CHAPTER 3

RETRIAL INVENTORY SYSTEM WITH SINGLE AND MODIFIED MULTIPLE VACATION FOR SERVER

3.1 Introduction

In this chapter, we consider a retrial inventory system with server vacation. Most of the inventory models considered in the literature assumed that the server is always available. However, in many real-life situations, server may become unavailable for a period of time due to lack of stock or customer or to do some secondary job like cheque clearing work in the bank, sorting out the cash in cash-counter, checking for maintenance on a production system and so on. In queueing models with server vacation, it is usually assumed that after the vacation period the server is either available thereafter (single vacation policy) or avail another vacations (multiple vacation policy) (see Doshi [1986], Tian and Zhang [2006]). Daniel and Ramanarayanan [1987, 1988] introduced the server vacation in an inventory system. They assumed that the server starts vacation, whenever the inventory level becomes zero. Recently, Sivakumar [2011] considered the multiple vacation policy for the server in retrial inventory system, server may go on another vacation until replenishment occurs.

Here, we introduce a new vacation policy called modified multiple vacation policy in a single server inventory system with constant retrial policy. We assume that the arrival process follows a Poisson process and the ordered items are replenished according to an \((s,S)\) ordering policy. Any arriving demand finds the server is on vacation or the inventory level is zero, enters an orbit of infinite size. After an exponential time, these orbiting demands retry for their requirements. Detailed review of server vacation and
retrial demands in inventory is given in chapter 1. This chapter extends the work of Sivakumar [2011] by incorporating a new feature, called *idle time* for server, in addition to vacation. At the time of stock depletion, the server idles for a random time before he goes for vacation, so that if the replenishment is received during the idle time, the server is immediately available, else goes for vacation. If the stock is replenished during the vacation, he is available only at the end of vacation else he starts his idle period, and possibly a subsequent vacation follows. This process of idle time followed by a vacation time is continued (see figure 3.1). The introduction of idle time makes it possible for the immediate availability of the server whereas, in the “only-vacation case” the server is available only when he completes his vacation. We call this as *modified multiple vacation policy*. In order to evaluate this policy, we also consider another model with *single vacation policy*, in which the server is available continuously after a vacation irrespective of the inventory level.

The rest of the chapter is organized as follows. In section 3.2, we describe two mathematical models. The steady state analysis is performed for both models and present some key system performance measures and the total expected cost rate in sections 3.3 and 3.4. In section 3.5, we present numerical comparison of these two models along with multiple vacation policy of Sivakumar [2011] and perform some sensitivity

![Figure 3.1: Various vacation policies for the server](image-url)
3.2 Problem Formulation

We consider a continuous review stochastic inventory system with a maximum stock of $S$ units. We assume that the demands arrive according to a Poisson process with rate $\lambda (> 0)$ and they demand only single unit at a time. As and when the on-hand inventory level drops to a prefixed level, say $s (\geq 0)$, an order for $Q = S - s > s$ units is placed. The condition $Q > s$ ensures that after replenishment, the inventory level is above $s$. The lead time for the order is exponentially distributed with parameter $\mu (> 0)$. The demands that occur during stock-out periods and or during server vacation periods enter an orbit of infinite size. These orbiting demands compete for their demands, and the time between two successive attempts has an exponential distribution with parameter $\theta (> 0)$. We consider two models, which are described below:

**Model I:** When the inventory level reaches to zero, the server starts his idle period, which is distributed as an exponential with parameter $\alpha (> 0)$. At the end of idle time, if the replenishment is not received, he starts his vacation, the duration of which has exponential distribution $\beta (> 0)$. On return from any vacation, if the stock is already replenished, he becomes available to the system, otherwise, his idle time starts, which may be followed by another vacation. Hence the idle time and vacation time alternate. At the time of replenishment if the server is in an idle state, he is immediately available, else he is available only at the end of the vacation. So, we call this policy as modified multiple vacation policy.

**Model II:** Whenever the inventory level drops to zero, the server leaves the system and goes for a vacation whose duration is exponentially distributed with parameter $\beta (> 0)$. After completing his vacation, server becomes available continuously irrespective of the replenishment. We call this policy as single vacation policy.

We assume constant retrial policy to the demands in the orbit, that is, the probability of a repeated attempt is independent of the number of demands in the orbit. We also assume that the inter-demand times between the demands, the lead times of the order, inter-retrial times, server idle times and server vacation times are mutually independent random variables.
3.3 Model I : Modified Multiple Vacation Policy

Let $X(t)$ and $L(t)$, respectively, denote the number of demands in the orbit and on-hand inventory level at time $t$. Further, let

$$Y(t) = \begin{cases} 
0, & \text{if server is on vacation} \\
1, & \text{if server is not on vacation}
\end{cases}$$

From our assumptions made on the input and output processes, it can be seen that the stochastic process $\{(X(t),Y(t),L(t)); t \geq 0\}$ is a continuous time Markov chain with state space

$$\Omega = \{(i,0,m) : i = 0,1,\ldots; m = 0,Q\} \cup \{(i,1,m) : i = 0,1,\ldots; m = 0,1,\ldots,S\}.$$

To determine the infinitesimal generator $P = (p((i,k,m),(j,l,n)))$, $(i,k,m),(j,l,n) \in E$ of this process, we use the following arguments:

- If the server is not on the vacation and the inventory level is positive,
  - an arriving demand takes the state of the process from $(i,1,m)$ to $(i,1,m-1)$ with the intensity $\lambda$, $i = 0,1,\ldots; m = 1,2,\ldots,S$.
  - an arriving retrial demand, if any, takes the state of the process from $(i,1,m)$ to $(i-1,1,m-1)$ and the intensity of this transition $\theta$, $i = 1,2,\ldots; m = 1,2,\ldots,S$.

- If the server is in idle and the inventory level is zero,
  - an arriving demand enters into the orbit and the orbit size increases by one. Hence the intensity of transition $p((i,1,0),(i+1,1,0))$, $i = 0,1,\ldots$ is given by $\lambda$.
  - at the end of the idle period, if the replenishment is not received, the server goes for vacation. The intensity of this transition is given by $p((i,1,0),(i,0,0)) = \alpha$, $i = 0,1,\ldots$.

- If the server is on vacation, any arriving demand joins the orbit with the intensity of transition $p((i,0,m),(i+1,0,m))$ is given by $\lambda$, $i = 0,1,\ldots; m = 0,Q$.

- $P$ transition from $(i,k,m)$ to $(i,k,m+Q)$ for $i = 0,1,\ldots; k = 0; m = 0$, or for $i = 0,1,\ldots; k = 1; m = 0,1,2,\ldots, s$ takes place with intensity $\mu$ as a replenishment occurs.
• The server terminates the vacation and the intensity of transition 
  \( p((i,0,m),(i,1,m)) \) is given by \( \beta \), for \( i = 0,1,\ldots;m = 0,Q \).

• For other transitions from \((i,k,m)\) to \((j,l,n)\), except \((j,l,n) \neq (i,k,m)\), the rates are zero.

• Finally, note that

\[
 p((i,k,m),(i,k,m)) = - \sum_j \sum_l \sum_n p((i,k,m),(j,l,n)).
\]

Hence, we have, \( p((i,k,m),(j,l,n)) = \)

\[
\begin{cases} 
\lambda, & j = i + 1, \quad i = 0,1,\ldots, \\
 & l = k, \quad k = 0, \\
 & n = m, \quad m = 0,Q, \\
 & \quad \text{or} \\
 & j = i + 1, \quad i = 0,1,\ldots, \\
 & l = k, \quad k = 1, \\
 & n = m, \quad m = 0, \\
 & \quad \text{or} \\
 & j = i, \quad i = 0,1,\ldots, \\
 & l = k, \quad k = 1, \\
 & n = m - 1, \quad m = 1,2,\ldots,S, \\
\theta, & j = i - 1, \quad i = 1,2,\ldots, \\
 & l = k, \quad k = 1, \\
 & n = m - 1, \quad m = 1,2,\ldots,S, \\
\alpha, & j = i, \quad i = 0,1,\ldots, \\
 & l = k - 1, \quad k = 1, \\
 & n = m, \quad m = 0, \\
\beta, & j = i, \quad i = 0,1,\ldots, \\
 & l = k + 1, \quad k = 0, \\
 & n = m, \quad m = 0,Q, \\
\mu, & j = i, \quad i = 0,1,\ldots, \\
 & l = k, \quad k = 0, \\
 & n = m + Q, \quad m = 0, \\
& \quad \text{or} \\
& j = i, \quad i = 0,1,\ldots, \\
& l = k, \quad k = 1, \\
& n = m + Q, \quad m = 0,1,\ldots,s, \\
-(\lambda + \mu \delta_{m0} + \beta), & j = i, \quad i = 0,1,\ldots, \\
& l = k, \quad k = 0, \\
& n = m, \quad m = 0,Q \\
-(\lambda + \alpha \delta_{m0} + \mu H(s - m) + \theta \tilde{\delta}_{i0} \tilde{\delta}_{m0}), & j = i, \quad i = 0,1,2,\ldots, \\
& l = k, \quad k = 1, \\
& n = m, \quad m = 0,1,\ldots,S, \\
0, & \text{Otherwise.}
\end{cases}
\]

