Chapter 4

Contra $\pi g\beta$-continuous functions

4.1 Introduction

In 1996, Dontchev [26] introduced a new class of functions called contra-continuous functions. He defined a function $f : X \to Y$ to be contra-continuous if the pre image of every open set of $Y$ is closed in $X$. In 2007, Caldas et al. [18] introduced and investigated the notion of contra $g$-continuity. In 1968, Zaitsev [105] introduced the notion of $\pi$-open sets as a finite union of regular open sets. This notion received a proper attention and some research articles came to existence. Dontchev and Noiri [28] introduced and investigated $\pi$-continuity and $\pi g$-continuity. Ekici and Baker [34] studied further properties of $\pi g$-closed sets and continuities. In 2007, Ekici [32] introduced and studied some new forms of continuities. In [54], Kalantan introduced and investigated $\pi$-normality. The
digital n-space is not a metric space, since it is not $T_1$. But recently Takigawa and Maki [97] showed that in the digital n-space every closed set is $\pi$-open. Recently, Ekici [31] introduced and studied contra $\pi g$-continuous functions. In 2010, Caldas et. al. [20] introduced and studied contra $\pi gp$-continuity.

In this chapter, we present a new generalization of contra-continuity called contra $\pi g\beta$-continuity. It turns out that the notion of contra $\pi g\beta$-continuity is a weaker form of contra $\pi gp$-continuity and contra $\pi g\gamma$-continuity [82].

4.2 Preliminaries

Definition 4.2.1 A subset $A$ of a space $X$ is said to be

1. $\gamma$-closed [36] if $\text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) \subseteq A$;

2. $gp$-closed [75] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$;

3. $g\beta$-closed [96] if $\beta \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$;

4. $\pi g\beta$-closed [95] if $\beta \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open in $X$.

The complements of the above closed sets are called their respective open sets.

The complements of the above open sets are called their respective closed sets.

The intersection of all $\beta$-closed sets containing $A$ is called $\beta$-closure of $A$ and is denoted by $\beta \text{cl}(A)$. 51
The family of all $\pi g \beta$-open (resp. $\pi g \beta$-closed, closed) sets of $X$ containing a point $x \in X$ is denoted by $\pi G \beta O(X, x)$ (resp. $\pi G \beta C(X, x)$, $C(X, x)$). The family of all $\pi g \beta$-open (resp. $\pi g \beta$-closed, closed, semi-open, $\beta$-open) sets of $X$ is denoted by $\pi G \beta O(X)$ (resp. $\pi G \beta C(X)$, $C(X)$, $SO(X)$, $\beta O(X)$).

Let $A$ be a subset of a space $(X, \tau)$. The set $\bigcap \{ U \in \tau : A \subseteq U \}$ is called the kernel of $A$ [68] and is denoted by $ker(A)$.

**Lemma 4.2.2** [50] The following properties hold for subsets $U$ and $V$ of a space $(X, \tau)$.

1. $x \in ker(U)$ if and only if $U \cap F \neq \emptyset$ for any closed set $F \in C(X, x)$;
2. $U \subseteq ker(U)$ and $U = ker(U)$ if $U$ is open in $X$;
3. If $U \subseteq V$, then $ker(U) \subseteq ker(V)$.

Recall that if $A$ is a subset of a space $X$, then $\beta cl(A) = A \cup int(cl(int(A)))$.

### 4.3 Contra $\pi g \beta$-continuous functions

**Definition 4.3.1** Let $A$ be a subset of a space $(X, \tau)$.

1. The set $\bigcap \{ F : F$ is $\pi g \beta$-closed in $X : A \subseteq F \}$ is called the $\pi g \beta$-closure of $A$ and is denoted by $\pi g \beta - cl(A)$.
2. The set $\bigcup \{ F : F$ is $\pi g \beta$-open in $X : A \supseteq F \}$ is called the $\pi g \beta$-interior of $A$ and is denoted by $\pi g \beta - int(A)$. 

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Lemma 4.3.2 Let $A$ be a subset of a space $(X, \tau)$, then

1. $\pi g\beta -\text{cl}(X - A) = X - \pi g\beta -\text{int}(A)$;

2. $x \in \pi g\beta -\text{cl}(A)$ if and only if $A \cap U \neq \emptyset$ for each $U \in \pi G\beta O(X, x)$;

3. If $A$ is $\pi g\beta$-closed in $X$, then $A = \pi g\beta -\text{cl}(A)$.

Remark 4.3.3 If $A = \pi g\beta -\text{cl}(A)$, then $A$ need not be a $\pi g\beta$-closed.

