CHAPTER 8

FINITE SOURCE INVENTORY SYSTEM WITH POSTPONED DEMANDS AND MODIFIED M VACATION POLICY

8.1 Introduction

All the models considered so far in this thesis, we assume the size of the population that generates demand is infinite. But in many practical situations such as the collection of similar electrical power supply units, spare parts for failed machines etc., needs finite number of units. If the population size is small, it is important to take into account that the rate of generation of new demands decreases as far as the number of demands in the system increases. Here each individual generates his own flow of primary demands. Hence, the probability that any particular source generates a demand in any interval \((t, t + dt)\) is \(\alpha dt + o(dt)\) as \(dt \to 0\) if the source is idle at time \(t\), and zero if the source is being in pool at time \(t\), independently of the behaviour of any other sources. In this case, the arrival process do not form a Poisson process because the arrivals may depend on the number of demands already in the system. This process is called Quasi-random process.

system with service facility. Shophia Lawrence et al. [2012] considered a perishable inventory system with service facility and finite source.

This chapter considers a finite sources inventory system with postponed demands and server vacation. Here, we deal an \((s,S)\) inventory system with single server, who avails the modified \(M\) vacation policy. In section 8.2 of this chapter, we describe the mathematical model under study. The analysis and steady state analysis of the model are made in section 8.3. In section 8.4, we calculate the performance measures of the system in steady-state case and derive the total expected cost rate using these measures. Finally, the convexity of the total expected cost rate is presented and illustrate some numerical examples.

### 8.2 Model Description

Consider a finite source \((s,S)\) inventory system with postponed demands and server vacation, where the arrival process of primary demands follows a quasi-random process with parameter \((\alpha)\). The lead time has phase type distribution with representation \((\tau_1,T_1)\) of order \(m_1\). The mean lead time rate is \(-\tau_1T_1^{-1}\mathbf{e}\) and the absorption vector \(T_1^0 = -T_1\mathbf{e}\). Whenever the stock-out period occurs, server goes for an inactive period, which comprises the inactive idle period and vacation period. This idle period and vacation time are independently distributed as Phase type distribution with representation \((\tau_2,T_2)\) and \((\tau_3,T_3)\) of order \(m_2,m_3\) respectively. Let \(T_i^0 = -T_i\mathbf{e}\) for \(i = 2, 3\) and thus the mean idle time \(\beta_2^{-1} = -\tau_2T_2^{-1}\mathbf{e}\) and the mean vacation time \(\beta_3^{-1} = -\tau_3T_3^{-1}\mathbf{e}\). The demand which occurs during the stock out periods or server inactive period, enters into the pool. These pooled demands are permitted one-after-another, only when the on-hand inventory level is greater than \(s\). The inter-selection time between any two successive demands is assumed to follow exponential distribution with parameter \(\theta(>0)\). We also assume that inter-demand times between primary demands, lead times, vacation time and idle times are mutually independent random variables.

**Notation** :

\[
N_0 : (s + 1)(N + 1)m_1 \\
N_1 : Q(N + 1) \\
N_2 : (N + 1)m_1m_2 \\
N_3 : (N + 1)m_1m_3
\]
\[ N_4 : (N + 1)m_3 \]
\[ N_5 : N_0 + N_1 \]
\[ N_6 : N_2 + N_3 + N_4 \]

8.3 Analysis

Let \( L(t), Y(t), J_1(t), J_2(t), J_3(t) \) respectively, denote the on-hand inventory level, the number of demands in the pool, phase of the lead time, phase of inactive-idle time and phase of the vacation at time \( t \). Further, let

\[
X_1(t) = \begin{cases} 
0, & \text{if server is in active} \\
\kappa, & \text{if server is in \( k \)th inactive period, } k = 1, 2, \ldots, M
\end{cases}
\]
\[
X_2(t) = \begin{cases} 
0, & \text{if server is in inactive-idle period} \\
1, & \text{if server is on vacation period}
\end{cases}
\]

From our assumptions made on the input and output processes, it can be seen that the stochastic process \( Z(t) = \{ (X_1(t), X_2(t), L(t), Y(t), J_1(t), J_2(t), J_3(t)), t \geq 0 \} \) is a continuous time Markov chain with state space

\[
\Omega = \{ (0, l, y, j_1) : l \in E_0^S; y \in E_0^N; j_1 \in E_{1}^{m_1} \}
\]
\[
\cup \{ (0, l, y) : l \in E_{s+1}^S; y \in E_0^N \}
\]
\[
\cup \{ (x_1, 0, 0, y, j_1, j_2) : x_1 \in E_1^M; y \in E_0^N; j_1 \in E_{1}^{m_1}; j_2 \in E_{1}^{m_2} \}
\]
\[
\cup \{ (x_1, 1, 0, y, j_1, j_3) : x_1 \in E_1^M; y \in E_0^N; j_1 \in E_{1}^{m_1}; j_3 \in E_{1}^{m_3} \}
\]
\[
\cup \{ (x_1, 1, Q, y, j_3) : x_1 \in E_1^M; y \in E_0^N; j_3 \in E_{1}^{m_3} \}.
\]

