CHAPTER - I

1.1 INTRODUCTION TO THE FIXED POINT THEORY

Fixed point theory is central to many existence theorems in Mathematics. One of the main tools in fixed point theory is the Banach contraction theorem (it is also called as Banach contraction principle) which states that every contraction mapping $F$ on a complete metric space $X$ has a unique fixed point. There are a lot of generalizations of this theorem in the literature. These generalizations were made in two typical ways. The first method is weakening / generalizing / the contractive condition $d(Fx,Fy) \leq kd(x,y)$ while the second is allowing $X$ to be a more general space than the metric space.

In fixed point theory the importance of various contractive inequalities cannot be over emphasized. Existence theorems of fixed points have been established for mappings defined on various types of spaces and satisfying different types of contractive inequalities.

A space $X$ is said to have the Fixed point property (f.p.p) if every continuous function $f$ from $X$ to $X$ has a fixed point. Whether if $X$ and $Y$ have the f.p.p then $X \times Y$ has the f.p.p. i.e., A point $x$ such that $x \in F(x)$.

**Theorem 1.1.1.** Let $f$ be a trace of a multi – valued function $F$ on $X$ to $Y$ and let $x$ be a fixed point of $f$. Then $x$ is a fixed point of $F$.

**Theorem 1.1.2.** A bounded closed interval $I$ of real numbers has the f.p.p.
1.2 About the Thesis

This Thesis is divided into 9 chapters.

In Chapter I, we give a brief history about the Fixed Point Theory and the Thesis organization.

In Chapter II, we give a brief history of Fixed point theory in Metric space, Fuzzy Metric Space and Dislocated Quasi Metric Space. We also give some basic definitions and fundamental results in the literature which we refer in subsequent chapters.

In Chapter III, the concept of fuzzy sets introduction by Zadeh[123], Butnariu[20], Chitra[23], Heilpern[38], Lee and Cho[60] and others introduced the concepts of fuzzy mappings and proved the existence of fixed points of fuzzy mappings. With this in view, R.K.Saini, Sanjeev Kumar and Peer Mohammed [94] have defined the coincidence point of a crisp mapping and a fuzzy mapping. They also introduced $R$ – weak commutativity for a pair of crisp mapping and fuzzy mapping. A common coincidence point theorem for the combination of crisp mappings and fuzzy mappings together using the notion of $R$– weakly commuting mappings.

Now, we introduce the notion of $\varphi$ - weakly commuting pair of self - maps on a metric space and prove a common fixed point theorem for such pairs of maps, satisfying a certain control condition, doing away with the implicit relations introduced by R.K.Saini, Sanjeev Kumar and Peer Mohammed [94].

The contents of this chapter have appeared in [95].

In Chapter IV, Gairola and Ram Krishan [33] presented fixed point results for three self maps satisfying a generalized weak contraction condition by using the concept of weakly compatible maps in a complete metric space. These results extend and generalize the results of Choudhary.et.al [26] and others. A generalized weak contraction condition involves an altering distance function which is non – decreasing.
and continuous. However, in dealing with problems, the altering distance function fails to be continuous throughout. There is a need to develop tools to handle such situations. Further, in many of the common fixed point results while dealing with generalized contractions involving an altering distance function $\psi$ and a deficit function $\varphi$, the absence of any link between $\psi$ and $\varphi$ may not give good results. Further, a suitable link between $\psi$ and $\varphi$ will lead to fruitful results. Keeping these things in view, we show a way to successfully avoid continuity of the altering distance function. Improving a suitable link between $\psi$ and $\varphi$, we prove common fixed point results, from which the results of earlier authors follow as corollaries.

**Now**, we introduce the notion of a Pseudo altering distance function (doing away with continuity) and obtain a common fixed point theorem. In the results of earlier authors, dealing with generalized contractions involving an altering distance function $\psi$ and a deficit function $\varphi$ no link is given between $\psi$ and $\varphi$. In this chapter, besides introducing a Pseudo altering distance function $\psi$, we also insist on a link between $\psi$ and the deficit function $\varphi$. We observe that, when $\psi$ is continuous, this link is satisfied automatically. With the aid of a Pseudo altering distance function $\psi$ and a link between $\psi$ and the deficit function $\varphi$, we prove a common fixed point theorem which extends the results of Gairola and Ram Krishan [33].

The contents of this Chapter form the content of the paper [101].