Example 4.3.4 Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Take $A = \{a, b\}$. Clearly $\pi g\beta -\text{cl}(A) = A$ but $A$ is not $\pi g\beta$-closed.

Definition 4.3.5 A function $f : X \to Y$ is called contra $\pi g\beta$-continuous if $f^{-1}(V)$ is $\pi g\beta$-closed in $X$ for every open set $V$ of $Y$.

Theorem 4.3.6 The following are equivalent for a function $f : X \to Y$:

1. $f$ is contra $\pi g\beta$-continuous;

2. The inverse image of every closed set of $Y$ is $\pi g\beta$-open in $X$;

3. For each $x \in X$ and each closed set $V$ in $Y$ with $f(x) \in V$, there exists a $\pi g\beta$-open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$;

4. $f(\pi g\beta -\text{cl}(A)) \subseteq \ker(f(A))$ for every subset $A$ of $X$;

5. $\pi g\beta -\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset $B$ of $Y$. 

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**Proof.** (1) ⇒ (2): Let U be any closed set of Y. Since Y/U is open, then by (1), it follows that \(f^{-1}(Y/U) = X/f^{-1}(U)\) is \(\pi g\beta\)-closed. This shows that \(f^{-1}(U)\) is \(\pi g\beta\)-open in X.

(1) ⇒ (3): Let \(x \in X\) and V be a closed set in Y with \(f(x) \in V\). By (1), it follows that \(f^{-1}(Y/V) = X/f^{-1}(V)\) is \(\pi g\beta\)-closed and so \(f^{-1}(V)\) is \(\pi g\beta\)-open. Take \(U = f^{-1}(V)\), we obtain that \(x \in U\) and \(f(U) \subseteq V\).

(3) ⇒ (2): Let V be a closed set in Y with \(x \in f^{-1}(V)\). Since \(f(x) \in V\), by (3) there exists a \(\pi g\beta\)-open set U in X containing x such that \(f(U) \subseteq V\). It follows that \(x \in U \subseteq f^{-1}(V)\). Hence \(f^{-1}(V)\) is \(\pi g\beta\)-open.

(2) ⇒ (4): Let A be any subset of X. Let \(y \notin \ker(f(A))\). Then by Lemma 4.2.2, there exist a closed set F containing y such that \(f(A) \cap F = \emptyset\). We have \(A \cap f^{-1}(F) = \emptyset\) and since \(f^{-1}(F)\) is \(\pi g\beta\)-open then we have \(\pi g\beta\)-cl(A) \(\cap f^{-1}(F) = \emptyset\). Hence we obtain \(f(\pi g\beta\text{-cl}(A)) \cap F = \emptyset\) and \(y \notin f(\pi g\beta\text{-cl}(A))\). Thus \(f(\pi g\beta\text{-cl}(A)) \subseteq \ker(f(A))\).

(4) ⇒ (5): Let B be any subset of Y. By (4), \(f(\pi g\beta\text{-cl}(f^{-1}(B))) \subseteq \ker(B)\) and \(\pi g\beta\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B))\).

(5) ⇒ (1): Let B be any open set of Y. By (5), \(\pi g\beta\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B)) = f^{-1}(B)\) and \(\pi g\beta\text{-cl}(f^{-1}(B)) = f^{-1}(B)\). So we obtain that \(f^{-1}(B)\) is \(\pi g\beta\)-closed in X.

**Definition 4.3.7** A function \(f : X \to Y\) is said to be

1. completely continuous [13] if \(f^{-1}(V)\) is regular open in X for every open set V of Y;
2. contra-continuous [26] (resp. contra pre-continuous [49], contra $\beta$-continuous [19]) if $f^{-1}(V)$ is closed (resp. pre-closed, $\beta$-closed) in $X$ for every open set $V$ of $Y$;

3. contra $g$-continuous [18] (resp. contra gp-continuous [20], contra $g\beta$-continuous) if $f^{-1}(V)$ is $g$-closed (resp. gp-closed, $g\beta$-closed) in $X$ for every open set $V$ of $Y$;

4. contra $\pi$-continuous [20] (resp. contra $\pi g$-continuous [31], contra $\pi gp$-continuous [20], contra $\pi g\gamma$-continuous [82]) if $f^{-1}(V)$ is $\pi$-closed (resp. $\pi g$-closed, $\pi gp$-closed, $\pi g\gamma$-closed) in $X$ for every open set $V$ of $Y$.