To induce an order on the state space we define the following ordered tuplets

\[
\langle 0, l \rangle = \begin{cases} 
(0, l, y, j_1) : & l \in E_0^S; \quad y \in E_0^N; \quad j_1 \in E_{1}^{m_1} \\
(0, l, y) : & l \in E_{s+1}^S; \quad y \in E_0^N
\end{cases}
\]

For \( x_1 = E_1^M \)

\[
\langle x_1, x_2, l \rangle = \begin{cases} 
(x_1, x_2, l, y, j_1, j_2) : & x_2 = 0; \quad l = 0; \quad y \in E_0^N; \quad j_1 \in E_{1}^{m_1}; \quad j_2 \in E_{1}^{m_2} \\
(x_1, x_2, l, y, j_1, j_3) : & x_2 = 1; \quad l = 0; \quad y \in E_0^N; \quad j_1 \in E_{1}^{m_1}; \quad j_3 \in E_{1}^{m_3} \\
(x_1, x_2, l, y, j_3) : & x_2 = 1; \quad l = Q; \quad y \in E_0^N; \quad j_3 \in E_{1}^{m_3}
\end{cases}
\]

\[
\ll 0 \gg = \left( \langle 0, 0 \rangle, \langle 0, 1 \rangle, \ldots, \langle 0, S \rangle \right)
\]

and

\[
\ll x_1 \gg = \left( \langle x_1, 0, 0 \rangle, \langle x_1, 1, 0 \rangle, \langle x_1, 1, Q \rangle \right), x_1 = 1, 2, \ldots, M
\]
By ordering the state space as $$\langle \langle 0 \rangle \rangle, \langle \langle 1 \rangle \rangle, \ldots, \langle \langle M \rangle \rangle$$, the infinitesimal generator $$P$$ can be conveniently expressed in a block partitioned matrix with entries

$$P = \begin{pmatrix}
\langle \langle 0 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 2 \rangle \rangle & \cdots & \langle \langle M-1 \rangle \rangle & \langle \langle M \rangle \rangle \\
\langle \langle 0 \rangle \rangle & P_0 & P_1 & 0 & \cdots & 0 & 0 \\
\langle \langle 1 \rangle \rangle & P_4 & P_2 & P_3 & \cdots & 0 & 0 \\
\langle \langle 2 \rangle \rangle & P_4 & 0 & P_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\langle \langle M-1 \rangle \rangle & P_4 & 0 & 0 & \cdots & P_2 & P_3 \\
\langle \langle M \rangle \rangle & P_5 & 0 & 0 & \cdots & 0 & P_2
\end{pmatrix}$$

$$[P_0]_{kl} = \begin{cases}
A_0, & l = k, \quad k = 0 \\
A_1, & l = k, \quad k = 1, 2, \ldots, s \\
A_2, & l = k, \quad k = s + 1, s + 2, \ldots, S \\
A_3, & l = k - 1, \quad k = 1, 2, \ldots, s \\
A_4, & l = k - 1, \quad k = s + 1 \\
A_5, & l = k - 1, \quad k = s + 2, s + 3, \ldots, S \\
I_{N+1} \otimes T_1^0, & l = k + Q, \quad k = 0, 1, \ldots, s \\
0, & \text{otherwise}
\end{cases}$$

$$A_0 = I_{N+1} \otimes T_1 - F_1 \otimes I_{m_1} + F_2 \otimes I_{m_1}, \quad A_2 = -(F_1 + H_1)$$

$$A_1 = I_{N+1} \otimes T_1 - F_1 \otimes I_{m_1}, \quad A_3 = F_1 \otimes I_{m_1}$$

$$A_4 = (F_1 + H_2) \otimes \tau_1, \quad A_5 = (F_1 + H_2) \otimes \tau_1$$

$$P_1 = \begin{pmatrix}
\langle 0, 0 \rangle & \langle 1, 0 \rangle & \langle 1, 1 \rangle & \langle 1, 1, Q \rangle \\
0 & 0 & 0 & 0 \\
0,1 & 0 & 0 & 0 \\
\langle 0, 2 \rangle & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\langle 0, S \rangle & 0 & 0 & 0
\end{pmatrix}, \quad B_0 = F_1 \otimes I_{m_1} \otimes \tau_2$$

For $$x_1 = 1, 2, \ldots, M$$

$$P_2 = \begin{pmatrix}
\langle x_1, 0, 0 \rangle & \langle x_1, 1, 0 \rangle & \langle x_1, 1, Q \rangle \\
\langle x_1, 0, 0 \rangle & C_0 & I_{(N+1)m_1} \otimes T_2^0 \otimes \tau_3 \\
\langle x_1, 1, 0 \rangle & 0 & C_1 & I_{(N+1)m_1} \otimes T_1^0 \otimes I_{m_3} \\
\langle x_1, 1, Q \rangle & 0 & 0 & C_2
\end{pmatrix}$$

