6.1 Introduction

In the previous three chapters, we assumed that the time axis was continuous. Although the continuous time models have received wide attention, many situations, there is also a need for discrete time models. In modern telecommunication system, time can be discretized into time slots, which can be taken as one unit. All the system events such as arrival, departure, vacation, etc., may occur at the boundaries of the slot \( t \) \((t = 0, 1, 2, \ldots)\). All the above activities occur between epoch \( t \) and \( t + 1 \) is assumed to occur at epoch \( t + 1 \), and they may occur at the same time. Hunter [1983] and Takagi [1993b] described the possible arrangements of the system events. For a detailed review of discrete time inventory models was given in the chapter 1.

In this chapter, we conduct a comparative study on a discrete time \((s, S)\) inventory system with a single server to handle the postponed demands. Here we propose to have three different policies for availing vacation. These policies differ in having single (only one vacation) or multiple vacations (one-after-another) immediately or modified multiple vacations (after some random idle time). The postponed demands are permitted only when the stock is replenished and is sufficiently high. The models with different policies have been studied under the broad set up and justify the best vacation policy by using their total expected cost functions numerically.

The rest of the chapter is organized as follows. Mathematical formulations of all
the three models are described in Section 6.2. In Section 6.3, 6.4 and 6.5 respectively analyse each model in steady-state case and obtained total expected cost using system performance measures. Finally, we illustrate some numerical examples and justify the best vacation policy numerically.

6.2 Model Description

We consider the discrete time inventory system with postponed demands and three vacation policy. We assign inventory system with modified multiple vacation policy as model I, inventory system with single vacation policy as model II and inventory system with multiple vacation policy as model III. The assumptions of the system for all models are as follows

- The unit demands occurs according to an $DMAP (D_0, D_1)$ of order $n$.
- Orders are replenished according to $(s, S)$ ordering policy with geometrically distributed lead time with parameter $b(>0)$.
- Vacation time follows $DPH$ distribution with representation $(\tau_1, T_1)$ of order $m_1$ with vacation rate $\beta_1 = (\tau_1(I - T_1)^{-1}e)^{-1}$.
- Idle time follows $DPH$ distribution with representation $(\tau_2, T_2)$ of order $m_2$ with vacation rate $\beta_2 = (\tau_2(I - T_2)^{-1}e)^{-1}$ in case of modified vacation policy (only for Model I).
- Inter-selection time follows $DPH$ distribution with representation $(\tau_3, T_3)$ of order $m_3$ with vacation rate $\beta_3 = (\tau_3(I - T_3)^{-1}e)^{-1}$

Notation :

\[
\begin{align*}
N_0 & : 2nm_1 + nm_2 \\
N_1 & : Sn \\
N_2 & : sn + Qnm_3 \\
\bar{N}_0 & : 2nm_1 + n \\
\bar{N}_0 & : 2nm_1 \\
\xi_1(i) = \begin{pmatrix} e_2(i) \otimes I_{nm_1} \\ 0_{(nm_2,nm_1)} \end{pmatrix} & \quad \tilde{\xi}_1(i) = \begin{pmatrix} e_2(i) \otimes I_{nm_1} \\ 0_{(n,nm_1)} \end{pmatrix} \\
\xi_2(1) = \begin{pmatrix} 0_{(2nm_1,nm_2)} \\ I_{nm_2} \end{pmatrix} & \quad \tilde{\xi}_2(1) = \begin{pmatrix} 0_{(2nm_1,n)} \\ I_n \end{pmatrix}
\end{align*}
\]
\[ \xi_3(l) = \left( e_s(l) \otimes I_n \right) \quad \xi_4(l) = \left( 0_{(s_n.m_n)} \otimes e_Q(l) \otimes I_{m_3} \right) \]

### 6.3 Modified Multiple Vacation Policy - Model I

The state of the system at a time \( t^+ \) can be described by a process \( Z_t = \{(X_t, Y_t, L_t, J_1, J_2, J_3); t \in N_0\} \) where \( X_t, Y_t, L_t, J_1, J_2, J_3 \) denote, respectively, the number of demands in the pool, status of the server, the on-hand inventory level, phase of the arrival process, phase of the vacation process, phase of the idle time and phase of the inter-selection at time \( t \). Further, let

\[
Y_t = \begin{cases} 
0, & \text{if the server is on vacation} \\
1, & \text{if the server is not on vacation}
\end{cases}
\]

Because of the discrete time set up, multiple events can occur in a slot, for mathematical tractability, we assume the following order of occurrence: supply of an order (if any), termination of idle period, termination of vacation period, a primary demand, selection process is initiated, pooled demand is satisfied, idle process initiated (only for model I) and finally vacation process is initiated.

Let the Markov chain \( Z_t = \{(X_t, Y_t, L_t, J_1, J_2, J_3); t \in N_0\} \) be a discrete time quasi-birth-death (QBD) process with the state space \( \Omega \) is given by

\[
\Omega = \{ (x, 0, l, j, j_1): x \in E_0; l = 0, Q; j \in E_{1}^{m_1}; j_1 \in E_{2}^{m_1} \} \\
\cup \{ (x, 1, l, j, j_2): x \in E_0; l = 0; j \in E_1^n ; j_2 \in E_{m_2}^{m_2} \} \\
\cup \{ (x, 1, l, j): x = 0; l \in E_1^S ; j \in E_1^n \} \\
\cup \{ (x, 1, l, j): x \in E_1; l \in E_1^n ; j \in E_1^n \} \\
\cup \{ (x, 1, l, j, j_3): x \in E_1; l \in E_{x+1}^S ; j \in E_1^n ; j_3 \in E_{m_3}^{m_3} \}.
\]

Using the lexicographical order of the states, we get the transition probability matrix \( P \) as

\[
P = \begin{pmatrix}
B_1 & B_0 & 0 & 0 & \cdots \\
B_2 & A_1 & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]
where