Let

\[
\langle i \rangle = ((i,0,0),(i,0,Q),(i,1,0),(i,1,1),(i,1,2),\ldots,(i,1,S)), i \geq 0
\]
By ordering the state space as \((<0>, <1>, <2>, \ldots,)<\), the infinitesimal generator \(A\) can be conveniently expressed in a block partitioned matrix with entries

\[
P = \begin{pmatrix}
<0> & <1> & <2> & <3> & <4> & \cdots \\
B_1 & A_0 & 0 & 0 & 0 & \cdots \\
A_2 & A_1 & A_0 & 0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix}
\]

where

\[
A_0 = \begin{pmatrix}
A_{00}^{(2,S+1)} & \cdots & 0 \\
\cdots & \cdots & \cdots \\
0 & \cdots & A_{11}^{(0)} \\
\end{pmatrix}
\quad A_0 = \lambda I_{(2,2)} \quad A_0 = \lambda e_{S+1}(1) e^T_{S+1}(1)
\]

\[
A_2 = \begin{pmatrix}
0 & \cdots & 0 \\
\cdots & \cdots & \cdots \\
0 & \cdots & A_{11}^{(2)} \\
\end{pmatrix}
\quad [A_{11}^{(2)}]_{ij} = \begin{cases} 
\theta, & j = i - 1, i = 1, 2, \ldots, S \\
0, & \text{Otherwise}
\end{cases}
\]

\[
B_1 = \begin{pmatrix}
B_{00} & B_{01} \\
B_{10} & B_{11} \\
\end{pmatrix}
\quad A_1 = \begin{pmatrix}
B_{00} & B_{01} \\
B_{10} & A_{11}^{(1)} \\
\end{pmatrix}
\quad B_{00} = \begin{pmatrix}
0 & (-(\lambda + \mu + \beta) & \mu \\
Q & 0 & (-(\lambda + \beta)) \\
\end{pmatrix}
\]

\[
B_{01} = \begin{pmatrix}
\beta & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
Q & 0 & \cdots & 0 & \beta & 0 & \cdots & 0 \\
\end{pmatrix}
[b_{10}]_{ij} = \begin{cases} 
\alpha, & j = 0, i = 0 \\
0, & \text{Otherwise}
\end{cases}
\]

\[
[A_{11}^{(1)}]_{ij} = \begin{cases} 
\lambda, & j = i - 1, i = 1, 2, \ldots, S \\
\mu, & j = i + Q, i = 0, 1, \ldots, s \\
d_2, & j = i, i = 0 \\
d_1, & j = i, i = 1, 2, \ldots, s \\
-\lambda, & j = i, i = s + 1, s + 2, \ldots, S \\
\end{cases}
\]

\[
[A_{11}^{(1)}]_{ij} = \begin{cases} 
\lambda, & j = i - 1, i = 1, 2, \ldots, S \\
\mu, & j = i + Q, i = 0, 1, \ldots, s \\
d_3, & j = i, i = 0 \\
d_2, & j = i, i = 1, 2, \ldots, s \\
-\lambda, & j = i, i = s + 1, s + 2, \ldots, S \\
\end{cases}
\]

3.3.1 Stability Analysis

To discuss the stability condition of the process under study, we consider the generator matrix \(A = A_0 + A_1 + A_2\), which is given by

\[
A = \begin{pmatrix}
F_{00} & B_{01} \\
B_{10} & F_{11} \\
\end{pmatrix}
\quad where \quad F_{00} = \begin{pmatrix}
0 & Q \\
0 & -(\mu + \beta) & \mu \\
\end{pmatrix}
\]

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and

\[
[F_{11}]_{ij} = \begin{cases} 
  d_4, & j = i - 1, \quad i = 1, 2, \ldots, S \\
  \mu, & j = i + Q, \quad i = 0, 1, \ldots, s \\
  -(\mu + \alpha), & j = i, \quad i = 0 \\
  -d_3, & j = i, \quad i = 1, 2, \ldots, s \\
  -d_4, & j = i, \quad i = s + 1, s + 2, \ldots, S 
\end{cases}
\]

Let \( \pi = (\pi^{(0,0)}, \pi^{(0,Q)}, \pi^{(1,0)}, \pi^{(1,1)}, \ldots, \pi^{(1,S)}) \) be the steady-state probability vector of the generator \( A \), which satisfy,

\[ \pi A = 0, \pi e = 1. \]

Lemma 3.3.1. The steady-state probability vector \( \pi \) corresponding to the generator \( A \) is given by

\[
\begin{align*}
\pi^{(i,j)} &= \begin{cases} 
  \pi^{(0,0)} \frac{\mu}{\beta}, & i = 0, j = Q, \\
  \pi^{(0,0)} \frac{\mu + \beta}{\alpha}, & i = 1, j = 0, \\
  \pi^{(0,0)} \frac{\mu d_{i-1}}{\alpha d_4} (\mu + \alpha + \beta), & i = 1, j = 1, 2, \ldots, s, \\
  \pi^{(0,0)} \frac{\mu d_i}{\alpha d_4} (\mu + \alpha + \beta), & i = 1, j = s + 1, s + 2, \ldots, Q, \\
  \pi^{(0,0)} \frac{\mu}{\alpha d_4} \left[ \frac{d_i}{d_4^2} - 1 \right] (\mu + \alpha + \beta), & i = 1, j = Q + 1, \\
  \pi^{(0,0)} \frac{\mu}{\alpha d_4} \left[ \frac{d_i}{d_4^2} - 1 - \sum_{k=0}^{i-Q-2} \frac{\mu d_{i-k}}{d_4^2} \right] (\mu + \alpha + \beta), & i = 1, j = Q + 2, Q + 3, \ldots, S,
\end{cases} \\
\pi^{(0,0)} &= \frac{1}{1 + \omega_1}
\end{align*}
\]

where

\[ \omega_1 = \frac{\mu}{\beta} + \frac{\mu + \beta}{\alpha} + \frac{\mu}{\alpha d_4} \left[ \sum_{k=1}^{s} \frac{d_{k-1}^2}{d_4^2} + Q \frac{d_{Q}^2}{d_4^2} - s \sum_{k=Q+2}^{s} \sum_{j=0}^{k-Q-2} \frac{\mu d_{i-k}}{d_4^2} \right] (\mu + \alpha + \beta). \]

Proof. We have

\[ \pi A = 0, \pi e = 1. \]

The first equation yields the set of equations, solving the those system of equations recursively and using the normalizing condition, we get the stated result.

Next, we derive the condition under which the system is stable.

Lemma 3.3.2. The stability condition of the system under study is given by

\[ \rho_1 = \frac{(\lambda + \theta)(\beta + \mu)(\beta + \alpha)}{\theta \alpha \beta (1 + \omega_1)} < 1. \] (3.1)
Proof. From the well known result of Neuts [1994] on the positive recurrence of $A$, we have

$$\pi A_0 e < \pi A_2 e$$

and by exploiting the structure of the matrices $A_0$ and $A_2$, and $\pi$ the stated result follows.

### 3.3.2 Steady State Analysis

It can be seen from the structure of the rate matrix $P$ and from the lemma 3.3.2, that the Markov process $\{(X(t), Y(t), L(t)), t \geq 0\}$ with the state space $\Omega$ is regular. Hence the limiting probability distribution

$$\phi^{(i,k,m)} = \lim_{t \to \infty} Pr \{X(t) = i, Y(t) = k, L(t) = m \mid X(0), Y(0), L(0)\},$$

exists and is independent of the initial state.