For $$x_1 = 1, 2, \ldots, M - 1$$

$$P_3 = \begin{pmatrix}
\langle x_1, 0, 0 \rangle & \langle x_1, 1, 0 \rangle & \langle x_1, 1, Q \rangle \\
\langle x_1, 0, 0 \rangle & 0 & 0 \\
\langle x_1, 1, 0 \rangle & I_{(N+1)m_1} \otimes T_3^0 \otimes \tau_2 & 0 & 0 \\
\langle x_1, 1, Q \rangle & 0 & 0 & 0
\end{pmatrix}$$
For $P$ exists. Let recurrent. The steady state probability distribution of this process, denoted by 

\[
\begin{pmatrix}
\langle x_1, 0, 0 \rangle & \langle x_1, 1, 0 \rangle & \langle x_1, 1, Q \rangle \\
\langle x_1, 0, 0 \rangle & 0 & I_{N+1} \otimes T_1^0 \otimes e_{m_2} & 0 \\
\langle x_1, 1, 0 \rangle & 0 & 0 & 0 \\
\langle x_1, 1, Q \rangle & 0 & I_{N+1} \otimes T_3^0 & 0 \\
\langle 0, 0 \rangle & (0, 1) & (0, Q) & \cdots & (0, S) \\
\end{pmatrix}
\]

\[
P_4 = \begin{pmatrix}
\langle M, 0, 0 \rangle & \langle M, 1, 0 \rangle \\
\langle M, 0, 0 \rangle & 0 & 0 & \cdots & I_{N+1} \otimes T_1^0 \otimes e_{m_3} & \cdots & 0 \\
\langle M, 1, 0 \rangle & I_{N+1} \otimes T_3^0 & 0 & \cdots & 0 & \cdots & 0 \\
\langle M, 1, Q \rangle & 0 & 0 & \cdots & I_{N+1} \otimes T_3^0 & \cdots & 0 \\
\end{pmatrix}
\]

\[
P_5 = I_{N+1} \otimes T_1 \otimes I_{m_2} + I_{N+1} \otimes I_{m_1} \otimes T_2 + (F_2 - F_1) \otimes I_{m_{1m_2}}
\]

\[
C_1 = I_{N+1} \otimes T_1 \otimes I_{m_2} + I_{N+1} \otimes I_{m_1} \otimes T_2 + (F_2 - F_1) \otimes I_{m_{1m_2}}
\]

\[
C_2 = I_{N+1} \otimes T_3 + (F_2 - F_1) \otimes I_{m_3}
\]

\[
F_1 = \text{diag}(N\alpha, (N - 1)\alpha, \ldots, \alpha, 0)
\]

\[
[F_2]_{kl} = \begin{cases} 
(N - k)\alpha, & l = k + 1, k = 0, 1, \ldots, N - 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
[H_2]_{kl} = \begin{cases} 
k\theta, & l = k - 1, k = 1, 2, \ldots, N \\
0, & \text{otherwise}
\end{cases}
\]

It may be noted that the matrices and its orders are shown below in Table 8.1.

### Table 8.1: Dimension of sub-matrices of $P$

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Dimension</th>
<th>Matrix</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0$</td>
<td>$N_5 \times N_5$</td>
<td>$A_i$, $i = 2, 5$</td>
<td>$(N + 1) \times (N + 1)$</td>
</tr>
<tr>
<td>$P_{1, i = 2, 3}$</td>
<td>$N_6 \times N_6$</td>
<td>$A_4$</td>
<td>$(N + 1) \times N_1$</td>
</tr>
<tr>
<td>$P_{1, i = 4, 5}$</td>
<td>$N_6 \times N_5$</td>
<td>$B_0$</td>
<td>$(N_1 \times N_2)$</td>
</tr>
<tr>
<td>$F_{1, i = 1, 2}$</td>
<td>$(N + 1) \times (N + 1)$</td>
<td>$C_0$</td>
<td>$N_2 \times N_2$</td>
</tr>
<tr>
<td>$H_{1, i = 1, 2}$</td>
<td>$(N + 1) \times (N + 1)$</td>
<td>$C_1$</td>
<td>$N_3 \times N_3$</td>
</tr>
<tr>
<td>$A_{1, i = 0, 1, 3}$</td>
<td>$(N + 1)m_1 \times (N + 1)m_1$</td>
<td>$C_2$</td>
<td>$N_4 \times N_4$</td>
</tr>
</tbody>
</table>

#### 8.3.1 Steady State Analysis

It can be seen from the structure of $P$ that the homogeneous Markov Process $Z(t) = \{(X_1(t), X_2(t), L(t), Y(t), J_1(t), J_2(t), J_3(t)), t \geq 0\}$ on the finite state space $\Omega$ is positive recurrent. The steady state probability distribution of this process, denoted by

\[
\phi^i = \lim_{t \to \infty} Pr[Z(t) = j \mid Z(0) = i], i, j \in \Omega
\]

exists. Let

\[
\Phi^{(0,l,y)}_{(0,l,y)j_1} = \begin{cases} 
(\phi^{(0,l,y)j_1}) : & l \in E^y_0; \quad y \in E^N_0; \quad j_1 \in E^{m_1}_1 \\
(\phi^{(0,l,y)}) : & l \in E^y_{s+1}; \quad y \in E^N_0; 
\end{cases}
\]