\[
\begin{align*}
B_0 &= \begin{pmatrix} B_{00}^0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_{10}^1 & B_{11}^1 \\ B_{10}^1 & B_{11}^1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B_{22}^2 \end{pmatrix} \\
B_{00}^0 &= \begin{pmatrix} \bar{b}(D_1 \otimes T_1) & b(D_1 \otimes T_1) & \bar{b}(D_1 \otimes T_1^0 \otimes \tau_2) \\ 0 & D_1 \otimes T_1 & 0 \\ \bar{b}(D_1 \otimes T_2^0 \otimes \tau_1) & 0 & \bar{b}(D_1 \otimes T_2) \end{pmatrix}, \\
B_{10}^1 &= \begin{pmatrix} \bar{b}(D_0 \otimes T_1) & b(D_0 \otimes T_1) & \bar{b}(D_0 \otimes T_1^0 \otimes \tau_2) \\ 0 & D_0 \otimes T_1 & 0 \\ \bar{b}(D_0 \otimes T_2^0 \otimes \tau_1) & 0 & \bar{b}(D_0 \otimes T_2) \end{pmatrix}, \\
B_{11}^1 &= \begin{pmatrix} 0 & 0 & e(1) \otimes \bar{b}(D_1 \otimes \tau_2) \\ 0 & 0 & e(1) \otimes \bar{b}(D_1 \otimes \tau_2) \\ 0 & 0 & e(1) \otimes \bar{b}(D_1 \otimes \tau_2) \end{pmatrix}, \\
B_{01}^0 &= \begin{pmatrix} 0 & \cdots & 0 & b(D_1 \otimes T_1^0) & b(D_0 \otimes T_1^0) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & D_1 \otimes T_1^0 & D_0 \otimes T_1^0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b(D_1 \otimes e_{m_2}) & b(D_0 \otimes e_{m_2}) & 0 & \cdots & 0 \end{pmatrix}, \\
B_{10}^1 &= \begin{pmatrix} bD_0, & l = k, & k = 1, 2, \ldots, s \\ bD_1, & l = k - 1, & k = 2, 3, \ldots, s \\ bD_1, & l = k + Q - 1, & k = 1, 2, \ldots, s \\ bD_0, & l = k + Q, & k = 1, 2, \ldots, s \\ D_1, & l = k - 1, & k = s + 1, s + 2, \ldots, S \\ D_0, & l = k, & k = s + 1, s + 2, \ldots, S \\ 0, & \text{otherwise} \end{pmatrix}, \\
B_{11}^1 &= \begin{pmatrix} D_1 \otimes T_1^0 & l = k - 2, & k = s + 2, s + 3, \ldots, S \\ D_0 \otimes T_3 & l = k - 1, & k = s + 1, s + 2, \ldots, S \\ 0, & \text{otherwise} \end{pmatrix}, \\
A_0 &= \begin{pmatrix} B_{00}^0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} B_{00}^1 & A_{01}^1 \\ A_{10}^1 & A_{11}^1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & A_{22}^2 \end{pmatrix}, \\
A_{10}^1 &= \begin{pmatrix} 0 & 0 & e(1) \otimes \bar{b}(D_1 \otimes \tau_2) \\ 0 & 0 & e(1) \otimes \bar{b}(D_1 \otimes \tau_2) \\ 0 & 0 & e(1) \otimes \bar{b}(D_1 \otimes \tau_2) \end{pmatrix}, \\
A_{01}^1 &= \begin{pmatrix} 0 & \cdots & 0 & b(D_1 \otimes T_1^0 \otimes \tau_3) & b(D_0 \otimes T_1^0 \otimes \tau_3) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & D_1 \otimes T_1^0 \otimes \tau_3 & D_0 \otimes T_1^0 \otimes \tau_3 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b(D_1 \otimes e_{m_2} \otimes \tau_3) & b(D_0 \otimes e_{m_2} \otimes \tau_3) & 0 & \cdots & 0 \end{pmatrix}, \\
A_{11}^1 &= \begin{pmatrix} bD_0, & l = k, & k = 1, 2, \ldots, s \\ bD_1, & l = k - 1, & k = 2, 3, \ldots, s \\ b(D_1 \otimes \tau_3), & l = k + Q - 1, & k = 1, 2, \ldots, s \\ b(D_0 \otimes \tau_3), & l = k + Q, & k = 1, 2, \ldots, s \\ D_1 \otimes e_{m_2}, & l = k - 1, & k = s + 1 \\ D_1 \otimes T_3, & l = k - 1, & k = s + 2, s + 3, \ldots, S \\ D_0 \otimes T_3, & l = k, & k = s + 1, s + 2, \ldots, S \\ 0, & \text{otherwise} \end{pmatrix}, \\
A_{21}^2 &= \begin{pmatrix} D_0 \otimes T_3^0, & l = k - 1, & k = s + 1 \\ D_1 \otimes T_3^0, & l = k - 2, & k = s + 2 \\ D_0 \otimes T_3^0 \otimes \tau_3, & l = k - 1, & k = s + 2, s + 3, \ldots, S \\ D_1 \otimes T_3^0 \otimes \tau_3, & l = k - 2, & k = s + 3, s + 4, \ldots, S \\ 0, & \text{otherwise} \end{pmatrix}, \\
\end{align*}
\]

Dimension of sub-matrices $B_0$, $B_1$ and $B_2$ are $(N_0 + N_1) \times (N_0 + N_2)$, $(N_0 + N_1) \times (N_0 + N_1)$ and $(N_0 + N_2) \times (N_0 + N_1)$ respectively. Dimension of $A_0$, $A_1$ and $A_2$ are $(N_0 + N_2) \times (N_0 + N_2)$. 

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### 6.3.1 Stability Analysis

In order to obtain the stability condition of the process, let $A = A_0 + A_1 + A_2$ be the generator matrix and $\pi$ be the stationary probability vector of $A$.

$$A = \begin{pmatrix} F_{00} & A_{10}^T \\ A_{10} & F_{11} \end{pmatrix}$$

where

$$F_{00} = \begin{pmatrix} \bar{b}(D \otimes T_1) & b(D \otimes T_1) & \bar{b}(D \otimes T_1^0 \otimes \tau_2) \\ 0 & D \otimes T_1 & 0 \\ \bar{b}(D \otimes T_2^0 \otimes \tau_1) & 0 & \bar{b}(D \otimes T_2) \end{pmatrix}$$

$$[F_{11}]_{kl} = \begin{cases} \bar{b}D_0, & k = 1, 2, \ldots, s \\ bD_1, & l = k - 1, \quad k = 2, 3, \ldots, s \\ b(D_k \otimes \tau_2), & l = k + Q, \quad k = 1, 2, \ldots, s \\ b(D_1 \otimes \tau_3), & l = k + Q - 1, \quad k = 1, 2, \ldots, s \\ (D_0 \otimes T_3), & l = k, \quad k = s + 1, s + 2, \ldots, S \\ (D_1 \otimes e_{m_3}) + (D_0 \otimes T_3^0), & l = k - 1, \quad k = s + 1 \\ D_1 \otimes T_3^0, & l = k - 2, \quad k = s + 2 \\ (D_1 \otimes T_3) + (D_0 \otimes T_3^0 \tau_2), & l = k - 1, \quad k = s + 2, s + 3, \ldots, S \\ (D_1 \otimes T_3^0 \tau_3) & l = k - 2, \quad k = s + 3, s + 4, \ldots, S \end{cases}$$

otherwise

Let $\pi = (\pi^*, \pi^1)$, where $\pi^* = (\pi^{(0,0)}, \pi^{(0,Q)}, \pi^{(1,0)})$ and $\pi^1 = (\pi^{(1,1)}, \pi^{(1,2)}, \ldots, \pi^{(1,S)})$ which satisfy $\pi A = \pi$ and $\pi e = 1$.

From the well known result of Neuts [1994] on the positive recurrence of $A$, the Markov chain is positive recurrent if and only if $\pi A_0 e < \pi A_2 e$. Thus the stability condition of the process is

$$\pi^{(0,0)}[(D_1 \otimes T_1)e + \bar{b}(D_1 \otimes T_1^0 \otimes \tau_2)e] + \pi^{(0,Q)}[D_1 \otimes T_1]e + \pi^{(1,0)}[\bar{b}(D_1 \otimes T_2^0 \otimes \tau_1)e]$$

$$+ \bar{b}(D_1 \otimes T_2)e] < \pi^{(1,s+1)}(D_0 \otimes T_3^0)\pi e + \pi^{(1,s+2)}(D_1 \otimes T_3^0 + D_0 \otimes T_3^0 \tau_3)e$$

$$+ \sum_{k=s+3}^S \pi^{(1,k)}(D \otimes T_3^0 \tau_3)e$$

where

$$\pi^{(0,l)} = \left\{ (\pi^{(0,l,j_1 j_2 j_3)}) : l = 0, Q; j_1 \in E_1^n; j_1 \in E_1^{m_1} \right\}$$

$$\pi^{(1,0)} = \left\{ (\pi^{(1,0,l,j_2)}) : j \in E_1^n; j_2 \in E_1^{m_2} \right\}$$

$$\pi^{(1,l)} = \left\{ (\pi^{(1,l,j_3)}) : l \in E_1^s; j \in E_1^n \right\}$$

$$\left\{ (\pi^{(1,l,j_3)}) : l \in E_1^s; j \in E_1^n \right\}$$

$$\pi^{(1,3)} = \left\{ (\pi^{(1,l,j_3)}) : l \in E_1^s; j \in E_1^n \right\}$$

$$j_3 \in E_1^{m_3}$$

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6.3.2 Steady State Analysis

Let $\Phi = (\Phi(0), \Phi(1), \Phi(2), \ldots)$ be the steady state probability vector associated with the transition matrix $P$ such that $\Phi P = \Phi$, $\Phi e = 1$.

