Chapter 2

$\alpha C^*$-sets and $\alpha C^*$-continuous functions

2.1 Introduction

Erguang and Pengfei [37] have introduced the concepts of $C$-sets and $C$-continuous functions. Also, they have proved that a function is $A$-continuous if and only if it is $C$-continuous and semicontinuous. Noiri and Sayed [72] have studied the notions of $\eta$-sets, $\eta\zeta$-sets, $\eta$-continuity and $\eta\zeta$-continuity. By using these new concepts, they have obtained some new decompositions of continuous functions and $A$-continuous functions. In this chapter, we introduce the notions of $\alpha C^*$-sets and $\alpha C$-sets; and investigate their properties. By using these classes of sets, we obtain some new results.
2.2 Preliminaries

**Definition 2.2.1** A subset $A$ of a space $(X, \tau)$ is called:

1. regular open [94] if $A = \text{int}(\text{cl}(A))$ and regular closed [94] if $A = \text{cl}(\text{int}(A))$,

2. preopen [66] or nearly open [40] if $A \subseteq \text{int}(\text{cl}(A))$ and preclosed [66] if $\text{cl}(\text{int}(A)) \subseteq A$,

3. preregular [32] if it is both preopen and preclosed,

4. an $\alpha$-locally closed (briefly $\alpha$-lc) [42] if $A = G \cap F$, where $G$ is $\alpha$-open and $F$ is $\alpha$-closed in $X$,

5. an $\eta\varsigma$-set [72] if $A = G \cap F$, where $G$ is open and $F$ is clopen in $X$,

6. an $\alpha$ LC-set [7] if $A = G \cap F$, where $G$ is $\alpha$-open and $\text{cl}(F) = F$,

7. an $\alpha\mathcal{A}$-set [14] if $A = G \cap F$, where $G$ is $\alpha$-open and $F = \text{cl}(\text{int}(F))$,

8. a $\mathcal{A}$-set [100] if $A = G \cap F$, where $G$ is open and $F = \text{cl}(\text{int}(F))$.

The family of all $\alpha$-open sets of $(X, \tau)$ forms a topology and is denoted by $\tau^\alpha$.

A space $X$ is called extremally disconnected [17] if every open subset of $X$ has open closure or equivalently if every regular closed set is open.
2.3 Properties of \(\alpha C^*\)-sets and \(\alpha C\)-sets in topological spaces

**Definition 2.3.1** A subset \(A\) of a space \((X, \tau)\) is called

1. an \(\alpha C^*\)-set if \(A = G \cap R\), where \(G\) is \(\alpha\)-open and \(R\) is preregular,

2. an \(\alpha C\)-set if \(A = G \cap B\), where \(G\) is \(\alpha\)-open and \(B\) is preclosed,

3. an \(\alpha B\)-set if \(A = G \cap B\), where \(G\) is \(\alpha\)-open and \(B\) is \(t\)-set.

The collection of all \(\alpha C^*\)-sets (resp. \(\alpha C\)-sets) in \(X\) will be denoted by \(\alpha C^*(X)\) (resp. \(\alpha C(X)\)).

**Remark 2.3.2** The following diagram holds for a subset \(A\) of a space \(X\):

\[
\begin{array}{ccc}
\eta\zeta\text{-set} & \Rightarrow & \alpha C^*\text{-set} \Rightarrow \alpha C\text{-set} \\
\downarrow & & \uparrow \\
\alpha \mathcal{A}\text{-set} & \Rightarrow & \alpha LC\text{-set} \quad \alpha \text{-lc-set}
\end{array}
\]

None of these implications is reversible as shown in the following Examples.

**Example 2.3.3** Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}\). Then \(\{a, b, d\}\) is an \(\alpha C^*\)-set but it is not \(\eta\zeta\)-set.

**Example 2.3.4** In Example 2.3.3, \(\{c\}\) is an \(\alpha C\)-set but it is not \(\alpha C^*\)-set.

**Example 2.3.5** In Example 2.3.3, \(\{b, d\}\) is an \(\alpha \mathcal{A}\)-set but it is not \(\eta\zeta\)-set.
Example 2.3.6 In Example 2.3.3, \( |c| \) is an \( \alpha \) LC-set but it is not \( \alpha \mathcal{A} \)-set.