For $x_1 = 1, 2, \ldots M$

\[
\Phi^{(x_1,0,0,y)j_1} = \begin{cases} 
(\phi^{(x_1,0,0,y)j_1,j_2}) : & y \in E^N_1; \quad j_1 \in E^{m_1}_1; \quad j_2 \in E^{m_2}_1 
\end{cases}
\]
\[ \Phi(x_1, l, y) = \begin{cases} 
(\Phi(x_1, l, y, j_1, j_3)) & : l = 0; \ y \in E_0^N; \ j_1 \in E_1^{m_1}; \ j_3 \in E_1^{m_3} \\
(\Phi(x_1, l, y, j_3)) & : l = Q; \ y \in E_0^N; \ j_3 \in E_1^{m_3} 
\end{cases} \]

Then the vector of limiting probabilities \( \Phi \) can be obtained by solving \( \Phi P = 0 \) and \( \Phi e = 1 \)

\[
\Phi^{\langle 0 \rangle} P_0 + \sum_{i=1}^{M-1} \Phi^{\langle i \rangle} P_4 + \Phi^{\langle M \rangle} P_5 = 0 \quad (8.1)
\]

\[
\Phi^{\langle 0 \rangle} P_1 + \Phi^{\langle 1 \rangle} P_2 = 0 \quad (8.2)
\]

\[
\Phi^{\langle i-1 \rangle} P_3 + \Phi^{\langle i \rangle} P_2 = 0, \ i = 2, 3, \ldots, M \quad (8.3)
\]

and

\[
\sum_{i=0}^{M} \Phi^{\langle i \rangle} e = 1 \quad (8.4)
\]

From equation (8.1) to (8.3) gives

\[
\Phi^{\langle i \rangle} = \left\{ (-1)^i \Phi^{\langle 0 \rangle} P_1 P_2^{-1} (P_3 P_2^{-1})^{i-1}, \ i = 1, 2, \ldots, M \right\}
\]

\( \Phi^{\langle 0 \rangle} \) can be obtained, by solving following equations

\[
\Phi^{\langle 0 \rangle} \left[ P_0 + \sum_{k=1}^{M-1} (-1)^k (P_1 P_2^{-1}) (P_3 P_2^{-1})^{k-1} P_4 + (-1)^M (P_1 P_2^{-1}) (P_3 P_2^{-1})^M P_5 \right] = 0
\]

\[
\Phi^{\langle 0 \rangle} \left[ e + \sum_{k=1}^{M} (-1)^k (P_1 P_2^{-1}) (P_3 P_2^{-1})^k e \right] = 1
\]

### 8.4 System Performance Measures

In this section we derive some stationary performance measures of the system. These measures are used to bring out the qualitative behaviour of the inventory system under study. Using these measures, we can construct the total expected cost per unit time.

#### 8.4.1 Mean Inventory Level

Let \( \zeta \) denote the mean inventory level in the steady state. Since \( \Phi^{\langle 0,l,y \rangle} \) gives the steady state probability vector of the \( l \)th the inventory level with components specify \( y \), number of demand in the pool and 0, the server in active mode and \( \Phi^{\langle x_1,1,Q,y \rangle} \) gives the steady state probability vector of the \( Q \)th inventory level with components specify \( y \), the number of demands in the pool and 1, the server in \( x_1 \) inactive period, \( \Phi^{\langle 0,l,y \rangle} e \) and \( \Phi^{\langle x_1,1,Q,y \rangle} e \)
gives the probability that the inventory level is \( l \) and \( Q \), respectively in the steady state. Hence the expected inventory level

\[
\zeta_I = \sum_{l=1}^{S} \sum_{y=0}^{N} l \Phi^{(0,l,y)} e + \sum_{x_1=1}^{M} \sum_{y=0}^{N} Q \phi^{(x_1,1,Q,y)} e \tag{8.5}
\]

### 8.4.2 Mean Reorder Rate

Let \( \zeta_R \) be the mean reorder rate in the steady state. The order is triggered only when the inventory level is \( s \). Since a drop to \( s \) from \( s+1 \) either by an arrival of primary demand or a selection of pooled demand, we get the mean reorder rate as

\[
\zeta_R = \sum_{y=0}^{N} \Phi^{(0,s+1,y)} [(N - y)\alpha \tau_1 + y\theta \tau_1] e \tag{8.6}
\]

### 8.4.3 Mean Number of Demand in the Pool

Let \( \zeta_{PD} \) be the steady state probability of mean number of demand in the pool. It is given by

\[
\zeta_{PD} = \sum_{l=0}^{S} \sum_{x_1=1}^{M} \sum_{y=0}^{N} \Phi^{(x_1,0,0,y)} [I_{(N+1)} \otimes T_1^0 \otimes e + I_{(N+1)m_1} \otimes T_2^0 \otimes \tau_3] e + \sum_{x_1=1}^{M} \sum_{y=0}^{N} \Phi^{(x_1,1,Q,y)} e \tag{8.7}
\]