Let

$$\Phi^{(i)} = (\phi^{(i,0,0)}, \phi^{(i,0,0)}, \phi^{(i,1,0)}, \phi^{(i,1,1)}, \phi^{(i,1,2)}, \ldots, \phi^{(i,1,S)}), i = 0, 1, \ldots,$$

and

$$\Phi = (\Phi^{(0)}, \Phi^{(1)}, \ldots).$$

The vector $\Phi$ denotes the steady state probability vector of $P$. That is, $\Phi$ satisfies

$$\Phi P = 0, \text{ and } \Phi e = 1.$$

**Theorem 3.3.1.** When the stability condition (3.1) holds good, the steady-state probability vector $\Phi$ is given by

$$\Phi^{(i)} = \Phi^{(0)} R^i, i = 0, 1, \ldots$$

(3.2)

where the matrix $R$ satisfies the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

(3.3)

and the vector $\Phi^{(0)}$ is obtained by solving

$$\Phi^{(0)}(B_1 + RA_2) = 0$$

(3.4)

subject to the normalizing condition

$$\Phi^{(0)}(I - R)^{-1} e = 1.$$  

(3.5)

Proof. The theorem follows from the well-known result on matrix-geometric methods Neuts [1994].
3.3.3 Algorithmic Analysis

In this section, we present an efficient algorithmic procedure for the computation of the \( R \) matrix and the vector \( \Phi^{(0)} \), which are the main ingredients for discussing the qualitative behaviour of the model under study.

**Computation of \( R \) Matrix**

Due to the special structure of the coefficient matrices appearing in (3.3), the square matrix \( R \) of dimension \((S + 3)\) can be computed as follows: We note that as \( A_0 \) has non-zero entries in its first three rows only. Hence, we conclude that the matrix \( R \) also has non-zero entries in its first three rows only as shown below

\[
R = \begin{pmatrix}
R_0 & R_1 \\
0_{(S,3)} & 0_{(S,S)}
\end{pmatrix}
\]

where

\[
R_0 = \begin{pmatrix}
r_{10} & r_{1Q} & r_{10'} \\
r_{20} & r_{2Q} & r_{20'} \\
r_{30} & r_{3Q} & r_{30'}
\end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix}
r_{11} & r_{12} & \cdots & r_{1S} \\
r_{21} & r_{22} & \cdots & r_{2S} \\
r_{31} & r_{32} & \cdots & r_{3S}
\end{pmatrix}.
\]

Due to this special form of \( R \), we note

\[
R^k = \begin{pmatrix}
R_0^k & R_1^k - R_0^k R_1 \\
0_{(S,3)} & 0_{(S,S)}
\end{pmatrix}, \quad k = 1, 2, \ldots,
\]

This form is exploited in the computation of \( R \) using (3.3). The relevant equations are given in the Appendix A.

**Computation of the Vector \( \Phi^{(0)} \)**

Here, we present an algorithmic procedure for computation of the vector \( \Phi^{(0)} \). Due to the special structure of \( R \) matrix as well as the coefficient matrices \( B_1 \) and \( A_2 \), the vector \( \Phi^{(0)} \) is computed using the equations given in the Appendix B.

**Computation of the Vectors \( \Phi^{(i)} \)**

We partition the vector \( \Phi^{(i)} \) as

\[
\Phi^{(i)} = (\Phi_1^{(i)}, \Phi_2^{(i)})
\]

where

\[
\Phi_1^{(i)} = (\phi^{(i,0,0)}_i, \phi^{(i,0,Q)}_i, \phi^{(i,1,0)}_i)
\]

\[
\Phi_2^{(i)} = (\phi^{(i,1,1)}_i, \phi^{(i,1,2)}_i, \ldots, \phi^{(i,1,S)}_i)
\]
Note that $\Phi^{(i)}_1$ gives the steady-state probability that the system has $i$ demands in the orbit either the server is on vacation or the server is available with zero inventory and the $\Phi^{(i)}_2$ gives the steady-state probability that the system has $i$ demands in the orbit and the server is available with positive inventory. By observing the special structure of $R$, equation (3.2) can be written as

$$
\Phi^{(i)}_1 = \Phi^{(0)}_1 R^i_0,
$$
$$
\Phi^{(i)}_2 = \Phi^{(0)}_1 R^{i-1}_0 R_1.
$$

3.3.4 System Performance Measures

In this section, we derive some system performance measures that are useful in giving a qualitative interpretation of the model under study.

Expected Inventory Level

Let $\zeta_I$ denote the mean inventory level in the steady state. Since $\phi^{(i,k,m)}$ denotes the steady state probability vector when the number of demands in the orbit is $i$, the server status is $k$ and the inventory level is $m$, the mean inventory level is given by

$$
\zeta_I = \sum_{i=0}^{\infty} \left( Q\phi^{(i,0,Q)} + \sum_{m=1}^{S} m\phi^{(i,1,m)} \right).
$$

(3.6)

Expected Reorder Rate

Let $\zeta_R$ denote the expected reorder rate in the steady state. We note that an order is triggered when the inventory level drops from $s+1$ to $s$. We also note that reordering will occur only when the server is in working condition. The steady-state probability $\phi^{(i,1,s+1)}$ denotes the rate at which $s+1$ is visited. After the system reaches the inventory level $s+1$, either a demand or a retrial demand, if any, triggers the reorder event. This leads to

$$
\zeta_R = \sum_{i=0}^{\infty} \phi^{(i,1,s+1)} (\lambda + \bar{\delta}_0 \theta).
$$

(3.7)

Expected Number of Demands in the Orbit

Since $\Phi^{(i)}$ is the steady-state probability vector for $i$ demands in the orbit with each component specifying a particular combination of the server status and inventory level,
the quantity $\Phi^{(i)}e$ denotes the steady state probability vector when the number of demands in the orbit is $i$. The expected number of demands in the orbit $\zeta_O$ is given by

$$
\zeta_O = \sum_{i=1}^{\infty} i \Phi^{(i)}e.
$$

$$
= \Phi^{(0)} R(I - R)^{-2} e
$$

(3.8)

**Expected Length of Idle Period**

Let $\zeta_{IS}$ denote the expected length of the idle period in the steady state. We observe that an idle period is terminated when the replenishment occurs and the server ready to serve any arriving demands. Otherwise server remains idle for exponentially distributed duration. Hence, the expected idle rate of the server is given by

$$
\zeta_{IS} = (\alpha + \mu) \sum_{i=0}^{\infty} \phi^{(i,1,0)}
$$

(3.9)

**Overall Rate of Retrial**

The overall retrials at which the orbiting demands retry for their demand is given by

$$
\zeta_{OR} = \theta \sum_{i=1}^{\infty} \Phi^{(i)}e
$$

$$
= \theta \Phi^{(0)} R(I - R)^{-1} e.
$$

(3.10)

**Successful Rate of Retrials**

Let $\zeta_{SR}$ denote the successful rate of retrials in the steady state. We note that the orbiting demands are receiving their demands only when the server is in working condition. Hence, the successful rate of retrials is given by

$$
\zeta_{SR} = \theta \sum_{i=1}^{\infty} \sum_{m=1}^{S} \phi^{(i,1,m)}.
$$

$$
= \theta \Phi^{(0)}_1 (I - R_0)^{-1} R_1 e.
$$

(3.11)

**3.3.5 Total Expected Cost Rate**

The long-run total expected cost rate for this model is defined to be

$$
TC(s, S) = c_h \zeta_I + c_s \zeta_R + c_o \zeta_O + c_{is} \zeta_{IS}.
$$
From equations (3.6), (3.7), (3.8) and (3.9), we obtain

\[ TC(s, S) = c^n \sum_{i=0}^{\infty} \left( Q\phi^{(i,0,Q)} + \sum_{m=1}^{S} m\phi^{(i,1,m)} \right) + c^s \sum_{i=0}^{\infty} \left( \phi^{(i,1,s+1)}(\lambda + \delta_0 \theta) \right) 
+ c_o\Phi(0)R(I - R)^{-2}e + c_is(\alpha + \mu)\sum_{i=0}^{\infty} \phi^{(i,1,0)}. \]

### 3.4 Model II: Single Vacation Policy

Let \( \tilde{X}(t) \) and \( \tilde{L}(t) \), respectively, denote the number of demands in the orbit and the on-hand inventory level at time \( t \). Further, let

\[ \tilde{Y}(t) = \begin{cases} 
0, & \text{if server is on vacation} \\
1, & \text{if server is not on vacation} 
\end{cases} \]

From our assumptions made on the input and output processes, it can be seen that the stochastic process \( \{(\tilde{X}(t), \tilde{Y}(t), \tilde{L}(t)), t \geq 0\} \) is a continuous time Markov chain with state space

\[ \tilde{E} = \{(i,0,m) : i = 0,1,\ldots;m = 0,Q\} \cup \{(i,1,m) : i = 0,1,\ldots;m = 0,1,\ldots,S\}. \]

It may be noted that when the server is on vacation, the on hand inventory level is either 0 (if the order is not received) or \( Q \) (if the order is received).