Let us partition the vector $\Phi(x)$ into $(\Phi(x,0), \Phi(x,1), \Phi(x,2), \ldots)$ and the vector $\Phi(x,1)$ into $(\Phi(x,1), \Phi(x,1,2), \ldots)$. For $x \in E_0$, $\Phi(x,0)$ and $\Phi(x,1)$ are row $n$ vectors, $\Phi(x,1)$ is a row $n$ vector. For $x \in E_1$, $\Phi(x,1)$ is a row $n$ vector and $\Phi(x,1,1)$ is a row $nm$ vector.

Theorem 6.3.1. When the stability condition (6.1) holds good, the steady-state probability vector $\Phi$ is given by

$$\Phi(x) = \Phi(1)R^{x-1}, x = 1, 2, \ldots$$

where the matrix $R$ satisfies the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = R$$

and the vector $\Phi(0)$ and $\Phi(1)$ is obtained by solving

$$B[R] = (\Phi(0), \Phi(1))$$

subject to the normalizing condition

$$\Phi(0) e + \Phi(1) (I - R)^{-1} e = 1.$$  

Proof. The theorem follows from the well-known result on matrix-geometric methods Neuts [1994].

Due to the special structure of the coefficient matrices appearing in (6.3), the square matrix $R$ of dimension $(N_0 + N_2)$ can be computed as follows: We note that the matrix $R$ also has zero rows as number of zero rows in $A_0$.

$$R = \begin{pmatrix} R_0 & R_1 \\ 0 & 0 \end{pmatrix}, \quad R^k = \begin{pmatrix} R_0^k & R_0^{k-1} R_1 \\ 0 & 0 \end{pmatrix}, k = 1, 2, \ldots,$$

where

$$R_0 = \begin{pmatrix} R_{00} & R_{0Q} & R_{001} \\ R_{Q0} & R_{QQ} & R_{Q01} \\ R_{00'0} & R_{00'Q} & R_{00'01} \end{pmatrix}, \quad R_1 = \begin{pmatrix} R_{01} & R_{02} & \cdots & R_{0S} \\ R_{Q1} & R_{Q2} & \cdots & R_{QS} \\ R_{01'} & R_{02'} & \cdots & R_{0S'} \end{pmatrix}.$$
Computation of Steady State Probability Vector

From the equation (6.2) and by the structure of the $R$ matrix, we have

$$(\Phi(x,*), \Phi(x,1)) = (\Phi(1,*), \Phi(1,1)) R_{x}^{-1}, x \in E_2$$

Due to structure of $R$ matrix, we can written above equation as

For $x \in E_2$

$$\Phi(x,*') = \Phi(1,*') R_{x}^{-1}$$

and

$$\Phi(x,1) = \Phi(1,1) R_{x}^{-2} R_{1}$$

$$\Phi(x,0,Q) = \Phi(1,s) R_{0}^{-1} \xi(1)$$

$$\Phi(x,1,0) = \Phi(1,1) R_{0}^{-1} \xi(2)$$

6.3.3 System Performance Measures

In this section, we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

Expected Inventory Level

Let $\zeta_I$ denote the mean inventory level in the steady state. Since $\Phi(x,1,l)$ is a steady state probability vector of the $l$ inventory in the system with component specify $x$, the number of demands in the pool and 1, the server is not on vacation and $\Phi(x,0,Q)$ is a steady state probability vector of $Q$ inventory in the system with the component specify $x$, the number of demands in the pool and 0, the server is on vacation, the mean inventory level $\zeta_I$ in the steady state is given by

$$\zeta_I = \sum_{x=0}^{\infty} \sum_{l=1}^{S} l \Phi(x,1,l) e + \sum_{x=0}^{\infty} Q \Phi(x,0,Q) e$$

$$= Q \Phi(0,0,Q) e + \sum_{x=0}^{\infty} \sum_{l=1}^{S} l \Phi(x,1,l) e + Q \Phi(1,s) (I - R_0)^{-1} \xi(2) e$$

$$+ \Phi(1,s) (I - R_0)^{-1} R_1 \left[ \sum_{l=1}^{s} l \xi_3(l) + \sum_{l=1}^{Q} (s+l) \xi_4(l) \right] e$$

(6.6)

Expected Reorder Rate

Let $\zeta_R$ denote the expected reorder rate in the steady state. A reorder is triggered when the inventory level drops from $s+1$ to $s$ either by a primary demand or selection of a pooled demand.

$$\zeta_R = \Phi(0,1,s+1) D_1 e + \sum_{x=1}^{\infty} \Phi(x,1,s+1) (D_0 \otimes T_3^0 + D_1 \otimes e_{m_2}) e + \sum_{x=1}^{\infty} \Phi(x,1,s+2) (D_1 \otimes T_3^0) e$$
\[
= \Phi^{(0,1,s+1)}D_1e + \left[\Phi^{(1,1,s+1)} + \Phi^{(1,s)}(I - R_0)^{-1}R_1\xi_4(1)\right] (D_0 \otimes T_3^0 + D_1 \otimes e_{m_3})e
+ \left[\Phi^{(1,1,s+2)} + \Phi^{(1,s)}(I - R_0)^{-1}R_1\xi_4(2)\right] (D_1 \otimes T_3^0)e
\] (6.7)

**Expected Number of Demands in the Pool**

Let \( \zeta_P \) denote the expected number of pool demands in the steady state. This is given by

\[
\zeta_P = \sum_{x=1}^{\infty} x\Phi^{(x)} e = \Phi^{(1)} [I - R]^{-2} e
\] (6.8)

**Expected Length of the Idle Period**

Let \( \zeta_{IS} \) denote the length of the idle period of the server in the steady state case. Idle period is terminated either by the completion of idle period or the replenishment of ordered item. This leads \( \zeta_{IS} \) to be

\[
\zeta_{IS} = \sum_{x=0}^{\infty} \Phi^{(x,1,0)} \left( b(D \otimes T_2^0 \otimes \tau_1) e + \delta_{x0}b(D \otimes e_{m_2}) e + \delta_{x0}b(D \otimes e_{m_2} \otimes \tau_3) e \right) \\
= \Phi^{(x,1,0)} \left[ b(D \otimes T_2^0 \otimes \tau_1) + b(D \otimes e_{m_2}) \right] e \\
+ \Phi^{(1,s)} (I - R_0)^{-1}\xi_2(1) \left[ b(D \otimes T_2^0 \otimes \tau_1) + b(D \otimes e_{m_2} \otimes \tau_3) \right] e
\] (6.9)

**6.3.4 Total Expected Cost Rate**

The long-run total expected cost per unit time for this system in the steady state is given by

\[
TC(s, S) = c_h \zeta_I + c_s \zeta_R + c_{pd} \zeta_{PD} + c_{is} \zeta_{IS}
\]

By putting the values of \( \zeta \)'s from the above measures of system performance, we obtain

\[
TC(s, S) = c_h \left( Q\Phi^{(0,0,Q)} e + \sum_{x=0}^{\infty} \sum_{l=1}^{S} \Phi^{(x,1,l)} e + Q\Phi^{(1,s)} (I - R_0)^{-1}\xi_1(2) e \right) \\
+ \Phi^{(1,s)} (I - R_0)^{-1}R_1 \left[ \sum_{l=1}^{S} l\xi_3(l) + \sum_{l=1}^{Q} (s + l)\xi_4(l) \right] e \\
+ \Phi^{(1,s)} (I - R_0)^{-1}\xi_2(1) \left[ b(D \otimes T_2^0 \otimes \tau_1) e + b(D \otimes e_{m_2} \otimes \tau_3) e \right]
\]
6.4 Single Vacation Policy - Model II

