Chapter 1

Preliminaries

BASIC CONCEPTS AND RESULTS IN DOMINATION THEORY IN GRAPHS

In this chapter the basic definitions and theorems which are needed for the subsequent chapters are collected. For graph theoretic terminology we refer to Harary[**17**] .

Definition 1.0.1 A simple graph is a finite non-empty set of objects called **vertices** together with a set of un ordered pairs of distinct vertices of G, called **edges**. The vertex set and the edge set of G are respectively denoted by $V(G)$ and $E(G)$. A graph G with vertex set $V(G)$ and edge set $E(G)$ is denoted by $G = (V, E)$.

If $e = \{u, v\}$ is an edge, we write $e = uv$ and we say e joint the vertices u

and v; u and v are called **adjacent** vertices; u and v are said to **incident** with e. If two vertices are not joined by an edge, then they are said to be **non-adjacent**. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

Definition 1.0.2 The number of elements in the vertex set of a graph is called the order of G and is denoted by n. The number of elements in the edge set of a graph is called the **size** of G and is denoted by m. A graph with n vertices and m edges is called as (n, m)-graph. The $(1, 0)$-graph is called as **trivial graph**.

Definition 1.0.3 A graph G_1 is **isomorphic** to a graph G_2 if there is a bijection f from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. If G_1 is isomorphic to G_2, then it is denoted by $G_1 \cong G_2$.

Definition 1.0.4 A graph H is called a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A **spanning subgraph** of G is a subgraph H with $V(H) = V(G)$.

For any set S of vertices of G, the **induced subgraph** $< S >$ is the maximal subgraph of G with vertex set S. Therefore two vertices of S are adjacent in $< S >$ if and only if they are adjacent in G.

Definition 1.0.5 Let $G = (V, E)$ be a graph and let $v \in V$. $G - v$ is the induced subgraph $< V(G) - \{v\} >$ of G and it is obtained from G by removing v and the edges incident with v. If $e \in E(G)$, $G - e$ is the spanning subgraph with edge set $E(G) - \{e\}$ and it is obtained from G by removing the edge e from G.

Definition 1.0.6 The **degree** of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg_G(v)$ or $deg(v)$ or $d(v)$, when there is no ambiguity. The maximum and the minimum of the degrees of the vertices of G are respectively denoted by $\Delta(G)$ and $\delta(G)$. A vertex of degree 0 in G is called an **isolated vertex**, and a vertex of degree 1 is called a **pendent vertex** or an **end vertex** of G. Any vertex adjacent to a pendent vertex is called a **support**. A support is said to be trivial if it has only one pendent vertex adjacent to it. A support with two or more pendent vertices is called a non-trivial support. The set of all pendent vertices adjacent to a support is denoted by $N_0(u)$. A vertex of a graph G is said to be a **full degree vertex** or a **dominating vertex** or a link complete vertex if it is adjacent to all other vertices in G.

Definition 1.0.7 A graph G is said to be a **regular graph** if every vertex of G has the same degree. A graph is called an r**-regular graph** if the degree of every vertex is r. A 3-regular graph is called a **cubic graph**.

Definition 1.0.8 A graph G is **complete** if every pair of its vertices are adjacent. A complete graph on n vertices is denoted by K_n. A **clique** of a graph is a maximal complete subgraph.

Definition 1.0.9 The **Complement** $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Definition 1.0.10 A **bipartite** graph is a graph G whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge in G has one end vertex in V_1 and the other end vertex in V_2. (V_1, V_2) is called a bipartition of G. Further, if every vertex of V_1 is adjacent to every vertex of V_2, then G is called a **complete bipartite graph**. The complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = r$ and $|V_2| = s$ is denoted by $K_{r,s}$. $K_{1,r}$ is called a **star**. When $r \geq 2$ the vertices of degree 1 of a star are called **claws** of the star and the vertex of degree > 1 is called the **center** of the star. When $r = 1, K_{1,1}$ becomes K_2 and in this case any one of the two vertices of K_2 can be called a center. A **double star** is a graph obtained by taking two stars and joining the vertices of maximum degrees with an edge. If the stars are $K_{1,r}$ and $K_{1,s}$, then the double star is denoted by $D_{r,s}$.

An r**-partite graph** (or) **multipartite graph** is a graph G whose vertex set $V(G)$ can be partitioned into r subsets $V_1, V_2, \cdots, V_r$ such that every edge of G has one end vertex in V_i and the other end vertex in V_j, for $i \neq j$. $(V_1, V_2, \ldots, V_r)$ is called a r**-partition** of G. If every vertex of V_i is adjacent to every vertex of V_j, $i \neq j$ for every i and j, then G is called a **complete** r**-partite graph** (or) **complete multipartite graph**. The complete r-partite graph with r-partition $(V_1, V_2, \ldots, V_r)$ such that $|V_i| = n_i,\ 1 \leq i \leq r$, is denoted by $K_{n_1, n_2, \ldots, n_r}$.

