Chapter 3

Bi-Clique Coloring in graphs

In this chapter, a new type of coloring is defined and studied.

3.1 Introduction

Vertex coloring has attracted the attention of many mathematicians. Basically it started with coloring different regions of a map such that no two adjacent regions are colored the same. This led to proper coloring of the vertices of a graph. In this coloring, adjacent vertices receive different colors. The result that the vertices of a planar graph can be colored with atmost four colors is an interesting conjecture which remained unsolved for more than 100 years. Now it is a celebrated theorem. Proper vertex coloring induces a partition of the vertex set into independent sets. The chromatic number of a graph is the minimum cardinality of a partition of the vertex set into independent sets. Various types of coloring have been defined in the past like Achromatic coloring, Grundy coloring, B-coloring, Dominator coloring etc. In this chapter, a new type of coloring is introduced. In this coloring each color class is either a bi-clique or an independent set. Clearly, the bi-clique Chromatic number is less than or equal to the Chromatic number. The bi-clique chromatic number of standard graphs are found. Graphs with bi-clique number 2 are characterized.

3.2 Bi-clique Chromatic Number for Standard graphs

Definition 3.2.1 *A partition of $V(G)$ into sets $V_1, V_2, ..., V_k$ such that $< V_i >$ is a bi-clique or an independent set is called a bi-clique coloring of G. The existence is guaranteed since the partition containing singleton subsets for each vertex is a bi-clique coloring of G. The minimum cardinality of a bi-clique coloring of G is denoted by $\chi_{bc}(G)$ and is called the bi-clique chromatic number of G.*

Remark 3.2.2 *Any proper coloring of G is a bi-clique coloring of G.*

Hence $\chi_{bc}(G) \leq \chi(G)$.

Proposition 3.2.3 $1 \leq \chi_{bc}(G) \leq n$.

BI-CLIQUE CHROMATIC NUMBER OF SOME GRAPHS.

(a) $\chi_{bc}(K_n) = \lceil \frac{n}{2} \rceil \; \forall \; n \geq 1$.

(b) $\chi_{bc}(K_{m,n}) = 1 \; \forall \; m, n \geq 1$.

(c) $\chi_{bc}(P_n) = 2$.

(d) $\chi_{bc}(C_n) = \begin{cases} 2 & \text{if } n \textit{ is even} \\ 3 & \text{if } n \textit{ is odd.} \end{cases}$

(e) $\chi_{bc}(W_{1,n}) = \begin{cases} 2 & \text{if } n \textit{ is even} \\ 3 & \text{if } n \textit{ is odd.} \end{cases}$

(f) $\chi_{bc}(T) = 2$, where T is a tree.

(g) $\chi_{bc}(D_{r,s}) = 2 \; \forall \; r, s \geq 1$.

(h) $\chi_{bc}(\overline{K_n}) = 1 \; \forall \; n \geq 1$.

(i) $\chi_{bc}(P) = 3$, where P is a Peterson graph.

Proposition 3.2.4 *Let k be any positive integer. Then there exists a connected graph G such that* $\chi(G) - \chi_{bc}(G) = k$.

For: Case (i) k is even.

$\chi(K_{2k}) = 2k$ *and* $\chi_{bc}(K_{2k}) = \lceil \frac{2k}{k} \rceil = k$.

Therefore $\chi(K_{2k}) - \chi_{bc}(K_{2k}) = k$.

Case (ii) k is odd.

Let $k = 2a + 3$, $a \geq 0$

$\chi(K_{2k}) = 2k = 2(2a+3) = 4a+6$ *and* $\chi_{bc}(K_{2k}) = \lceil \frac{2k}{k} \rceil = k = 2a+3$.

Therefore $\chi(K_{2k}) - \chi_{bc}(K_{2k}) = 4a + 6 - 2a - 3 = 2a + 3 = k$.

Example 3.2.5 $\chi_{bc}(K_8) = 4$ *and* $\chi(K_8) = 8$.

Therefore $\chi_{bc}(K_8 \cup K_8) = 8 = \chi(K_8 \cup K_8) = \chi(K_8)$.

Remark 3.2.6 $\chi_{bc}(G \cup G) \leq \chi(G \cup G) = \chi(G)$.

Remark 3.2.7 *Suppose* $\chi_{bc}(G) = \chi(G)$. *Then* $\chi_{bc}(G \cup G) = \chi(G \cup G)$ $= \chi(G)$.