To determine the infinitesimal generator

\[ \tilde{P} = ((\tilde{p}((i,k,m),(j,l,n))), (i,k,m),(j,l,n) \in E) \]

of this process, we use the following arguments:

- If the server is not on the vacation and the inventory level is more than one,
  - an arriving demand takes the state of the process from \( (i,1,m) \) to \( (i,1,m-1) \) with the intensity of transition \( \lambda, i = 0,1,\ldots;m = 1,2,\ldots,S. \)
  - an arriving retrial demand, if any, takes the state of the process from \( (i,1,m) \) to \( (i-1,1,m-1) \) and the intensity of this transition is \( \theta, i = 1,2,\ldots;m = 1,2,\ldots,S. \).

- If the server is not on the vacation and only one item is available in the inventory,
  - a demand takes the inventory level to zero and the server starts the vacation.

The intensity of transition \( \tilde{p}((i,1,1),(i,0,0)), i = 0,1,\ldots \) is given by \( \lambda. \)
- an arriving retrial demand takes the inventory level to zero, so the server starts the vacation and the intensity of transition for $i = 1, 2, \ldots$, $\tilde{p}(i, 1, 1), (i - 1, 0, 0))$ is $\theta$.

- If the server is not on the vacation and the inventory level is zero, an arriving demand enters into the orbit and the orbit size increases by one, and process moves from $(i, 1, 0)$ to $(i + 1, 1, 0)$, $i = 0, 1, \ldots$. The intensity of this transition is $\lambda$.

- If the server is on vacation, any arriving demand joins the orbit with the intensity of transition $\tilde{p}(i, 0, m), (i + 1, 0, m))$ given by $\lambda$, $i = 0, 1, \ldots; m = 0, Q$.

- A transition from $(i, k, m)$ to $(i, k, m + Q)$ for $i = 0, 1, \ldots; k = 0$; $m = 0$, or for $i = 0, 1, \ldots; k = 1; m = 0, 1, 2, \ldots, s$, occurs with intensity $\mu$, as a replenishment occurs.

- The server terminates the vacation and hence the intensity of transitions $\tilde{p}(i, 0, m), (i, 1, m))$ is given by $\beta$, for $i = 0, 1, \ldots; m = 0, Q$.

- Any other transitions from $(i, k, m)$ to $(j, l, n)$, except $(j, l, n) \neq (i, k, m)$, rates are zero.

- Finally, we also have

$$
\tilde{p}(i, k, m), (i, k, m)) = - \sum_{j} \sum_{l} \sum_{n_{(j,l,n)\neq(i,k,m)}} \tilde{p}(i, k, m), (j, l, n)).
$$
Hence, $\tilde{p}(i, k, m), (j, l, n)) =$

\[
\begin{align*}
\lambda, & \quad j = i + 1, \quad i = 0, 1, \ldots, \\
& \quad l = k, \quad k = 0, \\
& \quad n = m, \quad m = 0, Q, \\
& \quad or \\
& \quad j = i + 1, \quad i = 0, 1, \ldots, \\
& \quad l = k, \quad k = 1, \\
& \quad n = m, \quad m = 0, \\
& \quad or \\
& \quad j = i, \quad i = 0, 1, \ldots, \\
& \quad l = k - 1, \quad k = 1, \\
& \quad n = m - 1, \quad m = 1, \\
& \quad or \\
& \quad j = i, \quad i = 0, 1, \ldots, \\
& \quad l = k, \quad k = 1, \\
& \quad n = m - 1, \quad m = 2, 3, \ldots, S, \\
& \quad \theta, \\
& \quad j = i - 1, \quad i = 1, 2, \ldots, \\
& \quad l = k - 1, \quad k = 1, \\
& \quad n = m - 1, \quad m = 1, \\
& \quad or \\
& \quad j = i - 1, \quad i = 1, 2, \ldots, \\
& \quad l = k, \quad k = 1, \\
& \quad n = m - 1, \quad m = 2, 3, \ldots, S, \\
& \quad \beta, \\
& \quad j = i, \quad i = 0, 1, \ldots, \\
& \quad l = k + 1, \quad k = 0, \\
& \quad n = m, \quad m = 0, Q, \\
& \quad \mu, \\
& \quad j = i, \quad i = 0, 1, \ldots, \\
& \quad l = k, \quad k = 0, \\
& \quad n = m + Q, \quad m = 0, \\
& \quad or \\
& \quad j = i, \quad i = 0, 1, \ldots, \\
& \quad l = k, \quad k = 1, \\
& \quad n = m + Q, \quad m = 0, 1, \ldots, s, \\
& \quad -(\lambda + \mu \delta_{m0} + \beta), \\
& \quad j = i, \quad i = 0, 1, \ldots, \\
& \quad l = k, \quad k = 0, \\
& \quad n = m, \quad m = 0, Q, \\
& \quad -(\lambda + \mu H (s - m) + \theta \bar{\delta}_{i0} \bar{\delta}_{m0}), \\
& \quad j = i, \quad i = 0, 1, 2, \ldots, \\
& \quad l = k, \quad k = 1, \\
& \quad n = m, \quad m = 0, 1, \ldots, S, \\
& \quad 0, \\
& \quad Otherwise.
\end{align*}
\]

Define

\[<i> = ((i, 0, 0), (i, 0, Q), (i, 1, 0), (i, 1, 1), (i, 1, 2), \ldots, (i, 1, S)), i = 0, 1, \ldots\]

By ordering the state space as \((<0>, <1>, <2>, \ldots)\), the infinitesimal generator $\tilde{P}$

can be conveniently expressed in a block partitioned matrix with entries
\[ \tilde{P} = \begin{pmatrix}
<0> & \tilde{B}_1 & A_0 & 0 & 0 & 0 & \cdots \\
<1> & \tilde{A}_2 & \tilde{A}_1 & A_0 & 0 & 0 & \cdots \\
<2> & 0 & \tilde{A}_2 & \tilde{A}_1 & A_0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots 
\end{pmatrix} \]

where

\[ \tilde{B}_1 = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \quad \tilde{A}_1 = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & \tilde{A}_{11}^{(1)} \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} 0_{(2,2)} & 0_{(2,s+1)} \\ \tilde{A}_{10}^{(2)} & \tilde{A}_{11}^{(2)} \end{pmatrix} \]

\[ \tilde{B}_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \tilde{A}_{10}^{(2)} = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ [\tilde{B}_{11}]_{ij} = \begin{cases} 
\lambda, & j = i - 1, \ i = 2,3,\ldots,S \\
\mu, & j = i + Q, \ i = 0,1,\ldots,s \\
-d_1, & j = i, \ i = 0,1,\ldots,s \\
-d_3, & j = i, \ i = 1,2,\ldots,s \\
-d_4, & j = i, \ i = s + 1,s + 2,\ldots,S 
\end{cases} \quad \text{where} \ d_1 = \lambda + \mu \]

\[ [\tilde{A}_{11}^{(1)}]_{ij} = \begin{cases} 
\lambda, & j = i - 1, \ i = 2,3,\ldots,S \\
\mu, & j = i + Q, \ i = 0,1,\ldots,s \\
-d_1, & j = i, \ i = 0,1,\ldots,s \\
-d_3, & j = i, \ i = 1,2,\ldots,s \\
-d_4, & j = i, \ i = s + 1,s + 2,\ldots,S 
\end{cases} \quad \text{where} \ d_3 = \lambda + \mu + \theta \]