For the functions defined above, we have the following implications:

\[
\begin{align*}
\text{contra } \pi\text{-continuity} & \downarrow \\
\text{contra-continuity} & \quad \rightarrow \quad \text{contra pre-continuity} \\
\downarrow & \quad \downarrow \\
\text{contra } g\text{-continuity} & \quad \rightarrow \quad \text{contra gp-continuity} \\
\downarrow & \quad \downarrow \\
\text{contra } \pi g\text{-continuity} & \quad \rightarrow \quad \text{contra } \pi gp\text{-continuity} \\
\downarrow & \quad \downarrow \\
\text{contra } \pi g\gamma\text{-continuity} & \quad \rightarrow \quad \text{contra } \pi g\beta\text{-continuity}
\end{align*}
\]

Remark 4.3.8 None of these implications is reversible as shown by the following Examples and the related papers [20, 82].
Example 4.3.9 Let $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, X, \{a\}, \{e\}, \{a, e\}, \{c, d\}, \{a, c, d\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$ and $\sigma = \{\emptyset, X, \{d\}, \{e\}, \{d, e\}\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra $\pi g\beta$-continuous but not contra $\pi g\gamma$-continuous.

Example 4.3.10 In Example 4.3.9, the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra $\pi g\beta$-continuous but not contra $\pi g\beta$-continuous.

Definition 4.3.11 A function $f : X \rightarrow Y$ is said to be

1. $\pi g\beta$-semiopen if $f(U) \in SO(Y)$ for every $\pi g\beta$-open set $U$ of $X$;

2. contra-I($\pi g\beta$)-continuous if for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in \pi G\beta O(X, x)$ such that $\text{int}(f(U)) \subseteq F$.

3. $\pi$-continuous [28] if $f^{-1}(F)$ is $\pi$-closed in $X$ for every closed set $F$ of $Y$;

4. $\pi g\beta$-continuous [95] if $f^{-1}(F)$ is $\pi g\beta$-closed in $X$ for every closed set $F$ of $Y$.

Theorem 4.3.12 If a function $f : X \rightarrow Y$ is contra-I($\pi g\beta$)-continuous and $\pi g\beta$-semiopen, then $f$ is contra $\pi g\beta$-continuous.

Proof. Suppose that $x \in X$ and $F \in C(Y, f(x))$. Since $f$ is contra-I($\pi g\beta$)-continuous, there exists $U \in \pi G\beta O(X, x)$ such that $\text{int}(f(U)) \subseteq F$. By hypothesis $f$ is $\pi g\beta$-semiopen, therefore $f(U) \in SO(Y)$ and $f(U) \subseteq \text{cl(\text{int}(f(U)))} \subseteq F$. This shows that $f$ is contra $\pi g\beta$-continuous.

Lemma 4.3.13 [95] For a subset $A$ of $(X, \tau)$, the following statements are equivalent.
1. A is $\pi$-open and $\pi\beta$-closed;

2. A is regular open.

**Lemma 4.3.14** [20] A function $f : X \to Y$ is $\pi$-continuous if and only if $f^{-1}(V)$ is $\pi$-open in $X$ for every open set $V$ of $Y$.

**Theorem 4.3.15** For a function $f : X \to Y$, the following statements are equivalent.

1. $f$ is contra $\pi\beta$-continuous and $\pi$-continuous;

2. $f$ is completely continuous.

**Proof.** (1) $\Rightarrow$ (2): Let $U$ be an open set in $Y$. Since $f$ is contra $\pi\beta$-continuous and $\pi$-continuous, $f^{-1}(U)$ is $\pi\beta$-closed and $\pi$-open, by Lemma 4.3.13, $f^{-1}(U)$ is regular open. Then $f$ is completely continuous.

(2) $\Rightarrow$ (1): Let $U$ be an open set in $Y$. Since $f$ is completely continuous, $f^{-1}(U)$ is regular open, by Lemma 4.3.13, $f^{-1}(U)$ is $\pi\beta$-closed and $\pi$-open. Then $f$ is contra $\pi\beta$-continuous and $\pi$-continuous.