Example 2.3.7 Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a, b\}\} \). Then \( |a| \) is an \( \alpha \mathcal{C} \)-set but it is not \( \alpha \)-lc-set.

Remark 2.3.8 Every \( \alpha \) LC-set is \( \alpha \)-lc-set.

Remark 2.3.9 The following Examples show that the concepts of \( \alpha \mathcal{C}^* \)-sets and \( \alpha \) LC-sets, the concepts of \( \alpha \mathcal{C}^* \)-sets and \( \alpha \)-lc-sets and the concepts of \( \alpha \mathcal{C}^* \)-sets and \( \alpha \mathcal{A} \)-sets are independent of each other.

Example 2.3.10 In Example 2.3.7, \( |a| \) is an \( \alpha \mathcal{C}^* \)-set but it is not \( \alpha \) LC-set and moreover, \( |c| \) is an \( \alpha \) LC-set but it is not \( \alpha \mathcal{C}^* \)-set.

Example 2.3.11 In Example 2.3.7, \( |a| \) is an \( \alpha \mathcal{C}^* \)-set but it is not \( \alpha \)-lc-set and moreover, \( |c| \) is an \( \alpha \)-lc-set but it is not \( \alpha \mathcal{C}^* \)-set.

Example 2.3.12 1. In Example 2.3.7, \( |a| \) is an \( \alpha \mathcal{C}^* \)-set but it is not \( \alpha \mathcal{A} \)-set.

2. In Example 2.3.3, \( \{b, d\} \) is an \( \alpha \mathcal{A} \)-set but it is not \( \alpha \mathcal{C}^* \)-set.

Remark 2.3.13 Every \( \alpha \)-open and every preregular set is an \( \alpha \mathcal{C}^* \)-set.

Remark 2.3.14 The converses of the implications in Remark 2.3.13 are not true in general as shown in the following Examples.

Example 2.3.15 In Example 2.3.7, \( \{a, b\} \) is an \( \alpha \mathcal{C}^* \)-set but it is not preregular.
Example 2.3.16 In Example 2.3.7, \( \{a\} \) is an \( \alpha C^* \)-set but it is not \( \alpha \)-open.

Theorem 2.3.17 For a subset \( A \) of a topological space \( (X, \tau) \), the following properties are equivalent.

1. \( A \) is an \( \alpha C \)-set and a semi-open set in \( X \).

2. \( A = L \cap \text{cl}(\text{int}(A)) \) for an \( \alpha \)-open set \( L \).

Proof. (1) \( \Rightarrow \) (2): Suppose that \( A \) is an \( \alpha C \)-set and a semi-open set in \( X \). Since \( A \) is \( \alpha C \)-set, then we have \( A = L \cap M \), where \( L \) is an \( \alpha \)-open set and \( M \) is a preclosed set in \( X \). We have \( A \subseteq M \), so \( \text{cl}(\text{int}(A)) \subseteq \text{cl}(\text{int}(M)) \). Since \( M \) is a preclosed set in \( X \), we have \( \text{cl}(\text{int}(M)) \subseteq M \). Since \( A \) is a semi-open set in \( X \), we have \( A \subseteq \text{cl}(\text{int}(A)) \). It follows that \( A = A \cap \text{cl}(\text{int}(A)) = L \cap M \cap \text{cl}(\text{int}(A)) = L \cap \text{cl}(\text{int}(A)) \).

(2) \( \Rightarrow \) (1): Let \( A = L \cap \text{cl}(\text{int}(A)) \) for an \( \alpha \)-open set \( L \). We have \( A \subseteq \text{cl}(\text{int}(A)) \). It follows that \( A \) is a semi-open set in \( X \). Since \( \text{cl}(\text{int}(A)) \) is a closed set, then \( \text{cl}(\text{int}(A)) \) is a preclosed set in \( X \). Hence, \( A \) is an \( \alpha C \)-set in \( X \).

Recall that a space \( X \) is called submaximal [17] if every dense subset of \( X \) is open.

Theorem 2.3.18 For a subset \( A \) of a submaximal and extremally disconnected space \( (X, \tau) \), the following properties are equivalent.

1. \( A \) is open in \( X \).