From our assumptions made on the input and output processes, it can be verified that the stochastic process \( \tilde{Z}_t = \{(X_t, Y_t, L_t, J_t, J^1_t, J^2_t): t \in \mathbb{N}_0 \} \) is a discrete time Markov chain with the state space

\[
\tilde{\Omega} = \{(x, 0, l, j, j_1): x \in E_0; l = 0, Q; j \in E^S_1; j_1 \in E^{m_1}_1 \} \\
\cup \{(x, 1, l, j): x = 0; l \in E^S_0; j \in E^1_1 \} \\
\cup \{(x, 1, l, j): x \in E_1; l \in E^S_0; j \in E^1_1 \} \\
\cup \{(x, 1, l, j, j_3): x \in E_1; l \in E^S_{s+1}; j \in E^1_1; j_3 \in E^{m_1}_1 \}.
\]

Using the lexicographical order of the states, we get the transition probability matrix \( \tilde{P} \) is given by

\[
\tilde{P} = \begin{pmatrix}
B_1 & B_0 & 0 & 0 & \cdots \\
B_2 & A_1 & A_0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & A_2 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

where

\[
\begin{align*}
\tilde{B}_0 &= \begin{pmatrix} B_{00} & 0 \\ 0 & 0 \end{pmatrix} \\
\tilde{B}_1 &= \begin{pmatrix} B_{10} & \tilde{B}_{01} \\ \tilde{B}_{10} & B_{11} \end{pmatrix} \\
\tilde{B}_2 &= \begin{pmatrix} 0 & 0 \\ 0 & B_{11} \end{pmatrix} \\
\tilde{B}^0_{00} &= \begin{pmatrix} b(D_1 \otimes T_1) & b(D_1 \otimes T_1) \\ 0 & D_1 \otimes T_1 \end{pmatrix} \\
\tilde{B}^1_{01} &= \begin{pmatrix} b(D_0 \otimes T_1) & b(D_0 \otimes T_1) \\ 0 & D_0 \otimes T_1 \end{pmatrix} \\
\tilde{A}_0 &= \begin{pmatrix} \tilde{B}_{00}^0 & 0 \\ 0 & \tilde{A}_{01} \end{pmatrix} \\
\tilde{A}_{10} &= \begin{pmatrix} e(1) \otimes b(D_0 \otimes \tau_1) & 0 \\ 0 & \tilde{A}_{11} \end{pmatrix} \\
\tilde{A}_{01} &= \begin{pmatrix} 0 & 0 \end{pmatrix}
\end{align*}
\]

Dimension of sub-matrices \( \tilde{B}_0, \tilde{B}_1 \) and \( \tilde{B}_2 \) are \((\tilde{N}_0 + N_1) \times (\tilde{N}_0 + N_2), (\tilde{N}_0 + N_1) \times (\tilde{N}_0 + N_1) \) and \((\tilde{N}_0 + N_2) \times (\tilde{N}_0 + N_1) \) respectively. The dimension of \( \tilde{A}_0, \tilde{A}_1 \) and \( \tilde{A}_2 \) are \((\tilde{N}_0 + N_2) \times (\tilde{N}_0 + N_2) \).


### 6.4.1 Stability Analysis

In order to obtain the stability condition of the process, let \( \tilde{A} = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 \) be the generator matrix and \( \tilde{\pi} \) be the stationary probability vector of \( \tilde{A} \).

\[
\tilde{A} = \begin{pmatrix}
\tilde{F}_{00} & \tilde{A}_{01} \\
\tilde{A}_{10} & \tilde{F}_{11}
\end{pmatrix}
\]

where

\[
\tilde{F}_{00} = \begin{pmatrix}
\bar{b}(D \otimes T_1) & b(D \otimes T_1) & b(D \otimes T_1^0 \otimes \tau_2) \\
0 & D \otimes T_1 & 0 \\
0 & 0 & \bar{b}D
\end{pmatrix}
\]

Let \( \tilde{\pi} = (\tilde{\pi}^*, \tilde{\pi}^1) \), where \( \tilde{\pi}^* = (\tilde{\pi}^{(0,0)}, \tilde{\pi}^{(0,Q)}, \tilde{\pi}^{(1,0)}) \) and \( \tilde{\pi}^1 = (\tilde{\pi}^{(1,1)}, \tilde{\pi}^{(1,2)}, \ldots, \tilde{\pi}^{(1,S)}) \) which satisfy \( \tilde{\pi} \tilde{A} = \tilde{\pi} \) and \( \tilde{\pi}e = 1 \).

From the well known result of Neuts [1994] on the positive recurrence of \( \tilde{A} \). We have that the Markov chain is positive recurrent if and only if \( \tilde{\pi} \tilde{A}_0 e < \tilde{\pi} \tilde{A}_2 e \). Thus the stability condition of the process is

\[
\tilde{\pi}^{(0,0)}[(D_1 \otimes T_1)e + \bar{b}(D_1 \otimes T_1^0)e] + \tilde{\pi}^{(0,Q)}[D_1 \otimes T_1]e + \tilde{\pi}^{(1,0)}bD_1e < \\
\tilde{\pi}^{(1,s+1)}(D_0 \otimes T_3^0)e + \tilde{\pi}^{(1,s+2)}(D_1 \otimes T_3^0 + D_0 \otimes T_3^0 \tau_3)e + \sum_{k=s+3}^{S} \tilde{\pi}^{(1,k)}(D \otimes T_3^0 \tau_3)e(6.10)
\]

where

\[
\tilde{\pi}^{(0,l)} = \begin{cases}
(\tilde{\pi}^{0,l,j,j_1}) & : l = 0, Q; j, j_1 \in E_1^n; j_1 \in E_1^{m_1}
\end{cases}
\]

\[
\tilde{\pi}^{(1,0)} = \begin{cases}
(\tilde{\pi}^{1,0,j,j_2}) & : j, j_2 \in E_1^{m_2}
\end{cases}
\]

\[
\tilde{\pi}^{(1,l)} = \begin{cases}
(\tilde{\pi}^{1,l,j_3}) & : l \in E_1^n; j, j_3 \in E_1^{m_3}
(\tilde{\pi}^{1,l,j_3}) & : l \in E_1^{m_3}
\end{cases}
\]

### 6.4.2 Steady State Analysis

Let \( \tilde{\Phi} = (\tilde{\Phi}^{(0)}, \tilde{\Phi}^{(1)}, \tilde{\Phi}^{(2)}, \ldots) \) be the steady state probability vector associated with the transition matrix \( \tilde{P} \) such that \( \tilde{\Phi} \tilde{P} = \tilde{\Phi}, \tilde{\Phi}e = 1 \).

Let us partition the vector \( \tilde{\Phi}^{(x)} \) into \( (\tilde{\Phi}^{(x,s)}, \tilde{\Phi}^{(x,1)}), x \in E_0 \), where each of the vector can further be partitioned, for computational purposes, the vector \( \tilde{\Phi}^{(x,s)} \) into \( (\tilde{\Phi}^{(x,0,0)}, \tilde{\Phi}^{(x,0,Q)}, \tilde{\Phi}^{(x,1,0)}) \) and the vector \( \tilde{\Phi}^{(x,1)} = (\tilde{\Phi}^{(x,1,1)}, \tilde{\Phi}^{(x,1,2)}, \ldots, \tilde{\Phi}^{(x,1,S)}) \). For \( x \in E_0 \), \( \tilde{\Phi}^{(x,0,0)} \) and \( \tilde{\Phi}^{(x,0,Q)} \) are row \( nm_1 \) vector. For \( x = 0 \) and \( l \in E_0^S \tilde{\Phi}^{(x,1,l)} \) is a row \( n \) vector. For \( x \in E_1 \), \( \tilde{\Phi}^{(x,1,l)} (l \in E_1^n) \) is a row \( n \) vector and \( \tilde{\Phi}^{(x,1,l)} (l \in E_1^{m_3}) \) is a row \( nm_3 \) vector.
Theorem 6.4.1. When the stability condition (6.10) holds good, the steady-state probability vector $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(x) = \tilde{\Phi}^{(1)} \tilde{R}^{x-1}, \ x = 1, 2, ...$$