**Definition 4.3.16** 1. A subset $A$ of a topological space $X$ is said to be Q-set [61] if $\text{int}(\text{cl}(A)) = \text{cl}(\text{int}(A))$.

2. Let $f : X \to Y$ be a function. Then $f$ is called Q-continuous [98] (resp. $\pi$-perfectly continuous) if $f^{-1}(U)$ is Q-set (resp. $\pi$-clopen) in $X$ for each open set $U$ of $Y$.

**Lemma 4.3.17** [95] For a subset $A$ of $X$, the following statements are equivalent:
1. $A$ is $\pi$-clopen,

2. $A$ is $\pi$-open, $Q$-set and $\pi g\beta$-closed.

**Theorem 4.3.18** For a function $f : X \to Y$, the following statements are equivalent.

1. $f$ is $\pi$-perfectly continuous;

2. $f$ is $\pi$-continuous, $Q$-continuous and contra $\pi g\beta$-continuous.

**Proof.** It is obtained from the above Lemma.

**Theorem 4.3.19** If a function $f : X \to Y$ is contra $\pi g\beta$-continuous and $Y$ is regular, then $f$ is $\pi g\beta$-continuous.

**Proof.** Let $x$ be an arbitrary point of $X$ and $U$ be an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $W$ in $Y$ containing $f(x)$ such that $\text{cl}(W) \subseteq U$. Since $f$ is contra $\pi g\beta$-continuous, there exists $V \in \pi G\beta O(X, x)$ such that $f(V) \subseteq \text{cl}(W)$. Then $f(V) \subseteq \text{cl}(W) \subseteq U$. Hence $f$ is $\pi g\beta$-continuous.

**Theorem 4.3.20** Let $\{X_i : i \in \Omega\}$ be any family of topological spaces. If a function $f : X \to \prod X_i$ is contra $\pi g\beta$-continuous, then $Pr_i \circ f : X \to X_i$ is contra $\pi g\beta$-continuous for each $i \in \Omega$, where $Pr_i$ is the projection of $\prod X_i$ onto $X_i$.

**Proof.** For a fixed $i \in \Omega$, let $V_i$ be any open set of $X_i$. Since $Pr_i$ is continuous, $Pr_i^{-1}(V_i)$ is open in $\prod X_i$. Since $f$ is contra $\pi g\beta$-continuous, $f^{-1}(Pr_i^{-1}(V_i)) = (Pr_i \circ f)^{-1}(V_i)$ is $\pi g\beta$-closed in $X$. Therefore, $Pr_i \circ f$ is contra $\pi g\beta$-continuous for each $i \in \Omega$.
Theorem 4.3.21 Let \( f : X \to Y \) and \( g : Y \to Z \) be a function. Then the following hold:

1. If \( f \) is contra \( \pi g \beta \)-continuous and \( g \) is continuous, then \( g \circ f : X \to Z \) is contra \( \pi g \beta \)-continuous;

2. If \( f \) is \( \pi g \beta \)-continuous and \( g \) is contra-continuous, then \( g \circ f : X \to Z \) is contra \( \pi g \beta \)-continuous;

3. If \( f \) is contra \( \pi g \beta \)-continuous and \( g \) is contra-continuous, then \( g \circ f : X \to Z \) is \( \pi g \beta \)-continuous.

Definition 4.3.22 A space \((X, \tau)\) is called \( \pi g \beta - T_{1/2}\) [95] if every \( \pi g \beta \)-closed set is \( \beta \)-closed.

Remark 4.3.23 Every contra \( \pi g \beta \)-continuous function defined on a \( \pi g \beta - T_{1/2}\) space is contra \( \beta \)-continuous.

Theorem 4.3.24 Let \( f : X \to Y \) be a function. Suppose that \( X \) is a \( \pi g \beta - T_{1/2} \) space. Then the following are equivalent.

1. \( f \) is contra \( \pi g \beta \)-continuous;

2. \( f \) is contra \( g \beta \)-continuous;

3. \( f \) is contra \( \beta \)-continuous.

Proof. Obvious.