### 8.4.4 Mean Length of the Inactive-idle Period

Let \( \zeta_{IS} \) denote the expected length of inactive idle period of the server in steady state case. Each inactive idle period of the server is terminated either by a replenishment of ordered items or by the completion of that inactive idle period. Thus \( \zeta_{IS} \) is obtained by

\[
\zeta_{IS} = \sum_{x_1=1}^{M} \Phi^{(x_1,0,0,y)} (I_{(N+1)} \otimes T_1^0 \otimes e + I_{(N+1)m_1} \otimes T_2^0 \otimes \tau_3) e \tag{8.8}
\]

### 8.5 Total Expected Cost

The expected total cost rate per unit time in the steady state for this model is defined to be

\[
TC(s, S) = c_h \zeta_I + c_S \zeta_R + c_w \zeta_{PD} + c_s \zeta_{IS}
\]

By putting the values of \( \zeta \)'s from the above measures of system performance, we obtain

\[
TC(s, S) = c_h \left[ \sum_{l=1}^{S} \sum_{y=0}^{N} l \Phi^{(0,l,y)} e + \sum_{i=1}^{M} \sum_{y=0}^{N} Q \phi^{(i,1,Q,y)} e \right]
\]

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\[ +c_s \sum_{y=0}^{N} \Phi^{(0,s+1,y)}[(N - y)\alpha\tau_1 + y\theta\tau_1]e \]
\[ +c_w \left[ \sum_{l=0}^{S} \sum_{y=0}^{N} y\Phi^{(0,l,y)}e + \sum_{x_1=1}^{M} \sum_{y=0}^{N} y\Phi^{(x_1,0,0,y)}e \right] \]
\[ + \sum_{x_1=1}^{M} \sum_{y=0}^{N} y\Phi^{(x_1,1,0,y)}e + \sum_{x_1=1}^{M} \sum_{y=0}^{N} y\Phi^{(x_1,1,Q,y)}e \]
\[ +c_{ct} \sum_{x_1=1}^{M} \Phi^{(x_1,1,0,y)}(I_{N+1} \otimes T_1^0 \otimes e + I_{N+1} \otimes T_2^0 \otimes \tau_3)e \]

### 8.6 Numerical Illustration

In this section, we discuss some interesting numerical examples that qualitatively describe the performance of the inventory system under study. Also, we propose a simple optimization problem among a class of optimization problems of interest. We consider the following three different phase-type distributions for the lead time, the idle time and the vacation time. These three phase-type distributions will be normalized so as to have a specific rates for lead time, idle time and vacation time respectively. We denote EXPR, EXPID and EXPV as exponential lead time, exponential idle time and exponential vacation time respectively. Similarly we also have for Erlang and hyper-exponential distribution.

1. **Exponential (EXP)**

   \[ \tau = (1) \quad T = (-1) \]

2. **Erlang (ERL)**

   \[ \tau = (1, 0, 0, 0) \quad T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

3. **Hyper-exponential (HEX)**

   \[ \tau = (0.9, 0.1) \quad T = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \]

**Example 8.4:** In this first example, we explore the behaviour of the long-run expected cost rate as a function of \( s \) and \( S \) with fixed cost and parameter values for
all three $PH$ distributions for lead time, idle time and vacation time. With a large number of numerical examples, we have found out that the total expected cost rate is either convex function or an increasing function of any one fixed variable. Figure 8.1 shows the convexity of the cost function for the combination of HEXL, EXPID, HEXV and the table 8.2 presents the optimal cost $TC^*(s, S)$ and its optimal $(s^*, S^*)$ for all combination of the phase type distribution. The optimal cost is minimum at $(6, 51)$, i.e, occurs $TC^*(6, 51) = 34.122977$ which occurs at HEXL, ERLID and ERLV and is maximum at $(9, 57)$, i.e, $TC^*(9, 57) = 37.678901$ which occurs at EXPL, HEXID and HEXV.

![Figure 8.1: Three dimension plot of the total expected cost function](image)

**Example 8.5:** Next, we study the impact of system costs, namely holding cost $c_h$, setup cost $c_s$, waiting cost $c_w$ and server idle cost $c_{is}$ on optimal total expected cost rate for all possible combinations of distributions of lead time, server idle time and vacation time and they are given in tables 8.3 - 8.6. In tables, the upper and the lower entries of each cell gives optimal cost rate $TC^*$ and the corresponding (local) optima $S^*$ and $s^*$ respectively. First, we fix $N = 20, M = 8, \theta = 5, \alpha = 2.5, \beta_1 = 5, \beta_2 = 7.9, \beta_3 = 6.8$. From tables 8.3 to 8.6, we observe the following

- As each cost is increasing, the optimal cost rate $TC^*$ is increasing for every combinations of distributions of lead time, server idle time and vacation time with increasing of each costs.