Theorem 3.2.8 *Let G be a graph. Then* $\chi_{bc}(G) = \chi(G) - k$ *where k is the minimum number of bi-clique sets in any* χ_{bc} *- partition of G.*

Proof:- Suppose in a χ_{bc} *- partition with minimum number of bi-clique sets, there are t independent sets*$(t \geq 0)$. *Therefore* $\chi(G) \leq 2k + t$.

Thus $\chi(G) - k \leq k + t = \chi_{bc}(G)$.

Hence $\chi_{bc}(G) \geq \chi(G) - k$.

Suppose $\chi_{bc}(G) > \chi(G) - k$. *Then* $\chi(G) < \chi_{bc}(G) + k = (k+t)+k =$

$2k + t$. Hence, there exists a chromatic partition which admits atmost $2k+t-1$ elements. This implies there are $k-1$ bi-clique sets in a χ_{bc} - partition such that there are $t+1$ independent sets, a contradiction. Hence $\chi_{bc}(G) = \chi(G) - k$ where k is the minimum number of bi-clique sets in any χ_{bc} - partition of G.

Remark 3.2.9 *Let G be a simple graph in which k the minimum number of bi-clique sets in any χ_{bc} - partition of G. Then $\chi_{bc}(G \cup G) = 2k + t$ where t is the number of independent sets in a χ_{bc} - partition containing k bi-clique sets.*

Remark 3.2.10 *Suppose $\chi_{bc}(G) < \chi(G)$. Then G admits a bi-clique partition in which not all of them are independent. Let the minimum number of bi-clique sets in any χ_{bc} - partition of G be k. Then $\chi_{bc}(G)$ $= \chi(G) - k$.*

Therefore $\chi_{bc}(G \cup G) = 2k + t = 2k + (\chi_{bc}(G) - k) = \chi_{bc}(G) + k = \chi(G)$.

Remark 3.2.11 *Combining Remarks 3.2.7 and 3.2.10, we get that $\chi_{bc}(G \cup G) = \chi(G)$.*

Theorem 3.2.12 *Given a positive integer $k \geq 3$, the problem of deciding whether $\chi_{bc}(G) \geq k$ is NP - Complete for any graph G with $\chi_{bc}(G) \geq 3$.*

Proof:- Let G be a graph with $\chi(G) \geq 3$. Let $G_1 = G \cup G$. Then $\chi_{bc}(G_1) = \chi_{bc}(G \cup G) = \chi(G)$.

Hence the theorem.

Definition 3.2.13 *Let G be a simple graph. A partition of $V(G)$ into subsets $V_1, V_2, ..., V_k$ such that each $< V_i >$ contains a bi-clique is called a weak bi-clique coloring of G. $\pi = \{V\}$ is a weak bi-clique coloring of G and hence the existence of n weak bi-clique coloring is guaranteed. The maximum cardinality of a weak bi-clique coloring partition of G is called weak bi-clique coloring number of G and is denoted by $\chi_{wbc}(G)$.*

Remark 3.2.14 *If $\pi = \{V_1, V_2, ..., V_k\}$ is a χ_{bc} - coloring of G and if V_i is independent and V_j is independent then there exists an edge between V_i and V_j and $V_i \cup V_j$ is not a bi-clique.*

Definition 3.2.15 *Let $G = (V, E)$ be a simple graph. Let $\pi = \{V_1, V_2, ..., V_k\}$ be a partition of $V(G)$ into subsets such that each $< V_i >$ is a bi-clique or independent and $V_i \cup V_j$ is neither bi-clique nor independent. When π*

has maximum cardinality, then π is called a complete bi-clique coloring of G. The cardinality of a complete bi-clique coloring of G is denoted by $\psi_{bc}(G)$.

Remark 3.2.16 *Every χ_{bc} - partition of G is a complete bi-clique partition in the sense that the union of any two classes is neither independent nor a bi-clique.*

Remark 3.2.17 *Given any positive integer k there exists a connected graph G such that $\psi_{bc}(G) - \chi_{bc}(G) = k$.*

For: Case(i) k is odd.

Let $G = W_{1,3k+1}$. Then $\chi_{bc}(G) = 2$ and $\psi_{bc}(G) = k + 2$.

Therefore $\psi_{bc}(G) - \chi_{bc}(G) = k + 2 - 2 = k$.