3.4.1 Stability Analysis

To discuss the stability condition of the process under study, we consider the generator matrix \( \tilde{A} = A_0 + \tilde{A}_1 + \tilde{A}_2 \), which is given by

\[ \tilde{A} = \begin{pmatrix} \tilde{F}_{00} & \tilde{F}_{01} \\ \tilde{F}_{10} & \tilde{F}_{11} \end{pmatrix} \quad \text{where} \ \tilde{F}_{10} = \begin{pmatrix} 0 & Q \\ 0 & 0 \\ 1 & (\lambda + \theta) & 0 \\ 2 & 0 & 0 \\ \vdots & \vdots & \vdots \\ S & 0 & 0 \end{pmatrix} \]
Let \( \Phi = (\varphi^{(0,0)}, \varphi^{(0,Q)}, \varphi^{(1,0)}, \varphi^{(1,1)}, \ldots, \varphi^{(1,S)}) \) denote the steady-state probability vector of the generator \( \tilde{A} \). That is,
\[
\Phi \tilde{A} = 0, \Phi e = 1.
\]

**Lemma 3.4.1.** The steady-state probability vector \( \Phi \) corresponding to the generator \( \tilde{A} \) is given by
\[
\varphi^{(l,i)} = \begin{cases} 
\varphi^{(0,0)} \beta, & l = 0, i = Q, \\
\varphi^{(0,0)} \frac{\beta}{\mu}, & l = 1, i = 0, \\
\varphi^{(0,0)} d_3^{i-1} (\mu + \beta), & l = 1, i = 1, 2, \ldots, s, \\
\varphi^{(0,0)} d_4^i (\mu + \beta), & l = 1, i = s + 1, s + 2, \ldots, Q, \\
\varphi^{(0,0)} \frac{1}{d_4} \left( d_3^i - 1 \right) (\mu + \beta), & l = 1, i = Q + 1, \\
\varphi^{(0,0)} \frac{1}{d_4} \left[ d_3^i - 1 - \sum_{k=0}^{i-Q-2} d_3^k \right] (\mu + \beta), & l = 1, i = Q + 2, Q + 3, \ldots, S, 
\end{cases}
\]

and \( \varphi^{(0,0)} = \frac{1}{1 + \omega_2} \)

where \( \omega_2 = \frac{\mu + \beta}{\beta} + \frac{\mu + \beta}{d_4} \left[ \sum_{k=1}^{s} d_3^{k-1} + Q d_3^s \right] d_3^{-1} - s - \sum_{k=Q+2}^{s} \sum_{j=0}^{k-Q-2} d_3^j d_4^{-1} \).

**Proof.** We have
\[
\Phi \tilde{P} = 0, \Phi e = 1.
\]
Solving these two equations, we get the stated result.

Next, we derive the condition under which the system is stable.

**Lemma 3.4.2.** The stability condition of the system under study is given by
\[
\rho_2 = \frac{(1 + \frac{\mu + \beta}{\beta})}{\theta(1 + \omega_2)} < 1.
\]  
(3.12)

**Proof.** From the well-known result of Neuts [1994] on the positive recurrence of \( \tilde{P} \), we have
\[
\Phi A_0 e < \Phi \tilde{A}_2 e
\]
and by exploiting the structure of the matrices \( A_0 \) and \( \tilde{A}_2 \), and \( \Phi \) the stated result follows.
3.4.2 Steady State Analysis

By considering the structure of the rate matrix $\tilde{P}$ and by assuming the condition stated in lemma 3.4.2, we conclude that the Markov process $\{(\tilde{X}(t), \tilde{Y}(t), \tilde{L}(t)), t \geq 0\}$ with the state space $\tilde{E}$ is regular. Hence, the limiting probability distribution

$$\psi^{(i,k,m)} = \lim_{t \to \infty} Pr \left[ \tilde{X}(t) = i, \tilde{Y}(t) = k, \tilde{L}(t) = m \mid \tilde{X}(0), \tilde{Y}(0), \tilde{L}(0) \right],$$

exists and is independent of the initial state.

Let

$$\Psi^{(i)} = \left( \psi^{(i,0,0)}, \psi^{(i,0,Q)}, \psi^{(i,1,0)}, \psi^{(i,1,1)}, \psi^{(i,1,2)}, \ldots, \psi^{(i,1,S)} \right), i = 0, 1, \ldots,$$

and

$$\Psi = (\Psi^{(0)}, \Psi^{(1)}, \ldots).$$

The vector $\Psi$ denotes the steady state probability vector of $\tilde{A}$. That is, $\Psi$ satisfies

$$\Psi \tilde{A} = 0, \text{ and } \Psi e = 1.$$

**Theorem 3.4.1.** When the stability condition (3.12) holds good, the steady-state probability vector $\Psi$ is given by

$$\Psi^{(i)} = \Psi^{(0)} \tilde{R}^i, i = 0, 1, \ldots \tag{3.13}$$

where the matrix $\tilde{R}$ satisfies the matrix quadratic equation

$$\tilde{R}^2 \tilde{A}_2 + \tilde{R} \tilde{A}_1 + A_0 = 0 \tag{3.14}$$

and the vector $\Psi^{(0)}$ is obtained by solving

$$\Psi^{(0)} (\tilde{B}_1 + \tilde{R} \tilde{A}_2) = 0 \tag{3.15}$$

subject to the normalizing condition

$$\Psi^{(0)} (I - \tilde{R})^{-1} e = 1. \tag{3.16}$$

**Proof.** The theorem follows from the well-known result on matrix-geometric methods Neuts [1994]. \qed

3.4.3 Algorithmic Analysis

In this section, we present an efficient algorithmic procedure for the computation of the $\tilde{R}$ matrix and the vector $\Psi^{(0)}$, which are the main ingredients for discussing the qualitative behaviour of the model under study.
Computation of $\tilde{R}$ Matrix

Due to the special structure of the coefficient matrices appearing in (3.14), the square matrix $\tilde{R}$ of dimension $(S+3)$ can be computed as follows: We note that as $A_0$ has non-zero entries in its first three rows only, the matrix $\tilde{R}$ also has non-zero entries in its first three rows only and we write

$$\tilde{R} = \begin{pmatrix} \tilde{R}_0 & \tilde{R}_1 \\ 0_{(S,3)} & 0_{(S,S)} \end{pmatrix}$$

with

$$\tilde{R}_0 = \begin{pmatrix} \tilde{r}_{10} & \tilde{r}_{1Q} & \tilde{r}_{10'} \\ \tilde{r}_{20} & \tilde{r}_{2Q} & \tilde{r}_{20'} \\ \tilde{r}_{30} & \tilde{r}_{3Q} & \tilde{r}_{30'} \end{pmatrix}$$ and

$$\tilde{R}_1 = \begin{pmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \cdots & \tilde{r}_{1S} \\ \tilde{r}_{21} & \tilde{r}_{22} & \cdots & \tilde{r}_{2S} \\ \tilde{r}_{31} & \tilde{r}_{32} & \cdots & \tilde{r}_{3S} \end{pmatrix}.$$.

Due to this special form of $\tilde{R}$, we note that

$$\tilde{R}^k = \begin{pmatrix} \tilde{R}_0^k & \tilde{R}_0^{k-1} \tilde{R}_1 \\ 0_{(S,3)} & 0_{(S,S)} \end{pmatrix}, \quad k = 1, 2, \ldots,$$

This form is exploited in the computation of $\tilde{R}$ using (3.14). The relevant equations are given in the Appendix C.

Computation of the Vector $\Psi^{(0)}$

Now, we present an algorithmic procedure for the computation of the vector $\Psi^{(0)}$. Due to the special structure of $\tilde{R}$ matrix as well as the coefficient matrices $\tilde{B}_1$ and $\tilde{A}_2$, the vector $\Psi^{(0)}$ is computed using the equations given in the Appendix D.

Computation of the Vectors $\Psi^{(i)}$

We partition the vector $\Psi^{(i)}$ as

$$\Psi^{(i)} = (\Psi_1^{(i)}, \Psi_2^{(i)})$$

where

$$\Psi_1^{(i)} = \left( \psi^{(i,0,0)}, \psi^{(i,0,Q)}, \psi^{(i,1,0)} \right)$$

$$\Psi_2^{(i)} = \left( \psi^{(i,1,1)}, \psi^{(i,1,2)}, \ldots, \psi^{(i,1,S)} \right)$$

Note that $\Psi_1^{(i)}$ gives the steady-state probability that the system has $i$ demands in the orbit either the server is on vacation or the server is available with zero inventory and the $\Psi_2^{(i)}$ gives the steady-state probability that the system has $i$ demands in the
orbit and the server is available with positive inventory. Using the special structure of \( \tilde{R} \), equation (3.13) can be written as

\[
\Psi_1^{(i)} = \Psi_1^{(0)} \tilde{R}_0^i
\]

\[
\Psi_2^{(i)} = \Psi_1^{(0)} \tilde{R}_0^{i-1} \tilde{R}_1.
\]

3.4.4 System Performance Measures

In this section, we derive some system performance measures that can be used in the qualitative interpretation of the model under study.