(6.11)

where the matrix $\tilde{R}$ satisfies the matrix quadratic equation

$$\tilde{R}^2 \tilde{A}_2 + \tilde{R} \tilde{A}_1 + \tilde{A}_0 = \tilde{R}$$

(6.12)

and the vector $\tilde{\Phi}^{(0)}$ and $\tilde{\Phi}^{(1)}$ is obtained by solving

$$(\tilde{\Phi}^{(0)}, \tilde{\Phi}^{(1)}) B[\tilde{R}] = (\tilde{\Phi}^{(0)}, \tilde{\Phi}^{(1)})$$

(6.13)

subject to the normalizing condition

$$\tilde{\Phi}^{(0)} e + \tilde{\Phi}^{(1)} (I - \tilde{R})^{-1} e = 1.$$  

(6.14)

Proof. The theorem follows from the well-known result on matrix-geometric methods Neuts [1994].

\[ \square \]

Computation of $\tilde{R}$ and $\tilde{\Phi}^{(i)}$ vector

Due to special structure of $\tilde{A}_0$, we note that

$$\tilde{R} = \begin{pmatrix} \tilde{R}_0 & \tilde{R}_1 \\ 0 & 0 \end{pmatrix}$$

and $\tilde{R}^k = \begin{pmatrix} \tilde{R}_0^k & \tilde{R}_0^{k-1} \tilde{R}_1 \\ 0 & 0 \end{pmatrix}, \ k = 1, 2, .. ,$

where

$$\tilde{R}_0 = \begin{pmatrix} \tilde{R}_{00} & \tilde{R}_{0Q} & \tilde{R}_{0Q0} \\ \tilde{R}_{Q0} & \tilde{R}_{QQ} & \tilde{R}_{QQ0} \\ \tilde{R}_{01} & \tilde{R}_{0Q1} & \tilde{R}_{0Q01} \end{pmatrix}$$

$$\tilde{R}_1 = \begin{pmatrix} \tilde{R}_{01} & \tilde{R}_{02} & ... & \tilde{R}_{0S} \\ \tilde{R}_{Q1} & \tilde{R}_{Q2} & ... & \tilde{R}_{QS} \\ \tilde{R}_{01} & \tilde{R}_{012} & ... & \tilde{R}_{01S} \end{pmatrix}.$$  

From equation (6.14), we have

For $i \in E_2$

$$\tilde{\Phi}^{(x,s)} = \tilde{\Phi}^{(1,s)} \tilde{R}_0^{x-1}$$

$$\tilde{\Phi}^{(x,0,0)} = \tilde{\Phi}^{(1,0,0)} \tilde{R}_0^{x-1} \xi_1(1)$$

$$\tilde{\Phi}^{(x,0,Q)} = \tilde{\Phi}^{(1,0,0)} \tilde{R}_0^{x-1} \xi_1(2)$$

$$\tilde{\Phi}^{(x,1,0)} = \tilde{\Phi}^{(1,1,0)} \tilde{R}_0^{x-1} \xi_2(1)$$

$$\tilde{\Phi}^{(x,1,l)} = \tilde{\Phi}^{(1,1,l)} \tilde{R}_0^{x-2} R_{1}\xi_3(l), \ l \in E_{1}^{s}$$

$$\tilde{\Phi}^{(x,1,l)} = \tilde{\Phi}^{(1,1,l)} \tilde{R}_0^{x-2} R_{1}\xi_4(l - s), \ l \in E_{1}^{S+1}$$

6.4.3 System Performance Measures

In this section, we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.
Expected Inventory Level

Let \( \tilde{\zeta}_I \) denote the mean inventory level in the steady state. Since \( \tilde{\Phi}^{(x,1,l)} \) is a steady state probability vector of the \( l \) inventory in the system with component specify \( x \), the number of demands in the pool and 1, the server is not on vacation and \( \tilde{\Phi}^{(x,0,Q)} \) is a steady state probability vector of \( Q \) inventory in the system with the component specify \( x \), the number of demands in the pool and 0, the server is on vacation, the mean inventory level \( \tilde{\zeta}_I \) in the steady state is given by

\[
\tilde{\zeta}_I = \sum_{x=0}^{\infty} Q \tilde{\Phi}^{(x,0,Q)} e + \sum_{l=1}^{S} l \tilde{\Phi}^{(x,1,l)} e
\]

\[
= Q \tilde{\Phi}^{(0,0,Q)} e + Q \tilde{\Phi}^{(1,*)}(I - \tilde{R}_0)^{-1} \tilde{\zeta}_I (2)e + \sum_{x=0}^{1} \sum_{l=1}^{S} l \tilde{\Phi}^{(x,1,l)} e
\]

\[
+ \tilde{\Phi}^{(1,*)}(I - \tilde{R}_0)^{-1} \tilde{R}_1 \left[ \sum_{l=1}^{s} l \xi_3 (l) + \sum_{l=1}^{Q} (s + l) \xi_4 (l) \right] e
\]

(6.15)

Expected Reorder Rate

Let \( \tilde{\zeta}_R \) denote the expected reorder rate in the steady state. A reorder is triggered when the inventory level drops from \( s + 1 \) to \( s \) either by a primary demand or selection of a pooled demand.

\[
\tilde{\zeta}_R = \tilde{\Phi}^{(0,1,s+1)} D_1 e + \sum_{x=1}^{\infty} \tilde{\Phi}^{(x,1,s+1)} (D_0 \otimes T_0^0 + D_1 \otimes e_{m_3}) e
\]

\[
+ \sum_{x=1}^{\infty} \tilde{\Phi}^{(x,1,s+2)} (D_1 \otimes T_3^0) e
\]

\[
= \tilde{\Phi}^{(0,1,s+1)} D_1 e + \tilde{\Phi}^{(1,1,s+1)} (I - \tilde{R}_0)^{-1} \tilde{R}_1 \xi_4 (1) (D_0 \otimes T_3^0 + D_1 \otimes e_{m_3}) e
\]

\[
+ \left[ \tilde{\Phi}^{(1,1,s+2)} + \tilde{\Phi}^{(1,*)}(I - \tilde{R}_0)^{-1} \tilde{R}_1 \xi_4 (2) \right] (D_1 \otimes T_3^0) e
\]

(6.16)

Expected Number of Demands in the Pool

Let \( \tilde{\zeta}_P \) denote the expected number of pool demands in the steady state. This is given by

\[
\tilde{\zeta}_P = \sum_{x=1}^{\infty} x \tilde{\Phi}^{(x)} e = \tilde{\Phi}^{(1)} [I - \tilde{R}]^{-2} e = \tilde{\Phi}^{(1)} [I - \tilde{R}]^{-2} e
\]

(6.17)

Expected Length of the Idle Period

Let \( \tilde{\zeta}_{IS} \) denote the length of the idle period of the server in the steady state case. Idle period is terminated only by the replenishment of ordered item, because of single
vacation after completing his vacation, server remain idle until replenishment occurs. \( \tilde{\zeta}_{IS} \) is given by

\[
\tilde{\zeta}_{IS} = \Phi^{(0,1,0)}bD\mathbf{e} + \sum_{x=1}^{\infty} \Phi^{(x,1,0)} \left( bD + b(D \otimes \tau_3) \right) \mathbf{e}
\]

\[
= \Phi^{(0,1,0)}bD\mathbf{e} + \Phi^{(1,s)}(I - \tilde{R})^{-1}\tilde{z}_2(1) \left( bD + b(D \otimes \tau_3) \right) \mathbf{e}
\]  
(6.18)