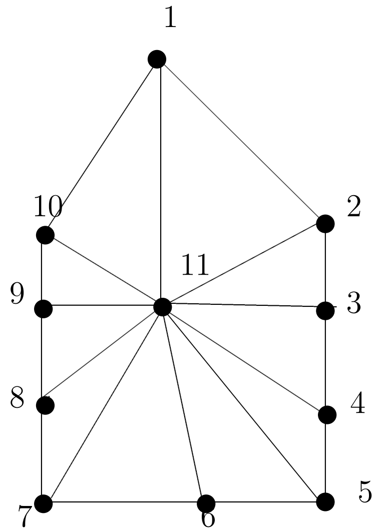
Case(ii) k is even.

Add a vertex u to $G = W_{1,3k+1}$ and make u adjacent with the central vertex of G and $(3k + 1)^{th}$ vertex of the cycle of G. Then $\chi_{bc}(G) = 3$ and $\psi_{bc}(G) = k + 3$.

Therefore $\psi_{bc}(G) - \chi_{bc}(G) = k + 3 - 3 = k$.

Illustration 3.2.18 *Case(i) $k = 3$(odd)*

Therefore $3k + 1 = 10$.

$G =$ 

$V_1' = \{1, 3, 5, 7, 9, 11\}$ *and* $V_2' = \{2, 4, 6, 8, 10\}$.

Therefore $\{V_1', V_2'\}$ *is a bi-clique coloring of* G.

Thus $\chi_{bc}(G) = 2$.

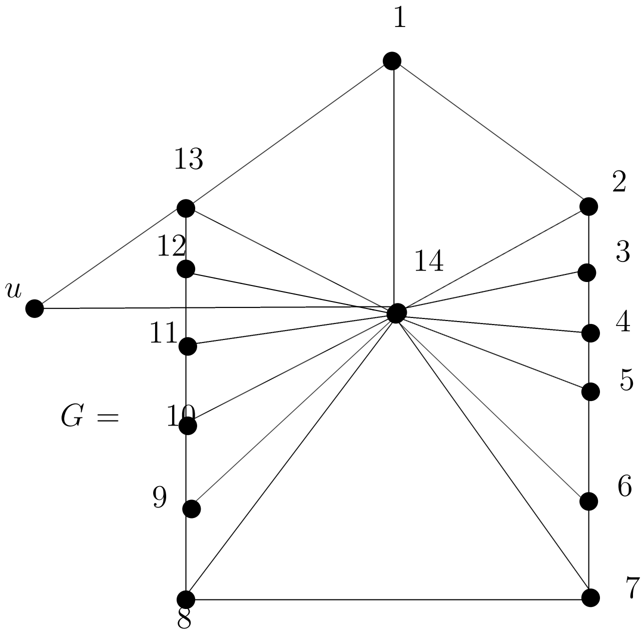
$V_1 = \{1, 2, 3\}$, $V_2 = \{4, 5, 6\}$, $V_3 = \{7, 8, 9\}$, $V_4 = \{10\}$ *and* $V_5 = \{11\}$.

Therefore $\{V_1, V_2, V_3, V_4, V_5\}$ *is a complete bi-clique coloring of* G. *Thus*

$\psi_{bc}(G) = 5$. $\psi_{bc}(G) - \chi_{bc}(G) = 5 - 2 = 3 = k$.

Case(ii) $k = 4$*(even)*

Therefore $3k + 1 = 13$.



$V_1' = \{1, 3, 5, 7, 9, 11, 14\}$, $V_2' = \{2, 4, 6, 8, 10, 12, u\}$ *and*

$V_3' = \{13\}$. *Therefore* $\{V_1', V_2', V_3'\}$ *is a bi-clique coloring of* G. *Hence*

$\chi_{bc}(G) = 3$.

$V_1 = \{1, 2, 3\}$, $V_2 = \{4, 5, 6\}$, $V_3 = \{7, 8, 9\}$, $V_4 = \{10, 11, 12\}$,

$V_5 = \{13\}$, $V_6 = \{14\}$, $V_7 = \{u\}$.

Therefore $\{V_1, V_2, V_3, V_4, V_5, V_6, V_7\}$ *is a complete bi-clique coloring of* G. *Hence* $\psi_{bc}(G) = 7$. $\psi_{bc}(G) - \chi_{bc}(G) = 7 - 3 = 4 = k$.