Expected Inventory Level

Let \( \tilde{\zeta}_I \) denote the mean inventory level in the steady state, which is given by

\[
\tilde{\zeta}_I = \sum_{i=0}^{\infty} \left( Q \psi^{(i,0,Q)} + \sum_{m=1}^{S} m \psi^{(i,1,m)} \right).
\]  

(3.17)

Expected Reorder Rate

The expected reorder rate in the steady state is

\[
\tilde{\zeta}_R = \sum_{i=0}^{\infty} \psi^{(i,1,s+1)} (\lambda + \delta_{0} \theta).
\]  

(3.18)

Expected Number of Demands in the Orbit

The expected number of demands in the orbit \( \tilde{\zeta}_O \) is given by

\[
\tilde{\zeta}_O = \sum_{i=1}^{\infty} i \psi^{(i)} \epsilon.
\]

\[
= \psi^{(0)} \tilde{R}(I - \tilde{R})^{-2} \epsilon
\]  

(3.19)

Expected Length of Idle Period

The expected idle rate of the server \( \tilde{\zeta}_{IS} \) in the steady state is given by

\[
\tilde{\zeta}_{IS} = \mu \sum_{i=0}^{\infty} \psi^{(i,1,0)}
\]  

(3.20)
Overall Rate of Retrial

The overall retrials $\tilde{\zeta}_{OR}$ at which the orbiting demands request their demand is given by

$$\tilde{\zeta}_{OR} = \theta \sum_{i=1}^{\infty} \Psi(i^{(i)}),$$

$$= \theta \Psi^{(0)} \tilde{R}(I - \tilde{R})^{-1} \mathbf{e}. \quad (3.21)$$

Successful Rate of Retrials

The successful rate of retrials is

$$\tilde{\zeta}_{SR} = \theta \sum_{i=1}^{\infty} \sum_{m=1}^{S} \psi^{(i,1,m)}.$$

$$= \theta \Psi^{(0)} (I - \tilde{R}_0)^{-1} \tilde{R}_1 \mathbf{e}. \quad (3.22)$$

3.4.5 Total Expected Cost Rate

The long-run total expected cost rate of single vacation policy model is defined to be

$$\tilde{T}\!C(s,S) = c_h \tilde{\zeta}_I + c_s \tilde{\zeta}_R + c_o \tilde{\zeta}_O + c_{is} \tilde{\zeta}_{IS}$$

From equations (3.17), (3.18), (3.19) and (3.20), we obtain

$$\tilde{T}\!C(s,S) = c_h \sum_{i=0}^{\infty} \left( Q \psi^{(i,0,Q)} + \sum_{m=1}^{S} m \psi^{(i,1,m)} \right) + c_s \sum_{i=0}^{\infty} \left( \psi^{(i,1,s+1)} (\lambda + \bar{\delta}_0') \right)$$

$$+ c_o \Psi^{(0)} \tilde{R}(I - \tilde{R})^{-2} \mathbf{e} + c_{is} \mu \sum_{i=0}^{\infty} \psi^{(i,1,0)}.$$

3.5 Numerical Illustration

First, we study the behaviour of the long-run expected cost rate with appropriate costs and parameters values in such a way that the stability conditions hold good. As the total cost functions are obtained in a complex form, we cannot study the qualitative behaviour of the total expected cost rates by analytical methods. Hence, we have used ‘simple’ numerical search procedures to find the “local” optimal values by considering the cost rate over a small set of integer values for the decision variables. With a large number of numerical examples, we have found out that the total expected cost rates are either convex function or an increasing function of any one fixed variable.
Example 3.1: A typical three-dimensional plot of total expected cost of modified multiple vacation policy $TC(s, S)$ and single vacation policy $\bar{TC}(s, S)$ are presented in figures 3.2 and 3.3 respectively. The optimal cost value for modified multiple vacation $TC^* = 4.188326.$ is obtained at optimal $(s^*, S^*) = (6, 48)$ and for single vacation $\bar{TC}^* = 4.1480439$ is obtained at optimal $(s^*, S^*) = (6, 48)$. First note that in this plot the domain for expected cost rates includes cases for $\rho_1 < 1$ and $\rho_2 < 1$.

$$\lambda = 7; \theta = 3; \alpha = 7; \mu = 3; \beta = 9; c_h = 0.09; c_s = 10; c_o = 2.8; c_{is} = 0.8$$

Figure 3.2: Three dimensional plot of the total expected cost rate $TC(s, S)$

$$\lambda = 7; \theta = 3; \mu = 3; \beta = 9; c_h = 0.09; c_s = 10; c_o = 2.8; c_{is} = 0.98$$

Figure 3.3: Three dimensional view of cost function $\bar{TC}(s, S)$ for single vacation policy

Example 3.2: In order to show how the modified multiple vacation policy is better than the multiple vacation policy and single vacation policy, we present figure 3.4. From the figure 3.4, for any value of $\alpha (> 0)$, modified multiple vacation policy is better than
multiple vacation policy in Sivakumar [2011]. Whereas compared to single vacation policy, our proposed policy is the best vacation policy when the value of \( \alpha \) lies between 0 to 0.5065. For \( \alpha > 0.5065 \), single vacation policy is better than the proposed policy, which can be observed from table 3.1. The optimal cost for single vacation policy is 4.14804391, which can be seen from the figure 3.3.

\[
\lambda = 7; \theta = 3; \mu = 3; \beta = 9; c_h = 0.09; c_s = 10; c_o = 2.8; c_{is} = 0.8
\]

![Figure 3.4: Three dimensional view of cost function \( TC(s, S) \) for different vacation policy](image)

\[
\lambda = 7; \theta = 3; \mu = 3; \beta = 9; c_h = 0.09; c_s = 10; c_o = 2.8; c_{is} = 0.8
\]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0.5063</th>
<th>0.5064</th>
<th><strong>0.5065</strong></th>
<th>0.5066</th>
<th>0.5067</th>
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<tbody>
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</tr>
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</table>

Table 3.1: Optimal cost of modified multiple vacation for various value of \( \alpha \)

**Example 3.3:** Next, we study the impact of varying primary demand rate \( \lambda \), vacation time rate \( \beta \), lead time rate \( \mu \) and retrial demand rate \( \theta \) on stability value

\[
\rho = \frac{(\lambda+\theta)(\beta+\mu)(\beta+\alpha)}{\theta \beta (1+\alpha)}
\]

for different values of \( \alpha \). From figure 3.5, we observe the following:

- As is to be expected, as \( \lambda \) increases, stability value \( \rho \) increases for different values of \( \alpha \).
• The stability value of $\rho$ decreases as the value of $\beta$ and $\mu$ increase. This is observed for different values of $\alpha$.

• The values of $\rho$ plotted against $\theta$ exhibits a convex form for different values of $\alpha$.

$S = 48, s = 6$

Figure 3.5: Effect of $\lambda, \beta, \mu$ and $\theta$ on stability $\rho$ for various server idle time rate $\alpha$

**Example 3.4:** In Table 3.2, the upper pair given in each cell gives the (local) optima $S^*$ and $s^*$ and the lower entry gives the corresponding optimal cost rate $TC^*$. This table presents a results for various holding cost $c_h$, setup cost $c_s$, demand waiting time cost $c_o$ and server idle time cost $c_{is}$ on $S^*, s^*$ and $TC^*$. For this, we first fix the parameter values as $\lambda = 7, \theta = 3, \alpha = 7, \mu = 3$, and $\beta = 9$. We have observed the following from the table 3.2.

• As is to be expected the optimal cost rate $TC^*$ increases with each of costs.

• As $c_h$ increases the optimal values $S^*$ and $s^*$ decreases. This is because as $c_h$ increases, we have to maintain a low stock only.