2. \( A \) is preopen and \( \alpha \)-open in \( X \).
3. A is preopen and an $\alpha\mathcal{B}$-set in $X$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): It is known that if $(X, \tau)$ is a submaximal and extremally disconnected space then $\tau = \tau^a$. By preopenness of $A$, $A \subseteq \text{int}(\text{cl}(A)) = \text{int}(\text{cl}(U \cap V))$, where $U$ is $\alpha$-open and $V$ is a t-set. Hence $A \subseteq U \cap A \subseteq U \cap \text{int}(\text{cl}(U)) \cap \text{int}(\text{cl}(V)) = U \cap \text{int}(V) = \text{int}(A)$. This shows that $A$ is open.

**Definition 2.3.19** 1. A subset $A$ of a space $X$ is called $\alpha$-generalized preclosed (briefly, $\alpha\text{gp}$-closed) in $X$ if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.

2. $A$ is called $\alpha\text{gp}$-open if its complement is $\alpha\text{gp}$-closed set or equivalently, if $N \subseteq \text{pint}(A)$ whenever $N \subseteq A$ and $N$ is $\alpha$-closed in $X$, where $\text{pint}(A) = A \cap \text{int}(\text{cl}(A))$.

**Theorem 2.3.20** For a subset $A$ of a topological space $(X, \tau)$, $A$ is $\alpha\text{gp}$-closed if and only if $\text{pcl}(A) \subseteq N$ whenever $A \subseteq N$ and $N$ is an $\alpha$-open set in $(X, \tau)$.

**Proof.** Let $A$ be an $\alpha\text{gp}$-closed set in $X$. Suppose that $A \subseteq N$ and $N$ is an $\alpha$-open set in $(X, \tau)$. Then $X \setminus A$ is $\alpha\text{gp}$-open and $X \setminus N \subseteq X \setminus A$ where $X \setminus N$ is $\alpha$-closed. Since $X \setminus A$ is $\alpha\text{gp}$-open, then we have $X \setminus N \subseteq \text{pint}(X \setminus A)$, where $\text{pint}(X \setminus A) = (X \setminus A) \cap \text{int}(\text{cl}(X \setminus A))$. Since $(X \setminus A) \cap \text{int}(\text{cl}(X \setminus A)) = (X \setminus A) \cap \text{cl}(\text{int}(A))$, $\text{cl}(\text{cl}(X \setminus A)) = X \setminus (A \cup \text{cl}(\text{int}(A)))$, then $(X \setminus A) \cap \text{int}(\text{cl}(X \setminus A)) = X \setminus \text{pcl}(A)$. It follows that $\text{pint}(X \setminus A) = X \setminus \text{pcl}(A)$. Thus $\text{pcl}(A) = X \setminus \text{pint}(X \setminus A) \subseteq N$ and hence $\text{pcl}(A) \subseteq N$. The converse is similar.
**Theorem 2.3.21** Let \((X, \tau)\) be a topological space and \(V \subseteq X\). Then \(V\) is an \(\alpha C\)-set in \(X\) if and only if \(V = G \cap \text{pcl}(V)\) for an \(\alpha\)-open set \(G\) in \(X\).

**Proof.** If \(V\) is an \(\alpha C\)-set, then \(V = G \cap M\) for an \(\alpha\)-open set \(G\) and a preclosed set \(M\). But then \(V \subseteq M\) and so \(V \subseteq \text{pcl}(V) \subseteq M\). It follows that \(V = V \cap \text{pcl}(V) = G \cap M \cap \text{pcl}(V) = G \cap \text{pcl}(V)\). Conversely, it is enough to prove that \(\text{pcl}(V)\) is a preclosed set. But \(\text{pcl}(V) \subseteq M\), for any preclosed set \(M\) containing \(V\). So, \(\text{cl}(\text{int}(\text{pcl}(V))) \subseteq \text{cl}(\text{int}(M)) \subseteq M\). It follows that \(\text{cl}(\text{int}(\text{pcl}(V))) \subseteq \cap V \subseteq M\), \(M\) is preclosed, \(M = \text{pcl}(V)\).

**Theorem 2.3.22** For a subset \(A\) of a topological space \((X, \tau)\), the following are equivalent:

1. \(A\) is preclosed,
2. \(A\) is an \(\alpha C\)-set and \(\alpha gp\)-closed.

**Proof.** (1) \(\Rightarrow\) (2). Follows from the fact that every preclosed set is an \(\alpha C\)-set and \(\alpha gp\)-closed.