Proposition 3.2.19 *For any bipartite graph* G *which is not complete,* $\chi_{bc}(G) = \chi(G) = 2$.

Proposition 3.2.20 $\chi_{bc}(G) = 1$ *if and only if* $G = \overline{K_n}$ *or* G *is a* *complete bipartite graph.*

Observation 3.2.21 *Let $G \neq \overline{K_n}$. Suppose $\chi_{bc}(G) \leq 2$. Then $\chi_{bc}(G)$ $= 1$ if G is complete bipartite and 2 otherwise [since $\chi_{bc} \leq \chi$].*

Proposition 3.2.22 $\chi_{bc}(G) \leq \lceil \frac{n}{2} \rceil$.

Proof: *Let $V(G) = \{u_1, u_2, ..., u_n\}$.*

Assume that G has no isolates.

When $n = 1$ or 2, $\chi_{bc}(G) = 1$. Hence $\chi_{bc}(G) = \lceil \frac{n}{2} \rceil$.

Induction hypothesis: Assume the result for all graphs G without isolates and for all order $\leq m$, $(m \geq 2)$.

Assume that $\mid V(G) \mid = m + 1$.

If G has a pendant vertex say u, then $G - u$ is a graph without isolates. If G has no pendant vertex, then $G - u$ is a graph without isolates. Let H be an induced subgraph of G such that $\mid V(H) \mid = m$. Since $m \geq 2$, by induction hypothesis, $\chi_{bc}(H) \leq \lceil \frac{m}{2} \rceil$.

If $\chi_{bc}(H) < \lceil \frac{m}{2} \rceil$ then $\chi_{bc}(G) \leq \lceil \frac{m+1}{2} \rceil$.

For: $\chi_{bc}(H) \leq \lceil \frac{m}{2} \rceil - 1$.

Therefore $\chi_{bc}(G) \leq \lceil \frac{m}{2} \rceil - 1 + 1 = \lceil \frac{m}{2} \rceil \leq \lceil \frac{m+1}{2} \rceil$.

Suppose $\chi_{bc}(H) = \lceil \frac{m}{2} \rceil$.

If m *is even, then* $\lceil \frac{m}{2} \rceil + 1 = \lceil \frac{m+1}{2} \rceil$.

Therefore $\chi_{bc}(G) \leq \lceil \frac{m}{2} \rceil + 1 = \lceil \frac{m+1}{2} \rceil$.

Suppose m *is odd. Let* $V(G) - V(H) = \{u\}$. *Since* m *is odd,* $\chi_{bc}(H) \geq 2$ *and hence there are atleast two classes in any* χ_{bc} *- partition of* H. *Since* $\chi_{bc}(H) = \lceil \frac{m}{2} \rceil$ *and* m *is odd, there exist atleast one class in any* χ_{bc} *- partition which is a singleton. Add* u *with that class which is a singleton. Therefore* $\chi_{bc}(G) = \lceil \frac{m}{2} \rceil = \lceil \frac{m+1}{2} \rceil$.

Thus in any case $\chi_{bc}(G) \leq \lceil \frac{m+1}{2} \rceil$. *By induction the proof is complete.*

Remark 3.2.23 $1 \leq \chi_{bc}(G) \leq \lceil \frac{n}{2} \rceil$ *and both bounds are sharp.*

Example 3.2.24 $\chi_{bc}(K_{m,n}) = 1$ *and* $\chi_{bc}(K_n) = \lceil \frac{n}{2} \rceil$.

Theorem 3.2.25 *Let* G *be a graph with* t *isolates. Then*

$\chi_{bc}(G) \leq \lceil \frac{n}{2} \rceil$.

Proof: Let H *be a graph obtained from* G *by removing all the isolates of* G. *Then order of* H *is* $n - t$. *Therefore* $\chi_{bc}(H) \leq \lceil \frac{(n-t)}{2} \rceil$.

If $t \geq 2$ *then* $\lceil \frac{(n-t)}{2} \rceil + 1 \leq \lceil \frac{n}{2} \rceil$.

Therefore $\chi_{bc}(G) \leq \chi_{bc}(H) + 1 \leq \lceil \frac{(n-t)}{2} \rceil + 1 \leq \lceil \frac{n}{2} \rceil$.

Suppose $t = 1$. *Therefore* $\chi_{bc}(H) \leq \lceil \frac{(n-1)}{2} \rceil$.