• When the setup cost $c_s$ increases, the value of $S^*$ increases and hence the value of $s^*$ decreases. This is because, as the setup cost increases, to avoid frequent order, we have to maintain more stock and low reorder point.
• As the waiting time cost of a demand in orbit $c_o$ increases, the value of $S^*$ as well as $s^*$ increases. This is because, if we maintain low stock, then the demand has to wait long time and hence the cost increases. To avoid such a situation, we have to maintain the more stock and high reorder point.

• When the idle time cost of the server $c_{is}$ increases both optimal values $S^*$ and $s^*$ also increase.

$$\lambda = 7; \theta = 3; \alpha = 7; \mu = 3; \beta = 9$$

<table>
<thead>
<tr>
<th>$c_h$</th>
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<th>$c_s$</th>
<th>$c_{is}$</th>
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</tr>
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Table 3.2: Impact of $c_h$, $c_s$, $c_o$ and $c_{is}$ on the optimal values

**Example 3.5:** Table 3.3 gives sensitivity analysis of the optimal values pertaining to different parameters and rates. Hence the upper entries in each cell correspond to the local optima $S^*$ and $s^*$ and the lower entry gives the optimal cost rate. For this we first fix the cost values as $c_h = 0.09, c_s = 10, c_o = 2.8, c_{is} = 0.8$. We notice the following from the table 3.3.

• As is to be expected the optimal cost rate $TC^*$ increases with the values of $\lambda$ and $\alpha$; and decreases against the values of $\mu, \beta$ and $\theta$.

• When the demand rate $\lambda$ increases, the value of $S^*$ and $s^*$ increase monotonically. This is because, more demands arrive to the system. Hence, we have to maintain more inventory.
• The value of $S^*$ decreases monotonically when each of the values of $\theta, \mu$ and $\beta$ increase.

• The value of $s^*$ decreases monotonically when each of the values of $\theta$ and $\mu$ increase.

**Example 3.6:** The fraction of successful rate of retrials $\tilde{\zeta}_{FSR}$ is $\frac{\tilde{\zeta}_{FSR}}{\tilde{\zeta}_{OR}}$. From the table 3.4, we observe that the fraction of successful rate of retrial increases when $\lambda, \mu$ and $\beta$ increase and decreases when $\alpha$ and $\theta$ increase.
Table 3.3: Effect of $\lambda$, $\mu$, $\alpha$, $\theta$ and $\beta$ on the optimal values

<table>
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<td>2.53013</td>
<td>2.53013</td>
</tr>
<tr>
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<td>1.74934</td>
<td>1.74934</td>
</tr>
<tr>
<td>6</td>
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<td>0.97875</td>
<td>0.97875</td>
</tr>
</tbody>
</table>

$c_h = 0.09$; $c_o = 10$; $c_o = 2.8$; $c_is = 0.8$
\[ S = 48, s = 6 \]

<table>
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<th>( \gamma )</th>
<th>( \alpha )</th>
<th>( \lambda )</th>
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<td>0.69985</td>
</tr>
</tbody>
</table>

Table 3.4: Fraction of successful rate of retrial

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Appendix A

To compute the $R$ matrix, we use the following set of non-linear equations. This can be solved by using Gauss-Seidel iterative process. These equations are derived by exploiting the coefficient matrices appearing in (3.3),

$$
\begin{align*}
    r_{00} \alpha - r_{00} (\lambda + \mu + \beta) + \lambda &= 0, \\
    r_{QQ} \alpha - r_{QQ} (\lambda + \mu + \beta) &= 0, \\
    r_{0Q} \alpha - r_{0Q} (\lambda + \mu + \beta) &= 0, \\
    r_{00} \mu - r_{00} (\lambda + \beta) &= 0, \\
    r_{Q0} \mu - r_{Q0} (\lambda + \beta) + \lambda &= 0, \\
    r_{00} \mu - r_{00} (\lambda + \beta) &= 0,
\end{align*}
$$

$$
\begin{align*}
    r_{00} r_{11} \theta + r_{00} r_{21} \theta + r_{00} r_{31} \theta + r_{00} \beta - d_2 r_{00} \theta + r_{11} \lambda &= 0, \\
    r_{QQ} r_{11} \theta + r_{QQ} r_{21} \theta + r_{QQ} r_{31} \theta + r_{Q0} \beta - d_2 r_{QQ} \theta + r_{21} \lambda &= 0, \\
    r_{00} r_{11} \theta + r_{00} r_{21} \theta + r_{00} r_{31} \theta + r_{00} \beta - d_2 r_{00} \theta + r_{31} \lambda + \lambda &= 0,
\end{align*}
$$

For $i = 1, 2, \ldots, s$,

$$
\begin{align*}
    r_{1(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{00} r_{2(i+1)} \theta + r_{00} r_{3(i+1)} \theta &= r_{1d3}, \\
    r_{2(i+1)} \lambda + r_{QQ} r_{1(i+1)} \theta + r_{QQ} r_{2(i+1)} \theta + r_{QQ} r_{3(i+1)} \theta &= r_{2d3}, \\
    r_{3(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{00} r_{2(i+1)} \theta + r_{00} r_{3(i+1)} \theta &= r_{3d3},
\end{align*}
$$

For $i = s + 1, s + 2, \ldots, Q - 1$,

$$
\begin{align*}
    r_{1(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{00} r_{2(i+1)} \theta + r_{00} r_{3(i+1)} \theta &= r_{1d4}, \\
    r_{2(i+1)} \lambda + r_{QQ} r_{1(i+1)} \theta + r_{QQ} r_{2(i+1)} \theta + r_{QQ} r_{3(i+1)} \theta &= r_{2d4}, \\
    r_{3(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{00} r_{2(i+1)} \theta + r_{00} r_{3(i+1)} \theta &= r_{3d4},
\end{align*}
$$

For $i = Q$,

$$
\begin{align*}
    r_{1(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{00} r_{2(i+1)} \theta + r_{00} r_{3(i+1)} \theta + r_{00} \beta &= r_{1d4}, \\
    r_{2(i+1)} \lambda + r_{QQ} r_{1(i+1)} \theta + r_{QQ} r_{2(i+1)} \theta + r_{QQ} r_{3(i+1)} \theta + r_{QQ} \beta + r_{QQ} \mu &= r_{2d4}, \\
    r_{3(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{QQ} r_{2(i+1)} \theta + r_{QQ} r_{3(i+1)} \theta + r_{QQ} \beta + r_{QQ} \mu &= r_{3d4},
\end{align*}
$$

For $i = Q + 1, Q + 2, \ldots, S - 1$,

$$
\begin{align*}
    r_{1(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{00} r_{2(i+1)} \theta + r_{00} r_{3(i+1)} \theta + r_{1(i-Q)} \mu &= r_{1d4}, \\
    r_{2(i+1)} \lambda + r_{QQ} r_{1(i+1)} \theta + r_{QQ} r_{2(i+1)} \theta + r_{QQ} r_{3(i+1)} \theta + r_{2(i-Q)} \mu &= r_{2d4}, \\
    r_{3(i+1)} \lambda + r_{00} r_{1(i+1)} \theta + r_{00} r_{2(i+1)} \theta + r_{00} r_{3(i+1)} \theta + r_{3(i-Q)} \mu &= r_{3d4},
\end{align*}
$$

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For $i = S$,

\begin{align*}
 r_{1(i-Q)} & = r_{1d_4}, \\
 r_{2(i-Q)} & = r_{2d_4}, \\
 r_{3(i-Q)} & = r_{3d_4},
\end{align*}

Appendix B

To compute the $\phi^{(0)}$ vector, we use the following set of non-linear equations.

\begin{align*}
 -\phi^{(0,0,0)}(\lambda + \mu + \beta) + \phi^{(0,1,0)}\alpha & = 0, \\
 \phi^{(0,0,0)}\mu - \phi^{(0,0,Q)}(\lambda + \beta) & = 0, \\
 \phi^{(0,0,0)}(\beta + r_{11}\theta) + \phi^{(0,0,Q)}r_{21}\theta + \phi^{(0,1,0)}(-d_2 + r_{31}\theta) + \phi^{(0,1,1)}\lambda & = 0,
\end{align*}

For $i = 1, 2, ..., s$,

\[ \phi^{(0,0,0)}r_{1(i+1)}\theta + \phi^{(0,0,Q)}r_{2(i+1)}\theta + \phi^{(0,1,0)}r_{3(i+1)}\theta - \phi^{(0,1,i)}d_1 + \phi^{(0,1,i+1)}\lambda = 0, \]