Definition 1.0.11 Let u and v (not necessarily distinct) be vertices of a graph G. A $u - v$ **walk** of G is a finite alternating sequence $u = u_0, e_1, u_1, e_2, \cdots, e_k,\ u_k = v$ of vertices and edges beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}u_i,\ i = 1, 2, \ldots, k$. The number k is called the **length of the walk**. A $u - v$ walk is said to be **closed** if $u = v$ and is **open** otherwise. A walk $u_0,\ e_1,\ u_1,\ e_2,\ u_2, \ldots,\ e_k,\ u_k$ is determined by the sequence $u_0,\ u_1,\ u_2,\ \cdots,\ u_k$ of its vertices and this walk is denoted by $u_0,\ u_1,\ u_2,\ \cdots,\ u_k$. A walk in which all the vertices are distinct is called a **path**. A closed walk $u_0,\ u_1,\ u_2,\ \ldots,\ u_k$ in which the vertices $u_0, u_1, u_2, \cdots, u_{k-1}$ are distinct is called a **cycle**. A path on k vertices is denoted by P_k and a cycle on k vertices is denoted by C_k.

Definition 1.0.12 A graph G is called a **hamiltonian graph** if it has a spanning subgraph isomorphic to a cycle.

Definition 1.0.13 A graph G is said to be **connected** if any two distinct vertices of G are joined by a path. A maximal connected subgraph of G is called a **component** of G. Thus a disconnected graph has at least two components.

Definition 1.0.14 The **distance** $d(u, v)$ between two vertices u and v in a graph G is the length of a shortest $u - v$ path in G. A shortest $u - v$ path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic. The diameter of G is denoted by $diam(G)$.

Definition 1.0.15 A **cut-vertex (cut-edge)** of a graph G is a vertex (edge) whose removal increases the number of components in G.

Definition 1.0.16 A graph is called **acyclic**, if it has no cycles. A connected acyclic graph is called a tree. A **spider** is a tree which has at most one vertex of degree ≥ 3. A tree is a **wounded spider** if the tree is $K_{1,r},\ r \geq 1$, in which at most $r - 1$ of the edges are subdivided. A star is also a wounded spider.

Definition 1.0.17 A **subdivision of an edge** uv of a graph G is obtained by introducing a new vertex w and replacing the edge uv with edges uw and wv. The graph obtained from G by subdividing each edge of G exactly once is called the **subdivision graph** of G and is denoted by $S(G)$.

Definition 1.0.18 *A graph H is said to be a subdivision of G, if it is obtained from G by subdividing each edge of G at most once. In other words, to obtain H, it is not necessary to subdivide each edge of G.*

Definition 1.0.19 *A graph G is said to be vertex transitive if given any two vertices $u, v (u \neq v)$ of G, there is an automorphism ϕ of G such that $\phi(u) = v$. If G is vertex transitive, then it is regular.*

Definition 1.0.20 A subset S of the vertex set in a graph G is said to be **independent** if no two vertices in S are adjacent. The maximum number of vertices in an independent set of a graph G is called the **independence number** of G and is denoted by $\beta_o(G)$ or $\beta(G)$.

Definition 1.0.21 A vertex and an edge are said to **cover** each other if they are incident. A set of vertices which cover all the edges of a graph is called a **vertex cover** of G. The smallest number of vertices in any vertex cover of G is called the **vertex covering number** of G and is denoted by $\alpha_o(G)$.

A set of edges which cover all the vertices of G is called an **edge cover** of G. The smallest number of edges in any edge cover of G is called the **edge covering number** of G and is denoted by $\alpha_1(G)$.

Definition 1.0.22 A set S of edges in a graph G is said to be **independent** if no two edges in S are adjacent in G.The **edge independence number** of a graph G is the size of its biggest edge independent set and it is denoted by $\beta_1(G)$.

Theorem 1.0.23 For any non-trivial connected graph G of order n, $\alpha_o + \beta_o = n = \alpha_1 + \beta_1$.

Definition 1.0.24 A graph G is a **split graph** if there is a partition $V = (I, K)$ of the vertices of G into an independent set I and a clique K.

Definition 1.0.25 A triangle T of a graph G is called **odd** if there is a vertex of G adjacent to odd number of vertices of T and is even otherwise.

Definition 1.0.26 For a graph G with edges, the **line graph** $L(G)$ is the graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

Theorem 1.0.27 A graph G is a **line graph** if and only if $K_{1,3}$ is not an induced subgraph of G or if two odd triangles have a common edge then the subgraph induced by their vertices is K_4.