If n *is odd then* $\lceil \frac{(n-1)}{2} \rceil = \frac{(n-1)}{2}$.

Therefore $\lceil\frac{(n-1)}{2}\rceil + 1 = \frac{(n-1)}{2} + 1 = \frac{(n+1)}{2} = \lceil\frac{n}{2}\rceil$.

Hence $\chi_{bc}(G) \leq \chi_{bc}(H) + 1 \leq \lceil\frac{(n-1)}{2}\rceil + 1 = \lceil\frac{n}{2}\rceil$.

Suppose n *is even. Suppose* $\chi_{bc}(H) < \lceil\frac{(n-1)}{2}\rceil$.

Then $\chi_{bc}(H) \leq \lceil\frac{(n-1)}{2}\rceil - 1 = \frac{n}{2} - 1$.

Therefore $\chi_{bc}(G) \leq \chi_{bc}(H) + 1 \leq \frac{n}{2} - 1 + 1 = \frac{n}{2} = \lceil\frac{n}{2}\rceil$.

Suppose n *is even. Suppose* $\chi_{bc}(H) = \lceil\frac{(n-1)}{2}\rceil$. *Therefore each color class in* χ_{bc} *- partition of* $V(H)$ *has two elements except one class which is a singleton. The isolate of* G *can be added with this singleton class.*

Therefore $\chi_{bc}(G) = \chi_{bc}(H) = \lceil\frac{(n-1)}{2}\rceil = \frac{n}{2} = \lceil\frac{n}{2}\rceil$.

Theorem 3.2.26 *Let* k_1 *be the minimum number of bi-clique sets in any* χ_{bc} *- partition of a connected graph* G *and* k_2 *be the minimum number of bi-clique sets in any* χ_{bc} *- partition of a connected graph* H. *Let* $\chi_{bc}(G) = k$ *and* $\chi_{bc}(H) = r$. *Then* $\chi_{bc}(G \cup H) = k_1 + k_2 + max\{k - k_1, r - k_2\}$.

Proof: Let k_1 *be the minimum number of bi-clique sets in any* χ_{bc} *- partition of a connected graph* G *and* k_2 *be the minimum number of bi-clique sets in any* χ_{bc} *- partition of a connected graph* H.

Let $\pi_1 = \{V_1, V_2, ..., V_k\}$ *and* $\pi_2 = \{W_1, W_2, ..., W_r\}$ *be two such* χ_{bc} *- partitions of* G , H *respectively.*

Let $\pi_3 = \{V_1, V_2, ..., V_{k_1}, W_1, W_2, ..., W_{k_2}, V_{k_1+1} \cup W_{k_2+1}, ...\}$ *where the independent sets in* π_1 *and* π_2 *are put together in* π_3. $|\pi_3| = k_1 + k_2 + max\{k - k_1, r - k_2\}$. π_3 *is a bi-clique coloring of* $G \cup H$.

Therefore $\chi_{bc}(G \cup H) \leq k_1 + k_2 + max\{k - k_1, r - k_2\}$.

Let π_4 *be a* χ_{bc} *- partition of* $G \cup H$. *Any element* X *of* π_4 *which contains vertices from both* G *and* H *must be the join of two independent sets one each in* G , H. *Any element* Y *of* π_4 *which is a bi-clique set, is a subset of* $V(G)$ *or* $V(H)$ *and not both.*

Therefore the elements in π_4 *which are bi-clique sets in* G *must appear in a bi-clique coloring of* G. *The same is true for* H *also.*

Therefore the number of bi-clique sets in π_4 *which are bi-clique sets in* G *is* $\geq k_1$ *and the number of bi-clique sets in* π_4 *which are bi-clique sets in* H *is* $\geq k_2$.

Therefore $|\pi_4| \geq k_1 + k_2 +$ *number of independent sets in* $G \cup H$, *appearing in* π_4.

Suppose there are t *independent sets in* π_4 *and let them be* $Y_1, Y_2, ..., Y_t$. *Then* $\{Y_1 \cup G, Y_2 \cap G, ..., Y_t \cap G; Z_1, Z_2, ..., Z_l\}$ *where* $Z_1, Z_2, ..., Z_l$ *are the bi-clique sets in* G *appearing in* π_4 *is a bi-clique coloring of* G. *Note that some* $Y_i \cap G$ *may be empty.*

Therefore $\chi_{bc}(G) \leq t+l$. *Similarly* $\chi_{bc}(H) \leq t+l_1$, $l \geq k$ *and* $l_1 \geq k_1$, $t \geq max\{k-k_1, r-k_2\}$.