- Among all the possible combinations of distributions of lead time, server idle time and vacation time, the total expected cost is minimum at HEXL, ERLID and ERLV and is maximum at EXPL, HEXID and HEXV.
\[ c_h = 0.76, c_s = 9, c_w = 1.235, c_{is} = 7.17 \]

\[ N = 20, M = 8, \theta = 5, \alpha = 2.5, \beta_1 = 5, \beta_2 = 7.9, \beta_3 = 6.8 \]

\[
\begin{array}{|l|l|l|l|}
\hline
PH_V & EXPV & ERLV & HEXV \\
\hline
PH_L & PH_ID & \quad & \\
\hline
EXPL & EXPID & 35.644142 & 35.025813 & 36.733021 \\
& & 54 & 7 & 53 & 7 & 55 & 8 \\
& ERLID & 35.265640 & 34.717445 & 36.232100 \\
& & 53 & 7 & 52 & 6 & 55 & 8 \\
& HEXID & 36.354645 & 35.610279 & 37.678901 \\
& & 55 & 8 & 54 & 7 & 57 & 9 \\
ERLL & EXPID & 35.512763 & 34.949165 & 36.125620 \\
& & 57 & 11 & 56 & 10 & 59 & 12 \\
& ERLID & 35.080259 & 34.621842 & 35.550507 \\
& & 56 & 10 & 55 & 9 & 57 & 11 \\
& HEXID & 36.060631 & 35.363752 & 36.821564 \\
& & 59 & 12 & 57 & 11 & 60 & 13 \\
HEXL & EXPID & 34.728513 & 34.332775 & 35.387597 \\
& & 52 & 6 & 51 & 6 & 53 & 7 \\
& ERLID & 34.442145 & 34.122977 & 35.023630 \\
& & 51 & 6 & 51 & 6 & 52 & 6 \\
& HEXID & 35.146260 & 34.664927 & 35.940480 \\
& & 53 & 7 & 52 & 6 & 53 & 7 \\
\hline
\end{array}
\]

Table 8.2: Optimal cost of total expected cost rate
$c_s = 8.5, c_w = 1.24, c_{is} = 7.15$

<table>
<thead>
<tr>
<th>$c_h$</th>
<th>0.74</th>
<th>0.76</th>
<th>0.78</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{HV}$</td>
<td>EXPV</td>
<td>ERLV</td>
<td>HEXV</td>
</tr>
<tr>
<td>$PH_{LV}$</td>
<td>34.7249</td>
<td>34.1077</td>
<td>35.2990</td>
</tr>
<tr>
<td>$PH_{ID}$</td>
<td>53 7</td>
<td>53 7</td>
<td>55 8</td>
</tr>
<tr>
<td>$c_h$</td>
<td>35.4192</td>
<td>34.6907</td>
<td>36.7276</td>
</tr>
<tr>
<td>$c_h$</td>
<td>55 8</td>
<td>54 7</td>
<td>57 9</td>
</tr>
<tr>
<td>$c_s$</td>
<td>34.7249</td>
<td>34.1077</td>
<td>35.2990</td>
</tr>
<tr>
<td>$c_s$</td>
<td>53 7</td>
<td>53 7</td>
<td>55 8</td>
</tr>
<tr>
<td>$c_w$</td>
<td>35.4192</td>
<td>34.6907</td>
<td>36.7276</td>
</tr>
<tr>
<td>$c_w$</td>
<td>55 8</td>
<td>54 7</td>
<td>57 9</td>
</tr>
</tbody>
</table>

Table 8.3: Effect of holding cost $c_h$ on total expected cost

- As the holding cost $c_h$ increases, the optimal values $S^*$ and $s^*$ decreases. This is because as the inventory holding cost increases, the inventory is to be maintained with minimum stock.
- When the setup cost $c_s$ increases, the value of $S^*$ increases and the value of $s^*$ decreases. This is because, as the setup cost increases, to avoid frequent order, we have to maintain more stock and low reorder point.
- As the waiting time cost of a demand in pool $c_w$ and the idle time cost of the server $c_{is}$ increases, the value of $TC^*(s, S)$ increases.

**Example 8.6:** Here, we study the sensitivity of system parameters and rates such as quasi input parameter $\alpha$, replenishment rate $\beta_1$, idle rate $\beta_2$, vacation rate $\beta_3$ and inter-selection rate $\theta$ on the optimal total expected cost rate for all possible combinations of distributions of lead time, server idle time and vacation time. In tables, the upper and the lower entries of each cell gives optimal cost rate $TC^*$ and the corresponding (local) optima $S^*$ and $s^*$, respectively. We first fix $N = 20, M = 8, c_h = 0.76, c_s = 8.5, c_w = 1.235, c_{is} = 7.17$ and vary any one of rates and fix all other rates to be constant. From tables 8.7 to 8.10, we observe the following

- As is to be expected, the optimal cost rate $TC^*$ increases with increasing of both the arrival parameter $\alpha$ and idle rate $\beta_2$.
- $TC^*$ is decreasing when the inter-selection rate $\theta$ and the lead time $\beta_1$ increases.
\( c_h = 0.74, c_w = 1.235, c_{is} = 7.17 \)