Definition 4.3.25 For a space \((X, \tau)\), \( \pi T^\beta = \{U \subseteq X : \pi g \beta - cl(X \setminus U) = X \setminus U\} \).
**Theorem 4.3.26** Let \((X, \tau)\) be a space. Then

1. Every \(\pi g \beta\)-closed set is \(\beta\)-closed (i.e. \((X, \tau)\) is \(\pi g \beta\)-\(T_{1/2}\)) if and only if \(\pi \tau^\beta = \beta O(X)\);

2. Every \(\pi g \beta\)-closed set is closed if and only if \(\pi \tau^\beta = \tau\).

**Proof.** (1) Let \(A \in \pi \tau^\beta\). Then \(\pi g \beta\)-\text{cl}(X \setminus A) = X \setminus A\). By hypothesis, \(\beta\text{cl}(X \setminus A) = \pi g \beta\)-\text{cl}(X \setminus A) = X \setminus A\) and hence \(A \in \beta O(X)\).

Conversely, let \(A\) be a \(\pi g \beta\)-closed set. Then \(\pi g \beta\)-\text{cl}(A) = A\) and hence \(X \setminus A \in \pi \tau^\beta = \beta O(X)\), i.e. \(A\) is \(\beta\)-closed.

(2) Similar to (1).

**Theorem 4.3.27** If \(\pi \tau^\beta = \tau\) in \(X\), then for a function \(f : X \to Y\) the following are equivalent:

1. \(f\) is contra \(\pi g \beta\)-continuous;

2. \(f\) is contra \(\pi g \gamma\)-continuous;

3. \(f\) is contra \(\pi g\)-continuous;

4. \(f\) is contra \(g\)-continuous;

5. \(f\) is contra-continuous.

**Proof.** Obvious.
4.4 Properties of Contra $\pi g\beta$-continuous functions

**Definition 4.4.1** A space $X$ is said to be $\pi g\beta$-$T_1$ if for each pair of distinct points $x$ and $y$ in $X$, there exist $\pi g\beta$-open sets $U$ and $V$ containing $x$ and $y$ respectively, such that $y \notin U$ and $x \notin V$.

**Definition 4.4.2** [83] A space $X$ is said to be $\pi g\beta$-$T_2$ if for each pair of distinct points $x$ and $y$ in $X$, there exist $U \in \pi G\beta O(X, x)$ and $V \in \pi G\beta O(X, y)$ such that $U \cap V = \emptyset$.

**Theorem 4.4.3** Let $X$ be a topological space. Suppose that for each pair of distinct points $x_1$ and $x_2$ in $X$, there exists a function $f$ of $X$ into a Urysohn space $Y$ such that $f(x_1) \neq f(x_2)$. Moreover, let $f$ be contra $\pi g\beta$-continuous at $x_1$ and $x_2$. Then $X$ is $\pi g\beta$-$T_2$.

**Proof.** Let $x_1$ and $x_2$ be any distinct points in $X$. Then suppose that there exist an Urysohn space $Y$ and a function $f : X \rightarrow Y$ such that $f(x_1) \neq f(x_2)$ and $f$ is contra $\pi g\beta$-continuous at $x_1$ and $x_2$. Let $w = f(x_1)$ and $z = f(x_2)$. Then $w \neq z$. Since $Y$ is Urysohn, there exist open sets $U$ and $V$ containing $w$ and $z$, respectively such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since $f$ is contra $\pi g\beta$-continuous at $x_1$ and $x_2$, then there exist $\pi g\beta$-open sets $A$ and $B$ containing $x_1$ and $x_2$, respectively such that $f(A) \subseteq \text{cl}(U)$ and $f(B) \subseteq \text{cl}(V)$. So we have $A \cap B = \emptyset$ since $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Hence, $X$ is $\pi g\beta$-$T_2$.

**Corollary 4.4.4** If $f$ is a contra $\pi g\beta$-continuous injection of a topological space $X$ into a Urysohn space $Y$, then $X$ is $\pi g\beta$-$T_2$. 
Proof. For each pair of distinct points $x_1$ and $x_2$ in $X$ and $f$ is contra $\pi g\beta$-continuous function of $X$ into a Urysohn space $Y$ such that $f(x_1) \neq f(x_2)$ because $f$ is injective. Hence by Theorem 4.4.3, $X$ is $\pi g\beta$-$T_2$.

Definition 4.4.5 A space $(X, \tau)$ is said to be $\pi g\beta$-connected if $X$ cannot be expressed as the disjoint union of two non-empty $\pi g\beta$-open sets.