### 6.4.4 Total Expected Cost Rate

The total expected cost per unit time for model II in the steady state is given by

\[
\tilde{TC}(s, S) = c_h \tilde{\zeta}_I + c_s \tilde{\zeta}_R + c_w \tilde{\zeta}_{PD} + c_{is} \tilde{\zeta}_{IS}
\]

By putting the values of \( \tilde{\zeta} \)'s from the above measures of system performance, we obtain

\[
\tilde{TC}(s, S) = c_h \left( Q \Phi^{(0,0,Q)} \mathbf{e} + Q \Phi^{(1,s)}(I - \tilde{R}_0)^{-1} \tilde{\zeta}_I(2) \mathbf{e} + \sum_{x=0}^{S} \sum_{l=1}^{l} \Phi^{(x,1,l)} \mathbf{e} \right.
\]

\[
+ \Phi^{(1,s)}(I - \tilde{R}_0)^{-1} \tilde{R}_1 \left[ \sum_{l=1}^{s} l \xi_3(l) + \sum_{l=1}^{Q} (s + l) \xi_4(l) \right] \mathbf{e} + c_s \left( \Phi^{(0,1,s+1)} D_1 \mathbf{e} \right.
\]

\[
+ \Phi^{(1,s+1)} + \Phi^{(1,s)}(I - \tilde{R}_0)^{-1} \tilde{R}_1 \xi_4(1) \left( (D_0 \otimes T_3^0) + (D_1 \otimes \mathbf{e}_{m_3}) \right) \mathbf{e}
\]

\[
+ \left( \Phi^{(1,s+2)} + \Phi^{(1,s)}(I - \tilde{R}_0)^{-1} \tilde{R}_1 \xi_4(2) \left( D_1 \otimes T_3^0 \right) \mathbf{e} \right. + c_w \Phi^{(1)} [I - \tilde{R}]^{-2} \mathbf{e} + \Phi^{(0,1,0)} bD \mathbf{e} + \Phi^{(1,s)}(I - \tilde{R})^{-1} \tilde{z}_2(1) \left( bD + b(D \otimes \tau_3) \right) \mathbf{e}
\]

### 6.5 Multiple Vacation Policy - Model III

From our assumptions made on the input and output processes, it can be verified that the stochastic process \( \tilde{Z}_t = \{(X_t, Y_t, L_t, J_t, J^1_t, J^2_t); t \in \mathbb{N}_0 \} \) is a discrete time Markov chain with the state space

\[
\Omega = \{(x, 0, l, j, j_1) : x \in E_0; l = 0, Q; j \in E_1^n; j_1 \in E_1^{m_1}\} \\
\cup \{(x, 1, l, j) : x = 0; l \in E_1^n; j \in E_0^n\} \\
\cup \{(x, 1, l, j) : x \in E_1; l \in E_1^n; j \in E_0^n\} \\
\cup \{(x, 1, l, j, j_3) : x \in E_1; l \in E_0^{n_1}; j \in E_1^n; j_3 \in E_1^{m_3}\}.
\]

Using the lexicographical order of the states, we get the transition probability matrix \( \tilde{P} \) is given by

\[
\tilde{P} = \begin{pmatrix}
\tilde{B}_1 & \tilde{B}_0 & 0 & 0 & \cdots \\
\tilde{B}_2 & \tilde{A}_1 & \tilde{A}_0 & 0 & \cdots \\
0 & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \cdots \\
0 & 0 & \tilde{A}_2 & \tilde{A}_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]

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where

\[ B_0 = \begin{pmatrix} B_{00}^0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_{00}^1 & B_{01}^1 \\ B_{10}^1 & B_{11}^1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B_{11}^2 \end{pmatrix} \]

\[ \bar{B}_{00} = \begin{pmatrix} \bar{b}(D_1 \otimes T_1) + \bar{b}(D_1 \otimes T_0^0 \tau_1) \\ 0 \end{pmatrix}, \quad \bar{B}_{01} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ \bar{B}_{10}^1 = (\mathbf{e}(1) \otimes \bar{b}(D_1 \otimes \tau_1) \cdot 0) \]

\[ \bar{B}_{01}^1 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}, \quad \bar{A}_0 = \begin{pmatrix} B_{00}^0 & \bar{A}_{01}^1 \\ 0 & 0 \end{pmatrix}, \quad \bar{A}_1 = \begin{pmatrix} B_{10}^1 & \bar{A}_{11}^1 \\ 0 & 0 \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} 0 & 0 \\ 0 & A_{11}^2 \end{pmatrix} \]

The dimension of sub-matrices \( \bar{B}_0, \bar{B}_1 \) and \( \bar{B}_2 \) are \((\bar{N}_0 + N_1) \times (\bar{N}_0 + N_2)\), \((\bar{N}_0 + N_1) \times (\bar{N}_0 + N_2)\) and \((\bar{N}_0 + N_2) \times (\bar{N}_0 + N_2)\) respectively. Dimension of \( \bar{A}_0, \bar{A}_1 \) and \( \bar{A}_2 \) are \((\bar{N}_0 + N_2) \times (\bar{N}_0 + N_2)\).

### 6.5.1 Stability Analysis

In order to obtain the stability condition of the process, let \( \bar{A} = \bar{A}_0 + \bar{A}_1 + \bar{A}_2 \) be the generator matrix and \( \pi \) be the stationary probability vector of \( \bar{A} \).

\[
\bar{A} = \begin{pmatrix} F_{00} & A_{01}^1 \\ A_{10} & F_{11} \end{pmatrix}
\]

where

\[
F_{00} = \begin{pmatrix} \bar{b}(D \otimes T_1) + \bar{b}(D \otimes T_0^0 \tau_1) \\ 0 \end{pmatrix}, \quad b(D \otimes T_1)
\]

Let \( \bar{\pi} = (\pi^*, \bar{\pi}^1) \), where \( \pi^* = (\pi^{(0,0)}, \bar{\pi}^{(0,Q)}, \bar{\pi}^{(1,0)}) \) and \( \bar{\pi}^1 = (\bar{\pi}^{(1,1)}, \bar{\pi}^{(1,2)}, \ldots, \bar{\pi}^{(1,S)}) \) which satisfy \( \bar{\pi} \bar{A} = \bar{\pi} \) and \( \bar{\pi} \mathbf{e} = 1 \).

From the well known result of Neuts [1994] on the positive recurrence of \( \bar{A} \). We have that the Markov chain is positive recurrent if and only if \( \pi \bar{A}_0 \mathbf{e} < \pi \bar{A}_2 \mathbf{e} \). Thus the stability condition of the process is

\[
\pi^{(0,0)} [\{D_1 \otimes T_1\} \mathbf{e} + \bar{b}(D_1 \otimes T_0^0 \tau_1) \mathbf{e}] + \pi^{(0,Q)} [D_1 \otimes T_1] \mathbf{e} < \pi^{(1,s+1)} (D_0 \otimes T_3^0) \mathbf{e} + \pi^{(1,s+2)} (D_1 \otimes T_3^0 + D_0 \otimes T_0^0 \tau_3) \mathbf{e} + \sum_{k=s+3}^{S} \pi^{(1,k)} (D \otimes T_0^0 \tau_3) \mathbf{e} \quad (6.19)
\]

where

\[
\pi^{(0,l)} = \left\{ (\pi^{(0,l,j,j)}) : l = 0, Q; j \in E_{1}^{0}; j_1 \in E_{1}^{m1} \right\}
\]
6.5.2 Steady State Analysis

Let \( \Phi = (\Phi(0), \Phi(1), \Phi(2), \ldots) \) be the steady state probability vector associated with the transition matrix \( \overline{P} \) such that \( \Phi \overline{P} = \Phi, \Phi e = 1. \)

Let us partition the vector \( \Phi(x) \) into \( (\Phi(x,0), \Phi(x,1)) \), \( x \in E_0 \), where each of the vector can further be partitioned, for computational purposes, the vector \( \Phi(x,0) \) into \( (\Phi(x,0,0), \Phi(x,0,Q)) \) and the vector \( \Phi(x,1) = (\Phi(x,1,1), \Phi(x,1,2), \ldots, \Phi(x,1,S)) \). For \( x \in E_0, \Phi(x,0,0) \) and \( \Phi(x,0,Q) \) are row \( nm_1 \) vector. For \( x = 0 \) and \( l \in E_1^s, \Phi(x,1,l) \) is a row \( n \) vector. For \( x \in E_1, \Phi(x,1,l) (l \in E_1^s) \) is a row \( n \) vector and \( \Phi(x,1,l) (l \in E_{s+1}^m) \) is a row \( nm_3 \) vector.