(2) \(\Rightarrow\) (1). Since \(A\) is an \(\alpha C\)-set, we have \(A = G \cap \text{pcl}(A)\) for an \(\alpha\)-open set \(G\) in \(X\). We obtain \(A \subseteq G\). Since \(A\) is \(\alpha gp\)-closed, then \(\text{pcl}(A) \subseteq G\). Hence, \(\text{pcl}(A) \subseteq G \cap \text{pcl}(A) = A\) and thus, \(A\) is preclosed.

Recall that a space \(X\) is called a partition space or locally indiscrete \([29, 70]\) if every open subset of \(X\) is closed.

**Theorem 2.3.23** Every an \(\alpha C\)-set of a locally indiscrete space \((X, \tau)\) is an \(\alpha C^*\)-set.
Proof. Let A be an $\alpha C$-set in X. Then there exist an $\alpha$-open set G and a preclosed set B such that $A = G \cap B$. It is well known that X is locally indiscrete if and only if every subset of X is preopen. So, B is also preopen. Thus, A is an $\alpha C^*$-set.

Theorem 2.3.24 [25] A subset A of a topological space $(X, \tau)$ is semi-closed if and only if A is a t-set.

Theorem 2.3.25 Let $(X, \tau)$ be a submaximal and extremally disconnected space and $A \subseteq X$. The following properties are equivalent.

1. $A$ is an open set in X.
2. $A$ is an $\alpha$-open set and a $\mathcal{A}$-set.
3. $A$ is a preopen and an $\alpha \mathcal{A}$-set.

Proof. (1) $\Rightarrow$ (2): It follows from the fact that every open set is an $\alpha$-open set and a $\mathcal{A}$-set.

(2) $\Rightarrow$ (3): It follows from the fact that every $\alpha$-open set is preopen and every $\mathcal{A}$-set is $\alpha \mathcal{A}$-set.

(3) $\Rightarrow$ (1): Suppose that $A$ is a preopen set and an $\alpha \mathcal{A}$-set. Since $A$ is an $a \mathcal{A}$-set, then we have $A = L \cap M$, where $L$ is an $\alpha$-open set and $M = \text{cl}(\text{int}(M))$. It follows that $\text{int}(\text{cl}(M)) \subseteq \text{cl}(M) \subseteq \text{cl}(M) = \text{cl}(\text{int}(M)) = M$. Since $\text{int}(\text{cl}(M)) \subseteq M$, then $M$ is a semi-closed set. By Theorem 2.3.24, $M$ is a t-set. Hence, $A$ is an $\alpha B$-set. Since $A$ is an $\alpha B$-set and a preopen set, then by Theorem 2.3.18, $A$ is an open set in X.
**Theorem 2.3.26** Let \((X, \tau)\) be a submaximal and extremally disconnected space and \(A \subseteq X\). The following properties are equivalent.

1. \(A\) is an \(\alpha\)-open set in \(X\).
2. \(A\) is a preopen and an \(\alpha A\)-set.

**Proof.** (1) \(\Rightarrow\) (2): It follows from the fact that every \(\alpha\)-open set is preopen and every \(\alpha\)-open set is an \(\alpha A\)-set.

(2) \(\Rightarrow\) (1): Suppose that \(A\) is a preopen set and an \(\alpha A\)-set. Since \(A\) is an \(\alpha A\)-set, then we have \(A = L \cap M\), where \(L\) is an \(\alpha\)-open set and \(M = \text{cl}(\text{int}(M))\). It follows that \(\text{int}(\text{cl}(M)) \subseteq \text{cl}(M) \subseteq \text{cl}(M) = \text{cl}(\text{int}(M)) = M\). Since \(\text{int}(\text{cl}(M)) \subseteq M\), then \(M\) is a semi-closed set. By Theorem 2.3.24, \(M\) is a \(t\)-set. Hence, \(A\) is an \(\alpha B\)-set. Since \(A\) is an \(\alpha B\)-set and a preopen set, then by Theorem 2.3.18, \(A\) is an \(\alpha\)-open set in \(X\).