Definition 1.0.28 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs Then their **Cartesian Product** $G_1 \square G_2$ is defined to be the graph whose vertex set is $V_1 \square V_2$ and edge set is $\{((u_1, v_1), (u_2, v_2))$: either $u_1 = u_2$ and $v_1 v_2 \in E_2$ or $v_1 = v_2$ and $u_1 u_2 \in E_1\}$.

Definition 1.0.29 *Let G, H be two graphs. The Kronecker product [direct,tensor,cross product] denoted by GXH and is defined as the graph whose vertex set is $V(G)XV(H)$ and two vertices $(u_1, v_1), (u_2, v_2)$ are adjacent if and only if u_1 is adjacent with u_2 in G and v_1 is adjacent with v_2 in H.*

Definition 1.0.30 A proper coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. A k-coloring of a graph is a coloring with k colors. The chromatic number $\chi(G)$ of a graph G is defined to be the minimum k for which G has a k-coloring. A graph G is **uniquely k-colorable** if $\chi(G) = k$ and there exists a unique partition of $V(G)$ into k independent sets.

Theorem 1.0.31 *[17]* For any graph G, $\chi(G) \leq \Delta(G) + 1$.

Theorem 1.0.32 *[17]* If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Theorem 1.0.33 *[17]* For any graph of order n, the sum and product of $\chi(G)$ and $\chi(\overline{G})$ satisfy the following inequalities:

(i) $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.

(ii) $n \leq \chi(G)\chi(\overline{G}) \leq \frac{(n+1)}{2}^2$.

Definition 1.0.34 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Then their **union** $G_1 \cup G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$.

Definition 1.0.35 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be such that $V_1 \cap V_2 = \phi$ be any two graphs Then their **join** $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

Definition 1.0.36 Let G_1 be a (n_1, m_1)-graph and let G_2 be (n_2, m_2)-graph. Then the **corona** $G_1 o G_2$ is defined as the graph G obtained by taking one copy of G_1 and n_1 copies of G_2, and joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Definition 1.0.37 The graph $K_{1,3}$ is called a **claw** and the graph $K_3 \circ K_1$ is called a **net**.

Definition 1.0.38 For any real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Definition 1.0.39 The open **neighbourhood** $N(v)$ of a vertex v in a graph G is the set of all vertices adjacent to v in G. The **closed neighbourhood** $N[v]$ of v is the set $N(v) \cup \{v\}$.

Definition 1.0.40 The **open neighbourhood** $N(D)$ of a set D of vertices is the set of all vertices adjacent to vertices in D. The closed neighbourhood $N[D]$ of D is the set $N(D) \cup D$. If $x \in D$, a private neighbour of x with respect to D is a vertex v such that $v \in N[x] - N[D - \{x\}]$.

Definition 1.0.41 *Let S be a set of vertices. Let $u \in S$. A vertex $v \in V(G)$ is said to be a* **Private neighbour of** u *(with respect to S) if $N(v) \cap S = \{u\}$. Furthermore, the private neighbour of u , with respect to S, to be $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.*

Definition 1.0.42 Let $G = (V, E)$ be a graph and $u, v \in V$. If $uv \in E$, it is said that u and v **dominate each other**. A subset D of V is a **dominating set** of G if every vertex v in $V - D$ is dominated by some $u \in D$. The property of domination is a superhereditary property. That is, any superset of a dominating set of G is a dominating set of G.

Definition 1.0.43 *A dominating set D of G is called an* ***independent dominating set*** *if the vertices in D are independent.*

Definition 1.0.44 *A dominating set D of G is called a* ***minimal dominating set*** *if no proper subset of D is a dominating set of G. Since, domination is a superhereditary property , a dominating set D is minimal if and only if it is 1-minimal, that is, $D - \{u\}$ is not a dominating set, for any $u \in D$. Let $\gamma(G)$ and $\Gamma(G)$ denote the cardinality of a smallest and largest minimal dominating sets of G respectively. The number $\gamma(G)$ is called the* ***domination number*** *of G and $\Gamma(G)$ is called the* ***upper domination number*** *of G. The* ***independent dominating number*** *$i(G)$ of G is the cardinality of its smallest independent dominating set. A dominating set D of G is called a* ***minimum dominating set*** *or γ-* ***set*** *if D is a dominating set with cardinality $\gamma(G)$.*

Definition 1.0.45 *A set D is* ***irredundant*** *if every vertex $v \in D$ has at least one private neighbour with respect to D. That is for all $v \in D, N[v] - N[D - \{v\}] \neq \phi$. An irredundant set D is* ***maximal irredundant*** *if for* *every vertex $u \in V - D, D \cup \{u\}$ is not irredundant. Let $ir(G)$ and $IR(G)$ respectively denote the cardinality of a smallest and a largest maximal irredundant set of G.*