Remark 4.4.6 Every $\pi g\beta$-connected space is connected.

Theorem 4.4.7 For a space $X$, the following are equivalent:

1. $X$ is $\pi g\beta$-connected;

2. The only subsets of $X$ which are both $\pi g\beta$-open and $\pi g\beta$-closed are the empty set $\emptyset$ and $X$;

3. Each contra $\pi g\beta$-continuous function of $X$ into a discrete space $Y$ with at least two points is a constant function.

Proof. (1) $\Rightarrow$ (2): Suppose $S \subset X$ is a proper subset which is both $\pi g\beta$-open and $\pi g\beta$-closed. Then its complement $X - S$ is also $\pi g\beta$-open and $\pi g\beta$-closed. Then $X = S \cup (X - S)$, a disjoint union of two non-empty $\pi g\beta$-open sets which contradicts the fact that $X$ is $\pi g\beta$-connected. Hence, $S = \emptyset$ or $X$.

(2) $\Rightarrow$ (1): Suppose $X = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and $A$ and $B$ are $\pi g\beta$-open. Since $A = X - B$, $A$ is $\pi g\beta$-closed. But by assumption $A = \emptyset$ or $X$, which is a contradiction. Hence (1) holds.
(2) ⇒ (3): Let $f : X \to Y$ be contra $\pi g \beta$-continuous function where $Y$ is a discrete space with at least two points. Then $f^{-1}(\{y\})$ is $\pi g \beta$-closed and $\pi g \beta$-open for each $y \in Y$ and $X = \bigcup \{f^{-1}(y) : y \in Y\}$. By hypothesis, $f^{-1}(\{y\}) = \emptyset$ or $X$. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, then $f$ is not a function. Also there cannot exist more than one $y \in Y$ such that $f^{-1}(\{y\}) = X$. Hence there exists only one $y \in Y$ such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y_1\}) = \emptyset$ where $y \neq y_1 \in Y$. This shows that $f$ is a constant function.

(3) ⇒ (2): Let $P$ be a non-empty set which is both $\pi g \beta$-open and $\pi g \beta$-closed in $X$. Suppose $f : X \to Y$ is a contra $\pi g \beta$-continuous function defined by $f(P) = \{a\}$ and $f(X \setminus P) = \{b\}$ where $a \neq b$ and $a, b \in Y$. By hypothesis, $f$ is constant. Therefore $P = X$.

**Definition 4.4.8** A subset $A$ of a space $(X, \tau)$ is said to be $\pi g \beta$-clopen if $A$ is both $\pi g \beta$-open and $\pi g \beta$-closed.

**Theorem 4.4.9** If $f$ is a contra $\pi g \beta$-continuous function from a $\pi g \beta$-connected space $X$ onto any space $Y$, then $Y$ is not a discrete space.

**Proof.** Suppose that $Y$ is discrete. Let $A$ be a proper non-empty open and closed subset of $Y$. Then $f^{-1}(A)$ is a proper non-empty $\pi g \beta$-clopen subset of $X$ which is a contradiction to the fact that $X$ is $\pi g \beta$-connected.

**Theorem 4.4.10** If $f : X \to Y$ is a contra $\pi g \beta$-continuous surjection and $X$ is $\pi g \beta$-connected, then $Y$ is connected.
Proof. Suppose that $Y$ is not a connected space. There exist non-empty disjoint open sets $U_1$ and $U_2$ such that $Y = U_1 \cup U_2$. Therefore $U_1$ and $U_2$ are clopen in $Y$. Since $f$ is contra $\pi\beta$-continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $\pi\beta$-open in $X$. Moreover, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are non-empty disjoint and $X = f^{-1}(U_1) \cup f^{-1}(U_2)$. This shows that $X$ is not $\pi\beta$-connected. This contradicts that $Y$ is not connected assumed. Hence $Y$ is connected.

Definition 4.4.11 The graph $G(f)$ of a function $f : X \to Y$ is said to be contra $\pi\beta$-graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a $\pi\beta$-open set $U$ in $X$ containing $x$ and a closed set $V$ in $Y$ containing $y$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.4.12 A graph $G(f)$ of a function $f : X \to Y$ is contra $\pi\beta$-graph in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a $U \in \pi\beta G(O(X))$ containing $x$ and $V \in C(Y)$ containing $y$ such that $f(U) \cap V = \emptyset$.