**Remark 2.3.27** 1. The notions of preopen sets and \(\alpha A\)-sets are independent of each other, in general.

2. The notions of \(\alpha\)-open sets and \(A\)-sets are independent of each other, in general.

**Example 2.3.28** 1. In Example 2.3.7, \({a}\) is preopen but it is not \(\alpha A\)-set.

2. In Example 2.3.3, \({a, d}\) is \(\alpha A\)-set but it is not preopen.

**Example 2.3.29** In Example 2.3.3, \({a, b, d}\) is \(\alpha\)-open set but it is not \(A\)-set and moreover, \({b, c, d}\) is \(A\)-set but it is not \(\alpha\)-open.
**Theorem 2.3.30** Let $A$ be a subset of a topological space $(X, \tau)$. If $A \in \alpha C(X)$, then $\text{pcl}(A) - A$ is preclosed.

**Proof.** Let $A \in \alpha C(X)$. It follows from Theorem 2.3.21 that $A = G \cap \text{pcl}(A)$ for some $\alpha$-open set $G$. Thus, $\text{pcl}(A) - A = \text{pcl}(A) - (G \cap \text{pcl}(A)) = \text{pcl}(A) \cap (X - (G \cap \text{pcl}(A))) = \text{pcl}(A) \cap ((X - G) \cup (X - \text{pcl}(A))) = (\text{pcl}(A) \cap (X - G)) \cup (\text{pcl}(A) \cap (X - \text{pcl}(A))) = (\text{pcl}(A) \cap (X - G)) \cup \emptyset = \text{pcl}(A) \cap (X - G)$. Hence, $\text{pcl}(A) - A$ is preclosed.

**Theorem 2.3.31** Let $A$ be a subset of a topological space $(X, \tau)$. If $A \in \alpha C(X)$, then $A \cup (X - \text{pcl}(A))$ is preopen.

**Proof.** Let $A \in \alpha C(X)$. Since $\text{pcl}(A) - A$ is preclosed, it follows from Theorem 2.3.30 that $X - (\text{pcl}(A) - A)$ is preopen. Thus, $X - (\text{pcl}(A) - A) = X - (\text{pcl}(A) \cap (X - A)) = (X - \text{pcl}(A)) \cup A$. Therefore, $A \cup (X - \text{pcl}(A))$ is preopen.

**Theorem 2.3.32** Let $A$ be a subset of a topological space $(X, \tau)$. If $A \in \alpha C(X)$, then $A \subseteq \text{pint}(A \cup (X - \text{pcl}(A)))$.

**Proof.** Since $A \cup (X - \text{pcl}(A))$ is preopen, then it follows from Theorem 2.3.31 that $A \subseteq A \cup (X - \text{pcl}(A)) = \text{pint}(A \cup (X - \text{pcl}(A)))$.

**Lemma 2.3.33** [85] The following are equivalent for a topological space $(X, \tau)$:

1. $X$ is submaximal,
2. Every preopen set is open.

**Theorem 2.3.34** If $(X, \tau)$ is a submaximal and extremally disconnected space, then 
\[ \alpha C^*(X) = \eta_\zeta(X). \]

Where $\eta_\zeta(X)$ denotes the family of $\eta_\zeta$-sets of a space $X$.

**Proof.** Obvious.

**Theorem 2.3.35** The following hold for an extremally disconnected space $(X, \tau)$:

1. Every $\alpha \mathcal{A}$-set of $X$ is an $\alpha C^*$-set.
2. Every $\alpha \mathcal{B}$-set of $X$ is an $\alpha C$-set.

**Proof.** (1) Let $A \subseteq X$ be an $\alpha \mathcal{A}$-set. Then $A = G \cap P$, where $G$ is $\alpha$-open and $P$ is regular closed. Since $X$ is extremally disconnected, then $P$ is preopen. Therefore, $A$ is an $\alpha C^*$-set.

(2) Let $A \subseteq X$ be an $\alpha \mathcal{B}$-set. Then $A = G \cap P$, where $G$ is $\alpha$-open and $P$ is semi-closed. Since $X$ is extremally disconnected, then $P$ is preclosed. Thus, $A$ is an $\alpha C$-set.

**2.4 $\alpha C^*$-continuous**

**Definition 2.4.1** A function $f : (X, \tau) \to (Y, \sigma)$ is called $\alpha C^*$-continuous if $f^{-1}(G) \in \alpha C^*(X)$ for each $G \in \sigma$.
Definition 2.4.2 A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called

1. \( \eta_\zeta \)-continuous [72] if \( f^{-1}(G) \in \eta_\zeta(X) \) for each \( G \in \sigma \).

