in a closed walk it is called a cycle.

Example 1.2.13. In the digraph $D$ of Figure 1.3, $W : v_1, v_2, v_5$ is a $v_1 - v_5$ path.

\[ D : \]

\[ \begin{array}{c}
    v_1 \\
    v_4 \\
    v_2 \\
    v_3 \\
    v_5 \\
\end{array} \]

Figure 1.3: A digraph $D$

Definition 1.2.14. [7] The underlying graph of a digraph $D$ is obtained by removing all directions from the arcs of $D$ and replacing any resulting pair of parallel edges by a single edge.

Definition 1.2.15. [7] A digraph $D$ is said to be weakly connected or connected if the underlying graph of $D$ is connected. A digraph that is not connected is disconnected.
Definition 1.2.16. [7] A vertex \( v \) is said to be reachable from a vertex \( u \) in a digraph \( D \) if \( D \) contains a \( u - v \) walk or, equivalently, a \( u - v \) path. The digraph \( D \) is called unilaterally connected or simply unilateral if for every two distinct vertices of \( D \), atleast one of them is reachable from the other. The digraph \( D \) is called strongly connected or simply strong if for every two distinct vertices of \( D \), each vertex is reachable from the other.

Example 1.2.17.

![Diagram](image)

**Figure 1.4:** Disconnected, connected, unilateral and strong digraphs

Definition 1.2.18. [7] Two digraphs \( D_1 \) and \( D_2 \) are identical, written \( D_1 = D_2 \), if \( V(D_1) = V(D_2) \) and \( E(D_1) = E(D_2) \).
Definition 1.2.19. [7] A digraph $D_1$ is isomorphic to a digraph $D_2$ if there exists a one-to-one mapping $\phi$, called an isomorphism, from $V(D_1)$ onto $V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(\phi u, \phi v) \in E(D_2)$. If $D_1$ is isomorphic to $D_2$, then we say that $D_1$ and $D_2$ are isomorphic and write $D_1 \cong D_2$.

Definition 1.2.20. [7] For a pair $u, v$ of vertices in a digraph $D$ the directed distance or, more simply, the distance $d_D(u, v)$ is the length of a shortest (directed) $u - v$ path in $D$. If $D$ contains no $u - v$ path, then $d_D(u, v) = \infty$.

Definition 1.2.21. [7] Let $D$ be a strong digraph. The eccentricity $e(v)$ of a vertex $v$ of $D$ is defined as the maximum distance from $v$ to any vertex in $D$. The radius of $D$, $\text{rad}(D)$, is the minimum eccentricity of the vertices in $D$; the diameter $\text{diam}(D)$ is the maximum eccentricity of the vertices in $D$.

Example 1.2.22.

![Diagram](figure1.5.png)

Figure 1.5: Distance in strong digraphs

Definition 1.2.23. [7] For a digraph $D$, the antipodal digraph $A(D)$ of $D$ is the digraph with $V(A(D)) = V(D)$ and $E(A(D)) = \{(u, v)/u, v \in V(D) \text{ and } d_D(u, v) = \text{diam}(D)\}$. 