Definition 1.0.46 *The number $ir(G)$ is called the* ***irredundance number*** *and $IR(G)$ is called the* ***upper irredundance number****. The numbers $\gamma(G), i(G)$ and $ir(G)$ are called the* ***domination number*** *,**independent domination number**, and **irredundance number** respectively and $\Gamma(G), \beta_0(G)$ and $IR(G)$ are called* ***upper domination number****,* ***independence number*** *and* ***upper irredundance number*** *respectively .*

Theorem 1.0.47 [**42**]. A dominating set D of a graph G is minimal if and only if for every $u \in D$ one of the following conditions holds.

(i) $N(u) \cap D = \phi$

(ii) There is a vertex $v \in V - D$ such that $N(v) \cap D = \{u\}$.

Theorem 1.0.48 [**42**]. Every connected graph G of order $n \geq 2$ has a dominating set D whose complement $V - D$ is also a dominating set.

Theorem 1.0.49 [**42**]. If G is a graph with no isolated vertices, then the complement $V - D$ of every minimal dominating set D is a dominating set.

Corollary 1.0.50 If G is a graph on n vertices having no isolated vertices, then $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 1.0.51 For any graph G with even order n and no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.

Definition 1.0.52 ***Generalised Petersen Graphs*** $P(n, k)$ [47]

For each $n \geq 3$ *and* $0 < k < n$, $P(n, k)$ *denotes the generalised Petersen graph with vertex set*

$V(G) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ *and the edge set*

$E(G) = \{u_i u_{i+1(mod\ n)}, u_i v_i, v_i v_{i+k(mod\ n)}\}, 1 \leq i \leq n.$

Definition 1.0.53 ***Mycielski Graphs*** Let $G = (V, E)$ be a simple graph. The Mycielskian of G is the graph $\mu(G)$ with vertex set equal to the disjoint union $V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$ and the edge set $E \cup \{xy', x'y : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex and the vertex u is called the root of $\mu(G)$.

Definition 1.0.54 ***Shadow Graphs*** Let $G = (V, E)$ be a simple graph. The Shadow graph of G is the graph $\S(G)$ with vertex set equal to the disjoint union $V \cup V'$ where $V' = \{x' : x \in V\}$ and the edge set $E \cup \{xy', x'y : xy \in E\}$.

The vertex x' is called the twin of the vertex and the vertex u is called the root of $\S(G)$.

Definition 1.0.55 ***Harary graphs*** $H_{n,m}$ with n vertices and $m < n$ is defined as follows:

Case(i):

n is even and $m = 2r$. Then $H_{n,2r}$ has n vertices $0, 1, 2, \cdots, n-1$ and i, j are joined if $i - r \leq j \leq i + r$, where the addition is taken under modulo n.

Case(ii):

m is odd and n is even. Let $m = 2r + 1$.Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex i to the vertex $i + \frac{n}{2}$, for $0 \leq i \leq \frac{n}{2}$.

Case(iii):

m, n are odd. Let $m = 2r + 1$. Then $H_{n,2r+1}$ is constructed by drawing $H_{n,2r}$ and then adding edges joining vertex 0 to the vertices $\frac{n-1}{2}$ and $\frac{n-1}{2}$ and vertex i to $i + \frac{n+1}{2}$, for $1 \leq i \leq \frac{n-1}{2}$.

Definition 1.0.56 ***Kneser Graph*** *Let k, n be two positive integers, such that $2 \leq k \leq n$. Let M be a set with n elements. The Kneser graph $K(n, k)$ is defined as the graph with vertex set V to be the set of all subsets of n of cardinality k. Two vertices of $K(n, k)$ are adjacent if and only if the* *corresponding sets are disjoint. This concept was introduced by Kneser in 1978 when $n = 2k + 1$, the Kneser graph is also called odd by Mulder.*

The domination number of $K(n, 2)$ is 3 for every n.

Excellent,Just Excellent and Very Excellent graphs

N.Sridharan et al introduced the concept of Excellent, Just Excellent and Very Excellent graphs [**40**]

Definition 1.0.57 *A graph G is said to be γ-excellent if given any vertex x, then there is a γ-set of G containing x.*

Definition 1.0.58 *A graph G is said to be just γ-excellent if to each $u \in V$, there is a unique γ-set of G containing u.*

Definition 1.0.59 *An excellent graph G is said to be very γ- excellent if there is a γ-set S of G such that to each vertex $u \in V - S$, there exist a vertex $v \in S$ such that $(S - v) \cup \{u\}$ is a γ-set of G. A γ-set S of G satisfying this property is called a very excellent γ-set of G.*