**Theorem 6.5.1.** When the stability condition (6.19) holds good, the steady-state probability vector \( \Phi \) is given by

\[
\Phi(x) = \Phi(1) \overline{R}^{x-1}, \quad x = 1, 2, \ldots
\]

where the matrix \( \overline{R} \) satisfies the matrix quadratic equation

\[
\overline{R}^2 \overline{A}_2 + \overline{R} \overline{A}_1 + \overline{A}_0 = \overline{R}
\]

and the vector \( \Phi(0) \) and \( \Phi(1) \) is obtained by solving

\[
(\Phi(0), \Phi(1)) B[\overline{R}] = (\Phi(0), \Phi(1))
\]

subject to the normalizing condition

\[
\Phi(0) e + \Phi(1) (I - \overline{R})^{-1} e = 1.
\]

**Proof.** The theorem follows from the well-known result on matrix-geometric methods Neuts [1994]. \( \square \)

**Computation of \( \overline{R} \) and \( \Phi(i) \) vector**

Due to special structure of \( \overline{A}_0 \), we note that

\[
\overline{R} = \begin{pmatrix} \overline{R}_0 & \overline{R}_1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{R}^k = \begin{pmatrix} \overline{R}_0^k & \overline{R}_0^{k-1} \overline{R}_1 \\ 0 & 0 \end{pmatrix}, \quad k = 1, 2, \ldots
\]
From the equation (6.21), we have

\[
\bar{R}_0 = \begin{pmatrix} R_{00} & R_{00}Q \\ R_{0Q} & R_{QQ} \end{pmatrix}, \quad \bar{R}_1 = \begin{pmatrix} R_{01} & R_{02} & \cdots & R_{0S} \\ R_{Q1} & R_{Q2} & \cdots & R_{QS} \end{pmatrix},
\]

From the equation (6.21), we have

For \( i \in E_2 \)
\[
\Phi(x,0) = \Phi(1,0) \bar{R}_0^{x-1}
\]
\[
\Phi(x,0,0) = \Phi(1,0) \bar{R}_0^{x-1}(e_2(1) \otimes I_{nm_1})
\]
\[
\Phi(x,0,Q) = \Phi(1,0) \bar{R}_0^{x-1}(e_2(2) \otimes I_{nm_1})
\]

6.5.3 System Performance Measures

In this section, we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

Expected Inventory Level

Let \( \bar{\zeta}_l \) denote the mean inventory level in the steady state. Since \( \Phi^{(x,1,l)} \) is a steady state probability vector of the \( l \) inventory in the system with component specify \( x \), the number of demands in the pool and 1, the server is not on vacation and \( \Phi^{(x,0,Q)} \) is a steady state probability vector of \( Q \) inventory in the system with the component specify \( x \), the number of demands in the pool and 0, the server is on vacation, the mean inventory level \( \bar{\zeta}_l \) in the steady state is given by

\[
\bar{\zeta}_l = \sum_{x=0}^{\infty} Q\Phi^{(x,0,Q)}e + \sum_{l=1}^{S} l\Phi^{(x,1,l)}e = Q\Phi(0,0,Q)e + Q\Phi(1,0)(I - \bar{R}_0)^{-1}(e_2(2) \otimes I_{nm_1})e + \sum_{x=0}^{1} \sum_{l=1}^{S} l\Phi^{(x,1,l)}e
\]
\[
+ \Phi(1,0)(I - \bar{R}_0)^{-1}\bar{R}_1 \left[ \sum_{l=1}^{s} l\xi_3(l) + \sum_{l=1}^{Q} (s + l)\xi_4(l) \right] e \quad (6.24)
\]

Expected Reorder Rate

Let \( \bar{\zeta}_R \) denote the expected reorder rate in the steady state. A reorder is triggered when the inventory level drops from \( s + 1 \) to \( s \) either by a primary demand or selection of a pooled demand.

\[
\bar{\zeta}_R = \Phi(1,0,s + 1)D_1e + \sum_{x=1}^{\infty} \Phi^{(x,1,s + 1)}(D_0 \otimes T_3^0 + D_1 \otimes e_{m_3})e + \sum_{x=1}^{\infty} \Phi^{(x,1,s + 2)}(D_1 \otimes T_3^0)e
\]
\[
= \Phi(1,0,s + 1)D_1e + \Phi(1,0)(I - \bar{R}_0)^{-1}\bar{R}_1\xi_4(1)\left[ (D_0 \otimes T_3^0 + D_1 \otimes e_{m_3})e + \left[ \Phi(1,1,s + 2) + \Phi(1,0)(I - \bar{R}_0)^{-1}\bar{R}_1\xi_4(2) \right] (D_1 \otimes T_3^0)e \right] \quad (6.25)
\]
Expected Number of Demands in the Pool

Let \( \bar{\zeta}_P \) denote the expected number of pool demands in the steady state. This is given by

\[
\bar{\zeta}_P = \sum_{x=1}^{\infty} x \bar{\Phi}(x) e = \bar{\Phi}(1)[I - \bar{R}]^{-2} e
\]

(6.26)

6.5.4 Total Expected Cost Rate

The total expected cost per unit time for model III in the steady state is given by

\[
TC(s, S) = c_h \bar{\zeta}_I + c_s \bar{\zeta}_R + c_w \bar{\zeta}_{PD}
\]

By putting the values of \( \bar{\zeta} \)'s from the above measures of system performance, we obtain

\[
TC(s, S) = c_h \left( Q \bar{\Phi}^{(0,0)} e + Q \bar{\Phi}^{(1,0)} (I - \bar{R})^{-1} (e_2(2) \otimes I_{nm_1}) e + \sum_{x=0}^{1} \sum_{l=1}^{s} l \bar{\Phi}(x,l) e \right) + c_s \left( \bar{\Phi}^{(0,1,s+1)} D_1 e \right) + \left[ \bar{\Phi}^{(1,1,s+1)} + \bar{\Phi}^{(1,0)} (I - \bar{R})^{-1} \bar{R}_1 \xi_4(1) \right] (D_0 \otimes T_3^0 + D_1 \otimes e_{m_1}) e
\]

\[
+ \left[ \bar{\Phi}^{(1,1,s+2)} + \bar{\Phi}^{(1,0)} (I - \bar{R})^{-1} \bar{R}_1 \xi_4(2) \right] (D_1 \otimes T_3^0) e + c_w \bar{\Phi}^{(1)} [I - \bar{R}]^{-2} e
\]

6.6 Numerical Examples

In this section, we discuss some numerical examples for all three models. Our interest is to determine the best vacation policy using the corresponding optimal total expected cost function.