Theorem 4.4.13 If $f : X \to Y$ is contra $\pi\beta$-continuous and $Y$ is Urysohn, $G(f)$ is contra $\pi\beta$-graph in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. It follows that $f(x) \neq y$. Since $Y$ is Urysohn, there exist open sets $V$ and $W$ such that $f(x) \in V$, $y \in W$ and $\cl(V) \cap \cl(W) = \emptyset$.

Since $f$ is contra $\pi\beta$-continuous, there exist a $U \in \pi\beta G(O(X, x))$ such that $f(U) \subseteq \cl(V)$ and $f(U) \cap \cl(W) = \emptyset$. Hence $G(f)$ is contra $\pi\beta$-graph in $X \times Y$.

Theorem 4.4.14 Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function
of $f$, defined by $g(x) = (x, f(x))$ for every $x \in X$. If $g$ is contra $\pi g_\beta$-continuous, then $f$ is contra $\pi g_\beta$-continuous.

**Proof.** Let $U$ be an open set in $Y$, then $X \times U$ is an open set in $X \times Y$. It follows that $f^{-1}(U) = g^{-1}(X \times U) \in \pi G_\beta C(X)$. Thus $f$ is contra $\pi g_\beta$-continuous.

**Theorem 4.4.15** If $A$ and $B$ are $\pi g_\beta$-closed sets in submaximal and extremally disconnected space $(X, \tau)$, then $A \cup B$ is $\pi g_\beta$-closed.

**Proof.** Let $A \cup B \subseteq U$ and $U$ be $\pi$-open in $(X, \tau)$. Since $A, B \subseteq U$ and $A$ and $B$ are $\pi g_\beta$-closed, $\beta \text{cl}(A) \subseteq U$ and $\beta \text{cl}(B) \subseteq U$. Since $(X, \tau)$ is submaximal and extremally disconnected, $\beta \text{cl}(F) = \text{cl}(F)$ for any set $F \subseteq X$. Now $\beta \text{cl}(A \cup B) = \beta \text{cl}(A) \cup \beta \text{cl}(B) \subseteq U$. Hence $A \cup B$ is $\pi g_\beta$-closed.

**Lemma 4.4.16** Let $(X, \tau)$ be a topological space. If $U, V \in \pi G_\beta O(X)$ and $X$ is submaximal and extremally disconnected space, then $U \cap V \in \pi G_\beta O(X)$.

**Proof.** Let $U, V \in \pi G_\beta O(X)$. We have $X \setminus U, X \setminus V \in \pi G_\beta C(X)$. By Theorem 4.4.15, $(X \setminus U) \cup (X \setminus V) = X \setminus (U \cap V) \in \pi G_\beta C(X)$. Thus, $U \cap V \in \pi G_\beta O(X)$.

**Theorem 4.4.17** If $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are contra $\pi g_\beta$-continuous, $X$ is submaximal and extremally disconnected and $Y$ is Urysohn, then $K = \{x \in X : f(x) = g(x)\}$ is $\pi g_\beta$-closed in $X$.

**Proof.** Let $x \in X \setminus K$. Then $f(x) \neq g(x)$. Since $Y$ is Urysohn, there exist open sets $U$ and $V$ such that $f(x) \in U$, $g(x) \in V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since $f$ and $g$ are
contra $\pi g\beta$-continuous, $f^{-1}(\text{cl}(U)) \in \pi G\beta O(X)$ and $g^{-1}(\text{cl}(V)) \in \pi G\beta O(X)$. Let $A = f^{-1}(\text{cl}(U))$ and $B = g^{-1}(\text{cl}(V))$. Then $A$ and $B$ contains $x$. Set $C = A \cap B$. $C$ is $\pi g\beta$-open in $X$. Hence $f(C) \cap g(C) = \emptyset$ and $x \not\in \pi g\beta$-cl($K$). Thus $K$ is $\pi g\beta$-closed in $X$.

**Definition 4.4.18** A subset $A$ of a topological space $X$ is said to be $\pi g\beta$-dense in $X$ if $\pi g\beta$-cl($A$) = $X$.

**Theorem 4.4.19** Let $f : X \to Y$ and $g : X \to Y$ be contra $\pi g\beta$-continuous. If $Y$ is Urysohn and $f = g$ on a $\pi g\beta$-dense set $A \subseteq X$, then $f = g$ on $X$.