Therefore $\mid \pi_4 \mid \geq k_1+k_2+t \geq k_1+k_2+max\{k-k_1, r-k_2\}$.

That is $\chi_{bc}(G \cup H) \geq k_1+k_2+max\{k-k_1, r-k_2\}$.

Therefore $\chi_{bc}(G \cup H) = k_1+k_2+max\{k-k_1, r-k_2\}$.

Theorem 3.2.27 *Let* t_1 *be the maximum number of independent sets in any* χ_{bc} *- partition of* G *and* t_2 *be the maximum number of independent sets in any* χ_{bc} *- partition of* H. *Let* $t_1 \leq t_2$. *Then* $\chi_{bc}(G+H) = \chi_{bc}(G) + \chi_{bc}(H) - t_1$.

Proof: Let t_1 *be the maximum number of independent sets in any* χ_{bc} *- partition of* G *and* t_2 *be the maximum number of independent sets in any* χ_{bc} *- partition of* H. *Any bi-clique of* G *or bi-clique of* H *will remain a bi-clique of* $G+H$. *Any independent set of* G *and any independent set of* H *together form a bipartite set in* $G+H$. *Let* $t_1 \leq t_2$. *Then there exist* t_1 *independent sets of* G *which can be combined with* t_1 *independent sets of* H *to form* t_1 *bi-clique sets in* $G+H$.

Therefore $\chi_{bc}(G+H) = \chi_{bc}(G) + \chi_{bc}(H) - t_2 + t_2 - t_1 = \chi_{bc}(G) + \chi_{bc}(H) - t_1$.

Observation 3.2.28 *Let T be a tree. $\chi_{bc}(T) = \chi(T)$ or $\chi(T) - 1$.*

Proof: Since any tree is bipartite, T has two independent sets whose union is $V(T)$. Therefore $\chi_{bc}(T) \leq 2$.

If T is complete bipartite then $\chi_{bc}(T) = 1 = \chi(T) - 1$.

Otherwise $\chi_{bc}(T) = 2 = \chi(T)$.

Lemma 3.2.29 *Let G_1 and G_2 be two complete bipartite subgraphs of a graph with bipartitions (V_1, V_2) and (V_3, V_4) respectively. Then $G_1 \cup G_2$ is complete bipartite*

(i) if $V_1 \cup V_3$ is independent or $V_1 \cup V_4$ is independent.

(ii) if $V_1 \cup V_3$ is independent then $V_2 \cup V_4$ is independent or if $V_1 \cup V_4$ is independent then $V_2 \cup V_3$ is independent.

(iii) if $V_1 \cup V_3$ is independent then every vertex of V_1 is adjacent with V_4 and every vertex of V_2 is adjacent with V_3. If $V_1 \cup V_4$ is independent then every vertex of V_1 is adjacent with V_3 and every vertex of V_2 is adjacent with V_4.

Proof: Obvious.

Lemma 3.2.30 *Let G_1 be a complete bipartite subgraph of a graph with bipartition (V_1, V_2) and G_2 be a totally disconnected subgraph. Then*

$G_1 \cup G_2$ is complete bipartite if $V(G_2)$ is independent of V_1 and every vertex of $V(G_2)$ is adjacent with every vertex of V_2 or $V(G_2)$ is independent with V_2 and every vertex of $V(G_2)$ is adjacent with every vertex of V_1.

Proof: Obvious.

Observation 3.2.31 *$\chi_{bc}(G) = 2$ if and only if G is not complete bipartite nor $\overline{K}_n$ and there are two complete bipartite subgraphs whose vertex union is $V(G)$ or there is a complete bipartite subgraph and a totally disconnected subgraph whose vertex union is $V(G)$ or G is a tree which is not complete bipartite.*

Proof: If G satisfies any of the conditions of hypothesis, then $\chi_{bc}(G) = 2$.

Conversly, if $\chi_{bc}(G) = 2$ then G is not complete bipartite nor $\overline{K}_n$.

Any χ_{bc} - partition of G contains exactly two element which are both bi-clique of G or one of them is bi-clique and the other is independent or both of them are independent sets. In the last case G is a tree which is not complete bipartite.