For $i = s + 1, s + 2, ..., Q - 1$,

\[ \phi^{(0,0,0)}r_{1(i+1)}\theta + \phi^{(0,0,Q)}r_{2(i+1)}\theta + \phi^{(0,1,0)}r_{3(i+1)}\theta - \phi^{(0,1,i)}\lambda + \phi^{(0,1,i+1)}\lambda = 0, \]

For $i = Q$,

\[ \phi^{(0,0,0)}r_{1(i+1)}\theta + \phi^{(0,0,Q)}(r_{2(i+1)}\theta + \beta) + \phi^{(0,1,0)}(r_{3(i+1)}\theta + \mu) - \phi^{(0,1,i)}\lambda + \phi^{(0,1,i+1)}\lambda = 0, \]

For $i = Q + 1, Q + 2, ..., S - 1$,

\[ \phi^{(0,0,0)}r_{1(i+1)}\theta + \phi^{(0,0,Q)}r_{2(i+1)}\theta + \phi^{(0,1,0)}r_{3(i+1)}\theta - \phi^{(0,1,i)}\lambda + \phi^{(0,1,i+1)}\lambda + \phi^{(0,1,i-Q)}\mu = 0, \]

For $i = S$,

\[ -\phi^{(0,1,i)}\lambda + \phi^{(0,1,i-Q)}\mu = 0. \]
Appendix C

To compute the $\hat{R}$ matrix, we use the following set of non-linear equations. This can be solved by using Gauss-Seidel iterative process. These equations are derived by exploiting the coefficient matrices appearing in (3.14),

\[
\begin{align*}
\tilde{r}_0\tilde{r}_{11}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{21}\theta + \tilde{r}_0\tilde{r}_3\theta + \tilde{r}_1\lambda - \tilde{r}_0\lambda + \mu + \beta + \lambda &= 0, \\
\tilde{r}_Q\tilde{r}_{11}\theta + \tilde{r}_Q\tilde{Q}_2\tilde{r}_{21}\theta + \tilde{r}_Q\tilde{r}_3\theta + \tilde{r}_2\lambda - \tilde{r}_Q\lambda + \mu + \beta &= 0, \\
\tilde{r}_0\tilde{r}_{11}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{21}\theta + \tilde{r}_0\tilde{r}_3\theta + \tilde{r}_1\lambda - \tilde{r}_0\lambda + \mu + \beta &= 0, \\
\tilde{r}_0\mu - \tilde{r}_Q\lambda + \mu &= 0, \\
\tilde{r}_Q\mu - \tilde{r}_Q\lambda + \mu &= 0, \\
\tilde{r}_0\mu - \tilde{r}_0\lambda + \mu &= 0, \\
\tilde{r}_0\beta - f_1\tilde{r}_0 &= 0, \\
\tilde{r}_Q\beta - f_1\tilde{r}_Q &= 0, \\
\tilde{r}_0\beta - f_1\tilde{r}_0 &= 0.
\end{align*}
\]

For $i = 1, 2, ..., s$,

\[
\begin{align*}
\tilde{r}_{1(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta &= \tilde{r}_{1f}2, \\
\tilde{r}_{2(i+1)}\lambda + \tilde{r}_Q\tilde{r}_{1(i+1)}\theta + \tilde{r}_Q\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_Q\tilde{r}_3(i+1)\theta &= \tilde{r}_{2f}2, \\
\tilde{r}_{3(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta &= \tilde{r}_{3f}2.
\end{align*}
\]

For $i = s + 1, s + 2, ..., Q - 1$,

\[
\begin{align*}
\tilde{r}_{1(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta &= \tilde{r}_{1f}3, \\
\tilde{r}_{2(i+1)}\lambda + \tilde{r}_Q\tilde{r}_{1(i+1)}\theta + \tilde{r}_Q\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_Q\tilde{r}_3(i+1)\theta &= \tilde{r}_{2f}3, \\
\tilde{r}_{3(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta &= \tilde{r}_{3f}3.
\end{align*}
\]

For $i = Q$,

\[
\begin{align*}
\tilde{r}_{1(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta + \tilde{r}_0\mu &= \tilde{r}_{1f}3, \\
\tilde{r}_{2(i+1)}\lambda + \tilde{r}_Q\tilde{r}_{1(i+1)}\theta + \tilde{r}_Q\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_Q\tilde{r}_3(i+1)\theta + \tilde{r}_Q\mu &= \tilde{r}_{2f}3, \\
\tilde{r}_{3(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta + \tilde{r}_0\mu &= \tilde{r}_{3f}3.
\end{align*}
\]

For $i = Q + 1, Q + 2, ..., S - 1$,

\[
\begin{align*}
\tilde{r}_{1(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta + \tilde{r}_{1(-Q)}\mu &= \tilde{r}_{1f}3, \\
\tilde{r}_{2(i+1)}\lambda + \tilde{r}_Q\tilde{r}_{1(i+1)}\theta + \tilde{r}_Q\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_Q\tilde{r}_3(i+1)\theta + \tilde{r}_{2(-Q)}\mu &= \tilde{r}_{2f}3, \\
\tilde{r}_{3(i+1)}\lambda + \tilde{r}_0\tilde{r}_{1(i+1)}\theta + \tilde{r}_0\tilde{Q}_2\tilde{r}_{2(i+1)}\theta + \tilde{r}_0\tilde{r}_3(i+1)\theta + \tilde{r}_{3(-Q)}\mu &= \tilde{r}_{3f}3.
\end{align*}
\]

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For $i = S$,
\[
\tilde{r}_{1(i-Q)} = \tilde{r}_{1i} f_3,
\]
\[
\tilde{r}_{2(i-Q)} = \tilde{r}_{2i} f_3,
\]
\[
\tilde{r}_{3(i-Q)} = \tilde{r}_{3i} f_3,
\]

**Appendix D**

To compute the $\psi^{(0)}$ vector, we use the following set of non-linear equations.
\[
\psi^{(0,0,0)}(\tilde{r}_{11} \theta - (\lambda + \mu + \beta)) + \psi^{(0,0,Q)}\tilde{r}_{21} \theta + \psi^{(0,1,0)}\tilde{r}_{31} \theta + \psi^{(0,1,1)} \lambda = 0,
\]
\[
\psi^{(0,0,0)} \mu - \psi^{(0,0,Q)}(\lambda + \beta) = 0,
\]
\[
\psi^{(0,0,0)} \beta - \psi^{(0,1,0)} f_1 = 0,
\]

For $i = 1, 2, ..., s$,
\[
\psi^{(0,0,0)} \tilde{r}_{1(i+1)} \theta + \psi^{(0,0,Q)} \tilde{r}_{2(i+1)} \theta + \psi^{(0,1,0)} \tilde{r}_{3(i+1)} \theta - \psi^{(0,1,i)} f_1 + \psi^{(0,1,i+1)} \lambda = 0,
\]

For $i = s + 1, s + 2, ..., Q - 1$,
\[
\psi^{(0,0,0)} \tilde{r}_{1(i+1)} \theta + \psi^{(0,0,Q)} \tilde{r}_{2(i+1)} \theta + \psi^{(0,1,0)} \tilde{r}_{3(i+1)} \theta - \psi^{(0,1,i)} \lambda + \psi^{(0,1,i+1)} \lambda = 0,
\]

For $i = Q$,
\[
\psi^{(0,0,0)} \tilde{r}_{1(i+1)} \theta + \psi^{(0,0,Q)} (\tilde{r}_{2(i+1)} \theta + \beta) + \psi^{(0,1,0)} (\tilde{r}_{3(i+1)} \theta + \mu) - \psi^{(0,1,i)} \lambda + \psi^{(0,1,i+1)} \lambda = 0,
\]

For $i = Q + 1, Q + 2, ..., S - 1$,
\[
\psi^{(0,0,0)} \tilde{r}_{1(i+1)} \theta + \psi^{(0,0,Q)} \tilde{r}_{2(i+1)} \theta + \psi^{(0,1,0)} \tilde{r}_{3(i+1)} \theta - \psi^{(0,1,i)} \lambda + \psi^{(0,1,i+1)} \lambda + \psi^{(0,1,i-Q)} \mu = 0,
\]

For $i = S$,
\[
-\psi^{(0,1,i)} \lambda + \psi^{(0,1,i-Q)} \mu = 0.
\]