<table>
<thead>
<tr>
<th>( PH_L )</th>
<th>( PH_{ID} )</th>
<th>( c_s )</th>
<th>( 8 )</th>
<th>( 8.5 )</th>
<th>( 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPID</td>
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<td>33.6148</td>
<td>35.3117</td>
<td>34.7284</td>
<td>34.1120</td>
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<tr>
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<td>53.7</td>
<td>56.9</td>
<td>55.8</td>
<td>57.7</td>
</tr>
<tr>
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<td>34.3519</td>
<td>35.8056</td>
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<td>52.7</td>
<td>54.8</td>
<td>53.7</td>
<td>53.7</td>
</tr>
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<td>34.1995</td>
<td>36.2506</td>
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</tr>
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<td>55.8</td>
<td>54.8</td>
<td>56.9</td>
<td>55.8</td>
<td>54.7</td>
</tr>
</tbody>
</table>

Table 8.4: Effect of setup cost \( c_s \) on optimal expected cost

<table>
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<tr>
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<th>( 1.23 )</th>
<th>( 1.235 )</th>
<th>( 1.24 )</th>
</tr>
</thead>
<tbody>
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<td>( 8.5 )</td>
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<td>35.7601</td>
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<td>53.8</td>
<td>52.7</td>
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<tr>
<td>ERLID</td>
<td>34.7609</td>
<td>33.7188</td>
<td>35.2494</td>
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<tr>
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<td>52.7</td>
<td>52.7</td>
<td>54.8</td>
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<tr>
<td>HEXID</td>
<td>35.3737</td>
<td>34.6297</td>
<td>36.7124</td>
</tr>
<tr>
<td></td>
<td>54.8</td>
<td>54.8</td>
<td>56.9</td>
</tr>
</tbody>
</table>

Table 8.5: Effect of demand waiting cost \( c_w \) on \( TC^*(s, S) \)
$c_h = 0.74, c_s = 9, c_w = 1.23$

<table>
<thead>
<tr>
<th>$c_{is}$</th>
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<th>7.17</th>
<th>7.19</th>
</tr>
</thead>
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<td>$P_{HiD}$</td>
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<td>7.17</td>
</tr>
<tr>
<td>EXPID</td>
<td>36.0354</td>
<td>35.4092</td>
<td>37.1741</td>
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<td>35.6577</td>
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<td>HEXID</td>
<td>36.7612</td>
<td>36.0026</td>
<td>38.1079</td>
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<tr>
<td>ERL</td>
<td>36.9413</td>
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<td>36.5664</td>
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<tr>
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<tr>
<td>HEXL</td>
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<td>35.7703</td>
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<td>HEXID</td>
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<td>35.0327</td>
<td>36.3306</td>
</tr>
</tbody>
</table>

Table 8.6: Effect of server idle cost $c_{is}$ on total expected cost rate

Because, if $\theta$ increases, the demands in the pool are selected frequently. So the waiting time is reduced and the cost of waiting demand is minimized. When the lead time rate $\beta_1$ increases, the ordered items are replenished quickly. So the idle time cost and waiting time cost of pooled demands are minimized since the server returned and starts service immediately.

- Among all the possible combinations of distributions of lead time, server idle time and vacation time, the total expected cost is minimum at ERLL, ERLID and ERLV and is maximum at EXPL, HEXID and HEXV.

- As $\alpha$ and idle rate $\beta_2$ increases the optimal values $S^*$ and $s^*$ increases. This is because, the inventory is to be maximized when the arrival parameter is increasing. Otherwise, the demands enter the pool and the waiting time cost is increasing and also that the order is to be placed frequently, so the setup cost is also increasing.

- When the inter-selection parameter $\theta$ increases, the value of $S^*$ increases and the value of $s^*$ decreases. When lead time rate $\beta_1$ increases, the value of $S^*$ and $s^*$ decreases.

- When vacation rate $\beta_3$ increases, the value of $S^*$ increases and the value of $s^*$ decreases.
\( \beta_1 = 4; \beta_2 = 5; \beta_3 = 7 \)

**ERL - EXPID - HEXV**

\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
\theta \backslash \alpha & 1.5 & 2 & 2.5 & 3 & 3.5 \\
\hline
3 & 25.39977 & 29.46277 & 33.03060 & 36.23991 & 39.15611 \\
& 39 & 10 & 47 & 13 & 58 & 17 & 63 & 19 \\
& 39 & 10 & 46 & 12 & 53 & 15 & 58 & 17 & 63 & 19 \\
5 & 25.28423 & 29.30939 & 32.84851 & 36.03618 & 38.94193 \\
& 39 & 10 & 46 & 12 & 53 & 15 & 59 & 17 & 64 & 19 \\
6 & 25.25216 & 29.26198 & 32.79790 & 35.97704 & 38.87920 \\
& 39 & 10 & 47 & 12 & 53 & 14 & 59 & 17 & 64 & 19 \\
7 & 25.22757 & 29.22555 & 32.75822 & 35.93207 & 38.83146 \\
& 40 & 10 & 47 & 12 & 53 & 14 & 59 & 17 & 64 & 19 \\
\hline
\end{array} \]

Table 8.7: Corresponding optimal expected cost for various \( \alpha \) and \( \theta \)

\( \alpha = 2.5; \theta = 5; \beta_2 = 5; \beta_3 = 7 \)