**In Chapter V**, A generalization of the concepts of a metric space which was first introduced by Menger [64] in 1942 and following him is called a statistical (probabilistic) metric space. Later in 1965, Zadeh [123] introduced the concepts of fuzzy sets laid the foundation of fuzzy Mathematics. Later in 1975 Kramosil and Michalek[56] introduced the fuzzy metric space by generalizing the concept of probabilistic metric space to fuzzy situation. Later in 1980, Hadzic[37] studied the concepts of a generalization of the contraction principles in probabilistic metric spaces. Later in 1986, Jungck generalized the concepts of weak commuting mappings by introducing the notion of compatible maps. With this in view, Vasuki [120] proved a common fixed point theorem for two $R$ – weakly commutative self maps on a maps
on a complete fuzzy metric space with certain condition. Sarma.et.al [109] extended this result to three self maps.

Now, we introduce the notion of $\varphi$ – weakly commuting pair of maps and make use of contractive control function of type (AS) to prove a common fixed point theorem for three self maps on a continuous complete fuzzy metric space.

The contents of this Chapter form the content of the paper [97].

In Chapter VI, Most recently, Sintunavarat and Kumam [114] defined the notion of “Common limit in the range” property (or (CLR) property) in fuzzy metric spaces and improved the results of Mihet [65]. In [114], it is observed that the notion of (CLR) property never requires the condition of the closedness of the subspace while (E.A) property requires this condition for the existence of the fixed point. Chauhan.et.al.[22] introduced the notion of Joint common limit in the range of mappings property called (JCLR) property and proved a common fixed point theorem ([22], Theorem3.1) for a pair of weakly compatible mappings using (JCLR) property in fuzzy metric spaces (See Note 6.1.17). As an application of their result, a common fixed point theorem for two finite families of self mappings in fuzzy metric spaces, using the notion of pair wise commuting property is also given by Imdad[42]. Their results improve and generalize results of Cho.et.al [25], Abbas.et.al[4] and Kumar [57].

Now, we remove the restriction that $\alpha = 1$ in the result of Chauhan.et.al ([22], Theorem 3.1), by assuming that $\alpha \in (0,2)$; we also remove the restriction that $k \in (0,\frac{1}{2})$ and prove the result by taking $k \in (0,1)$. This amounts to obtaining a good improvement over Chauhan.et.al.[22] (See note 6.1.17).

The contents of this Chapter form the content of the paper [98].

Now, we prove a coincidence point theorem in dislocated metric spaces, provide a supporting example and extend the theorem to dislocated quasi metric spaces. We observe that the supporting example of a result of Rao and Ranga Swamy [84], on the existence of a coincidence point for four self maps on a dislocated metric space is not valid. We also make a modification of their result observe that (Theorem 2.1, [84]) is a special case of our result.

The contents of this Chapter form the content of the paper [96].

In Chapter VIII, The concept of dislocated metrics was studied under the name of metric domains in the context of domain theory in Hitzler [39]. As a generalization of metrics where the self distance for any point need not be equal to zero, Hitzler[39] and Hitzler and Seda[40], introduced the notion of dislocated metric spaces and generalized the celebrated Banach contraction principle to such spaces.

Now, we observe that Theorem 3.3 of Rajesh.et.al [82] is not meaningful and provide a revised statement to make this theorem meaningful. Further we prove a fixed point theorem for a self map on a complete dislocated quasi metric space, satisfying a condition similar to the one in Rajesh.et.al [82]. We also extend a fixed point theorem of Rajesh.et.al [82] for a A-contraction pair on a dislocated metric space (Theorem 3.7 of Rajesh.et.al [82]) to dislocated quasi metric spaces.

The contents of this Chapter form the content of the paper [102].

In Chapter IX, Ahmed and Zeyada [7]established the convergence of a sequence \( \{x_n\} \), in a dislocated - quasi metric space \((X,d)\) if a map \( T : X \rightarrow X \) is quasi - nonexpansive with respect to \( \{x_n\} \). We observe that the role played by the map \( T \) in proving the convergence of \( \{x_n\} \), is meagre.

Now, we introduce the notion of a quasi – nonexpansive sequence with respect to a non empty subset of a dislocated quasi metric space and establish the convergence of such sequences under certain conditions. These results extend the results of Ahmed and Zeyada[7].

The contents of this Chapter form the content of the paper [103].
A WORD ABOUT NOTATION:

To specify and to refer whenever necessary, items such as definitions, theorems etc., we follow decimal notation. In each chapter, such items are numbered serially. Accordingly 3.8 means eighth item of chapter 3. Important sub items in an item which require further referencing in the same or other items are further subdivided in decimal notation, but written in parenthesis.

Accordingly, (3.2.8) means: 3rd Chapter 2nd Section 8th item.

The following notation is employed throughout the thesis without explicit mention.

\[ \mathbb{R} : \text{Set of real numbers.} \]

\[ R^+ : \text{Set of non negative real numbers.} \]

\[ N : \text{Set of positive integrers.} \]