2. \( \alpha \mathrm{C} \)-continuous if \( f^{-1}(G) \in \alpha \mathrm{C}(X) \) for each \( G \in \sigma \).

3. \( \alpha \mathrm{LC} \)-continuous [7] if \( f^{-1}(G) \) is \( \alpha \mathrm{LC} \)-set in \( X \) for each \( G \in \sigma \).

4. \( \alpha \mathrm{lc} \)-continuous [42] if \( f^{-1}(G) \) is \( \alpha \mathrm{lc} \)-set in \( X \) for each \( G \in \sigma \).

5. \( \alpha \mathcal{A} \)-continuous [14] if \( f^{-1}(G) \) is \( \alpha \mathcal{A} \)-set in \( X \) for each \( G \in \sigma \).

Remark 2.4.3 The following diagram holds for a function \( f : (X, \tau) \rightarrow (Y, \sigma) \):

\[
\begin{array}{ccc}
\alpha \mathrm{C}^* \text{-continuous} & \rightarrow & \alpha \mathrm{C} \text{-continuous} \\
\uparrow & & \uparrow \\
\eta_\zeta \text{-continuous} & & \alpha \mathrm{lc} \text{-continuous} \\
\downarrow & & \\
\alpha \mathcal{A} \text{-continuous} & \rightarrow & \alpha \mathrm{LC} \text{-continuous}
\end{array}
\]

None of the implications in the above diagram is reversible as shown in the following Examples.

Example 2.4.4 Let \( X = Y = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X \} \) and \( \sigma = \{\emptyset, \{c\}, \{d\}, \{c, d\}, Y\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then \( f \) is an \( \alpha \mathrm{C} \)-continuous function but not an \( \alpha \mathrm{C}^* \)-continuous.

Example 2.4.5 Let \( X = Y = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X \} \) and \( \sigma = \{\emptyset, \{a, b, d\}, Y\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then \( f \) is an \( \alpha \mathrm{C}^* \)-continuous function but not an \( \eta_\zeta \)-continuous.
Example 2.4.6  Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, [a, b], X\}$ and $\sigma = \{\emptyset, [a], [b], [a, b], Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is an $\alpha C$-continuous function but not an $\alpha$-lc-continuous.

Example 2.4.7  In Example 2.4.5, $f$ is an $\alpha \mathcal{A}$-continuous function but not an $\eta \varsigma$-continuous.

Example 2.4.8  In Example 2.4.4, $f$ is an $\alpha LC$-continuous function but not an $\alpha \mathcal{A}$-continuous.

Definition 2.4.9  A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called PR-continuous if $f^{-1}(G)$ is preregular set in $X$ for each $G \in \sigma$.

Definition 2.4.10  A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\alpha$-continuous [67] iff $f^{-1}(G)$ is $\alpha$-open in $X$ for each $G \in \sigma$.

Remark 2.4.11  1. Every PR-continuous function is $\alpha C^\ast$-continuous.

2. Every $\alpha$-continuous function is $\alpha C^\ast$-continuous.

Proof. (1). Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a PR-continuous function and $G \in \sigma$. Since $f$ is PR-continuous, then $f^{-1}(G)$ is preregular set in $X$. By Remark 2.3.13, we have $f^{-1}(G) \in \alpha C^\ast(X)$. Hence, $f$ is $\alpha C^\ast$-continuous.

The proof of (2) is similar to that of (1).

Definition 2.4.12  A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\alpha \mathcal{B}$-continuous if $f^{-1}(G)$ is $\alpha \mathcal{B}$-set in $X$ for each $G \in \sigma$.  

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**Theorem 2.4.13** Let \((X, \tau)\) be an extremally disconnected space. Then

1. if \(f : (X, \tau) \to (Y, \sigma)\) is \(\alpha A\)-continuous, then \(f\) is \(\alpha C^*\)-continuous.

2. if \(f : (X, \tau) \to (Y, \sigma)\) is \(\alpha B\)-continuous, then \(f\) is \(\alpha C\)-continuous.

**Proof.** The proof is immediate from Theorem 2.3.35.

**Theorem 2.4.14** The following are equivalent for a function \(f : (X, \tau) \to (Y, \sigma)\) where \(X\) is locally indiscrete.

1. \(f\) is \(\alpha C^*\)-continuous.

2. \(f\) is \(\alpha C\)-continuous.

**Proof.** The proof is immediate from Theorem 2.3.23.