<table>
<thead>
<tr>
<th>( PH_L )</th>
<th>( \beta_1 )</th>
<th>EXPL</th>
<th>ERLL</th>
<th>HEXL</th>
</tr>
</thead>
<tbody>
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<td>( PH_V )</td>
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<td>4</td>
<td>6</td>
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<td>EXPID</td>
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<td>57</td>
<td>12</td>
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<td>56</td>
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<td>49</td>
<td>8</td>
<td>47</td>
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<td>35.26773</td>
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<td>47</td>
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<td>55</td>
<td>10</td>
<td>49</td>
<td>8</td>
<td>47</td>
</tr>
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<td>10</td>
<td>48</td>
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<td>11</td>
<td>49</td>
<td>8</td>
<td>47</td>
</tr>
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<tr>
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<td>50</td>
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<td>48</td>
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<td>33.84852</td>
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<td>9</td>
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<tr>
<td>57</td>
<td>13</td>
<td>50</td>
<td>9</td>
<td>49</td>
</tr>
</tbody>
</table>

Table 8.8: Influence of lead time \( \beta_1 \) on optimal cost
\[ \alpha = 2.5; \theta = 5; \beta_1 = 4; \beta_3 = 7 \]

<table>
<thead>
<tr>
<th>(PH_L)</th>
<th>(\beta_3)</th>
<th>(EXPL)</th>
<th>(ERL)</th>
<th>(HEXL)</th>
<th>(PH_ID)</th>
<th>(PH_v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPID</td>
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<td>35.70562</td>
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<td>33.50266</td>
<td>34.40470</td>
</tr>
<tr>
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</tr>
<tr>
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<td>38.64849</td>
<td>34.02918</td>
<td>35.12190</td>
<td>35.88288</td>
</tr>
</tbody>
</table>

| ERLID | EXPV | 33.40348 | 35.23463 | 36.44494 | 32.74616 | 33.87579 | 34.63564 | 32.77649 | 33.91124 | 34.64132 | 53 9 | 56 11 | 58 13 | 59 17 | 61 20 | 62 21 | 50 7 | 52 9 | 53 10 |
| ERLV | 33.14091 | 34.83544 | 35.94883 | 32.50649 | 33.50882 | 34.16802 | 32.69568 | 33.63864 | 34.29186 | 53 9 | 55 11 | 58 12 | 58 16 | 61 19 | 62 21 | 50 7 | 51 8 | 52 9 |
| HEXV | 34.04425 | 36.21177 | 37.60214 | 35.12108 | 34.46142 | 34.35239 | 34.13684 | 34.46511 | 34.34774 | 54 9 | 57 13 | 59 15 | 59 18 | 60 21 | 63 23 | 50 8 | 52 9 | 53 11 |

| HEXID | EXPV | 35.30358 | 36.72782 | 37.59664 | 34.31872 | 35.16992 | 35.69946 | 33.96322 | 34.88237 | 35.43574 | 56 11 | 58 13 | 60 14 | 62 21 | 63 22 | 64 23 | 52 9 | 53 10 | 54 11 |
| ERLV | 34.90119 | 36.21191 | 37.01441 | 33.90038 | 34.63794 | 35.06884 | 33.66320 | 34.49896 | 34.97978 | 56 10 | 58 12 | 59 13 | 61 20 | 62 21 | 62 22 | 53 10 | 53 9 | 53 11 |
| HEXV | 36.30467 | 37.99286 | 39.02676 | 35.01534 | 36.07106 | 36.74143 | 34.71553 | 35.07999 | 36.34717 | 57 13 | 59 15 | 61 17 | 63 22 | 64 24 | 65 25 | 52 10 | 54 12 | 55 13 |

Table 8.9: Effect of inactive server idle rate on optimal \(TC^*(s, S)\)

\[ \alpha = 2.5; \theta = 5; \beta_1 = 2; \beta_2 = 3 \]

<table>
<thead>
<tr>
<th>(PH_v)</th>
<th>(\beta_3)</th>
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<th>(ERL)</th>
<th>(HEXL)</th>
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<th>(PH_v)</th>
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<td>34.89764</td>
<td>34.92362</td>
<td>34.83316</td>
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</tbody>
</table>

| ERLID | EXPD | 33.28577 | 33.28361 | 33.33873 | 33.14261 | 33.00894 | 33.04211 | 32.68942 | 32.70090 | 32.73312 | 51 9 | 52 9 | 52 9 | 52 9 | 52 9 | 52 9 | 49 8 | 49 7 | 50 7 |
| HEXID | 32.76027 | 32.88141 | 33.01254 | 32.61713 | 32.62354 | 32.62809 | 32.37111 | 32.45153 | 32.53559 | 51 8 | 52 8 | 52 8 | 52 8 | 52 8 | 52 8 | 49 7 | 49 7 | 50 7 |

| HEXID | EXPD | 34.28032 | 34.11290 | 34.06876 | 33.7703 | 33.65737 | 33.60221 | 34.25307 | 34.17762 | 34.14560 | 52 10 | 52 10 | 52 10 | 52 10 | 52 10 | 52 10 | 49 8 | 50 8 | 50 8 |

Table 8.10: Sensitivity of vacation time \(\beta_3\) on optimal cost