Example 6.1: In this first example, we reveal the possible convexity of the total expected cost function of each model. We consider the following matrices. The arrival process is \( DMAP \) specified \((D_0, D_1)\) with the arrival rate \( \lambda = 0.9 \) where

\[
D_1 = \begin{pmatrix} 0.1 & 0.0 \\ 0.05 & 0.05 \end{pmatrix} \quad D_1 = \begin{pmatrix} 0.8 & 0.1 \\ 0.3 & 0.6 \end{pmatrix}
\]

The vacation time has the phase-type distribution \((\tau_1, T_1)\) with vacation rate \( \beta_1 = 0.2475 \) where

\[
\tau_1 = (0.3 \quad 0.1 \quad 0.6) \quad T_1 = \begin{pmatrix} 0.505 & 0.495 & 0 \\ 0 & 0.505 & 0.495 \\ 0 & 0 & 0.505 \end{pmatrix}
\]

The idle time has the phase-type distribution \((\tau_2, T_2)\) with idle rate \( \beta_2 = 0.9630 \) where

\[
\tau_2 = (0.3 \quad 0.4 \quad 0.3) \quad T_2 = \begin{pmatrix} 0.001 & 0.03 & 0.05 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0.005 \end{pmatrix}
\]

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The inter-selection time has the phase-type distribution $(\tau_3, T_3)$ with selection rate $\beta_3 = 0.9363$ where

\[
\begin{align*}
\tau_3 &= \begin{pmatrix} 0.4 & 0.6 \end{pmatrix} \\
T_3 &= \begin{pmatrix} 0.001 & 0.001 \\ 0.050 & 0.050 \end{pmatrix}
\end{align*}
\]

In order to show the convexity of each model, we have plotted three figures from 6.1 to 6.3, for fixed value of lead time parameter $b = 0.6$ and cost values to be $c_h = 0.095, c_s = 20, c_w = 3, c_{is} = 2.4$. Optimal total expected cost $\bar{TC}^*(s^*, S^*)$ of modified multiple vacation policy is 2.140065 at $(3, 24)$, $\bar{T\bar{C}}^*(s^*, S^*)$ of single vacation policy is 2.169107 at $(3, 24)$ and $\bar{T\bar{C}}^*(s^*, S^*)$ of multiple vacation policy is 2.188129 at $(3, 24)$.

**Example 6.2:** Here, we consider the six different discrete phase type distribution to idle time, for determining the best vacation policy, since the idle time plays a role only in the modified multiple vacation policy. The values for the matrices $D_0, D_1, T_1, T_3, \tau_1$ and $\tau_3$ are as given in example 6.1.
We assume following matrices for $DPH$ distributed idle time.

<table>
<thead>
<tr>
<th>Idle rate $\beta_2$</th>
<th>$TC(s, S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>2.124666174</td>
</tr>
<tr>
<td>0.318</td>
<td>2.107325471</td>
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<tr>
<td>0.3</td>
<td>2.088026167</td>
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<tr>
<td>0.23</td>
<td>2.075374076</td>
</tr>
<tr>
<td>0.13</td>
<td>2.052586057</td>
</tr>
<tr>
<td>0.1</td>
<td>2.050031112</td>
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</tbody>
</table>

Table 6.1: $TC(s, S)$ for various idle rate

We fix the other parameters and cost values to be $b = 0.6$, $c_h = 0.095$, $c_s = 20$, $c_w = 3$, $c_{is} = 2.4$. We present total expected cost rate $TC(s, S)$ for different idle rate in table 6.1. $TC(s, S)$ decreases as idle rate decreases, which is less than $\widetilde{TC}(s, S)$ and $\overline{TC}(s, S)$, that is $TC(s, S) < \widetilde{TC}(s, S) < \overline{TC}(s, S)$ for all idle rate. From this we conclude that the modified vacation policy is the best vacation policy.

**Example 6.3:** Now we analyse the sensitivity of arrival rate on optimal total expected cost of the each vacation policy for different arrival rate. Let us consider the
\[ b = 0.6, c_h = 0.075, c_s = 20, c_w = 3, c_{1s} = 2.4 \]

The other parameters and costs are same as in example 6.1.

We observe the following from the figure 6.4

- The optimal total expected cost rate is increasing for all vacation policy with each of arrival rates.
- When the \( \lambda \in (0.2, 0.5) \), \( \tilde{T}C^*(s, S) < TC^*(s, S) < TC^*(s, S) \). Hence, single vacation policy is the best vacation policy in this region.
- When the \( \lambda > 0.5 \), \( TC^*(s, S) < \tilde{T}C^*(s, S) < TC^{1s}(s, S) \), Hence in this range modified multiple vacation policy is the best policy in this region.
Example 6.4: We study the influence of vacation rate of each vacation policy on its the optimal total expected cost function for different vacation rate. For this, we consider the following 13 different initial probability vectors

<table>
<thead>
<tr>
<th>( \tau_1 )</th>
<th>( \beta_1 )</th>
<th>( \tau_2 )</th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0, 0)</td>
<td>0.1650</td>
<td>(0, 0.33, 0.67)</td>
<td>0.3722</td>
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<tr>
<td>(0.7, 0.1, 0.2)</td>
<td>0.198</td>
<td>(0, 0.2, 0.8)</td>
<td>0.4125</td>
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<tr>
<td>(0.5, 0.25, 0.25)</td>
<td>0.22</td>
<td>(0, 0.15, 0.85)</td>
<td>0.4304</td>
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<tr>
<td>(0.3, 0.38, 0.32)</td>
<td>0.25</td>
<td>(0, 0.075, 0.95)</td>
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<tr>
<td>(0.1, 0.56, 0.34)</td>
<td>0.2812</td>
<td>(0, 0.04, 0.96)</td>
<td>0.4759</td>
</tr>
<tr>
<td>(0, 0.59, 0.41)</td>
<td>0.3113</td>
<td>(0, 0, 1)</td>
<td>0.495</td>
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<tr>
<td>(0.45, 0.55)</td>
<td>0.3511</td>
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</table>

and its rates for matrix \( T_1 = \begin{pmatrix} 0.505 & 0.495 & 0.0 \\ 0.0 & 0.505 & 0.495 \\ 0.0 & 0.0 & 0.505 \end{pmatrix} \).

We fix the other parameters and costs as in example 6.1. We plot the figure 6.5 for three vacation policies for all the above 13 vacation parameters. The optimal total expected cost function decreases for all the vacation policies when \( \beta_1 \) increases. The modified multiple vacation policy is the best vacation when \( \beta_1 < 0.3472 \), the single vacation policy is the best when \( \beta_1 > 0.3472 \). When \( \beta_1 \in (0.3472, 0.4707) \) modified multiple vacation policy is better than multiple vacation. When \( \beta_1 > 0.4707 \), The multiple vacation policy is better than the modified multiple vacation policy.

Example 6.5: Finally, we study the sensitivity of cost rates namely, the holding cost, the setup cost, the waiting cost of a demand in the pool and the server idle cost on optimal cost rate \( TC^* \) and its optimal \( S^* \) and \( s^* \) for the modified multiple vacation policy. In Table 6.2, the upper entry gives optimal cost rate \( TC^* \) and the lower pair

Figure 6.5: Influence of vacation rate on total expected cost function
given in each cell gives the corresponding (local) optimal $S^*$ and $s^*$. We observe the following

- The optimal cost $TC^*$ appears to increase with each of costs.

- As $c_h$ increases the optimal values $S^*$ and $s^*$ decreases. Since the cost of holding an inventory is high, it is good to maintain low stock and low reorder point in order to optimize the total expected cost.

- When the setup cost $c_s$ increases, the value of $S^*$ increase and the value of $s^*$ decreases. This is because, as the setup cost increases, ordering cost is also increases. Hence in order to avoid the frequent ordering, we have to maintain more stock and low reorder point.

- When the waiting cost of a demand in the pool and the idle cost of server in the system increases, the values of $S^*$ and $s^*$ increases. Due to the out of stock, server becomes idle and arriving demands are sent to the pool, which will lead to increase the server idle period as well as waiting time of pool demands. In order to avoid the stock out period, we have to maintain more inventory and high reorder point in the system.
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