**Proof.** Since $f$ and $g$ are contra $\pi g\beta$-continuous and $Y$ is Urysohn, by Theorem 4.4.17, $K = \{x \in X : f(x) = g(x)\}$ is $\pi g\beta$-closed in $X$. We have $f = g$ on $\pi g\beta$-dense set $A \subseteq X$. Since $A \subseteq K$ and $A$ is $\pi g\beta$-dense set in $X$, then $X = \pi g\beta$-cl($A$) $\subseteq \pi g\beta$-cl($K$) = $K$. Hence, $f = g$ on $X$.

**Definition 4.4.20** A space $X$ is said to be weakly Hausdroff [90] if each element of $X$ is an intersection of regular closed sets.

**Theorem 4.4.21** If $f : X \to Y$ is a contra $\pi g\beta$-continuous injection and $Y$ is weakly Hausdroff, then $X$ is $\pi g\beta$-$T_1$.

**Proof.** Suppose that $Y$ is weakly Hausdroff. For any distinct points $x_1$ and $x_2$ in $X$, there exist regular closed sets $U$ and $V$ in $Y$ such that $f(x_1) \in U$, $f(x_2) \not\in U$, $f(x_1) \not\in V$ and $f(x_2) \in V$. Since $f$ is contra $\pi g\beta$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are
πgβ-open subsets of X such that \( x_1 \in f^{-1}(U), x_2 \notin f^{-1}(U), x_1 \notin f^{-1}(V) \) and \( x_2 \in f^{-1}(V) \). This shows that X is \( \pi g\beta-T_1 \).

**Theorem 4.4.22** Let \( f : X \to Y \) have a contra \( \pi g\beta \)-graph. If \( f \) is injective, then X is \( \pi g\beta-T_1 \).

**Proof.** Let \( x_1 \) and \( x_2 \) be any two distinct points of X. Then, we have \((x_1, f(x_2)) \in (X \times Y) \setminus G(f)\). Then, there exist a \( \pi g\beta \)-open set U in X containing \( x_1 \) and \( F \in C(Y, f(x_2)) \) such that \( f(U) \cap F = \emptyset \). Hence \( U \cap f^{-1}(F) = \emptyset \). Therefore, we have \( x_2 \notin U \). This implies that X is \( \pi g\beta-T_1 \).

**Definition 4.4.23** A topological space X is said to be Ultra Hausdroff [92] if for each pair of distinct points x and y in X, there exist clopen sets A and B containing x and y, respectively such that \( A \cap B = \emptyset \).

**Theorem 4.4.24** Let \( f : X \to Y \) be a contra \( \pi g\beta \)-continuous injection. If Y is an Ultra Hausdroff space, then X is \( \pi g\beta-T_2 \).

**Proof.** Let \( x_1 \) and \( x_2 \) be any distinct points in X, then \( f(x_1) \neq f(x_2) \) and there exist clopen sets U and V containing \( f(x_1) \) and \( f(x_2) \) respectively, such that \( U \cap V = \emptyset \). Since f is contra \( \pi g\beta \)-continuous, then \( f^{-1}(U) \in \pi G\beta O(X) \) and \( f^{-1}(V) \in \pi G\beta O(X) \) such that \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Hence, X is \( \pi g\beta-T_2 \).

**Definition 4.4.25** A topological space X is said to be
1. πgβ-normal if each pair of non-empty disjoint closed sets can be separated by disjoint πgβ-open sets.

2. Ultra normal [92] if for each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 4.4.26 If \( f : X \rightarrow Y \) is a contra πgβ-continuous, closed injection and \( Y \) is Ultra normal, then \( X \) is πgβ-normal.

Proof. Let \( F_1 \) and \( F_2 \) be disjoint closed subsets of \( X \). Since \( f \) is closed and injective, \( f(F_1) \) and \( f(F_2) \) are disjoint closed subsets of \( Y \). Since \( Y \) is Ultra normal, \( f(F_1) \) and \( f(F_2) \) are separated by disjoint clopen sets \( V_1 \) and \( V_2 \), respectively. Hence \( F_i \subseteq f^{-1}(V_i), f^{-1}(V_i) \in \piGBO(X, x) \) for \( i = 1, 2 \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \) and thus \( X \) is πgβ-normal.