Chapter 1

Preliminaries

1.1. Basic Definitions and Results

For Lattice theoretic notations and terminologies, we follow [2, 10, 30]

Definition 1.1. Let \( L \) be a nonempty set. Let \( R \) be a relation defined on \( L \). Then \( R \) is said to be a partial order if

i) \( R \) is reflexive, that is, \( xRx \) for all \( x \in L \)

ii) \( R \) is antisymmetric, that is, \( xRy \) and \( yRx \) \( \Rightarrow \) \( x = y \) for all \( x, y \in L \)

iii) \( R \) is transitive, that is, \( xRy \) and \( yRz \) \( \Rightarrow \) \( xRz \) for all \( x, y, z \in L \).

Definition 1.2. A nonempty set \( P \) together with a partial order relation \( \leq \) defined on it is called as a partially ordered set or a Poset and is denoted by \( (P, \leq) \).

Definition 1.3. A Poset \( (P, \leq) \) is called a chain if every pair of elements of \( P \) are comparable. That is, if \( a, b \in P \), then \( a \leq b \) or \( b \leq a \). A chain with \( n \) elements is denoted by \( C_n \).

Let \( a, b \in P \). If \( a \) and \( b \) are incomparable, then we denote it by \( a \parallel b \). Otherwise they are comparable.
Diagrammatic Representation of a Poset:

Let \((P, \leq)\) be a partially ordered set and let \(a, b \in P\), we say that \(a\) is covered by \(b\) written as \(a \prec b\) if \(a < b\) and there is no element \(c \in P\) such that \(a < c < b\). Every finite partially ordered set \((P, \leq)\) can be represented by a diagram (or Hasse diagram). The elements of \(P\) are represented with small circles and circles representing two elements \(a, b\) are connected by a straight line if \(a < b\) or \(b < a\). If \(a\) is covered by \(b\), then the circle representing \(a\) is placed lower than the circle representing \(b\).

Definition 1.4. Let \((P, \leq)\) be a Poset. Let \(A\) be a subset of \(P\). An element \(u \in P\) is said to be an upper bound of \(A\) if \(a \leq u\) for all \(a \in A\).

Definition 1.5. Let \((P, \leq)\) be a Poset. Let \(A\) be a subset of \(P\). An element \(u \in P\) is said to be a least upper bound of \(A\) or supremum of \(A\) if

i) \(a \leq u\) for all \(a \in A\) and

ii) if \(v \in P\) is such that \(a \leq v\) for all \(a \in A\), then \(u \leq v\).

Definition 1.6. Let \((P, \leq)\) be a Poset. Let \(A\) be a subset of \(P\). An element \(l \in P\) is said to be a lower bound of \(A\) if \(l \leq a\), for all \(a \in A\).

Definition 1.7. Let \((P, \leq)\) be a Poset. Let \(A\) be a subset of \(P\). An element \(l \in P\) is said to be a greatest lower bound of \(A\) or infimum of \(A\) if

i) \(l \leq a\), for all \(a \in A\) and

ii) if \(v \in P\) is such that \(v \leq a\) for all \(a \in A\), then \(v \leq l\).
Definition 1.8. A Poset \((L, \leq)\) is a lattice in which every pair of elements \(a\) and \(b\) have the greatest lower bound (g.l.b.) and the least upper bound (l.u.b.). If every subset of \(L\) has a supremum and an infimum in it, then \(L\) is said to be a complete lattice.

Another definition of lattice in algebraic form is as follows.

Definition 1.9. A nonempty set \(L\) together with two binary operations \(\lor\) and \(\land\) is said to be a lattice if for all \(a, b, c \in L\) it satisfies

i) Idempotency: \(a \land a = a, a \lor a = a\)

ii) Commutativity: \(a \land b = b \land a, a \lor b = b \lor a\)

iii) Associativity: 
\[
(a \land b) \land c = a \land (b \land c),
(a \lor b) \lor c = a \lor (b \lor c)
\]

iv) Absorbtion identities: \(a \lor (a \land b) = a, a \land (a \lor b) = a\).

It is usually denoted by \((L, \lor, \land)\). The above two definitions of the lattice are equivalent.

For a given lattice \((L, \leq)\), the join (\(\lor\)) and meet (\(\land\)) operations are defined as \(a \lor b = \text{l.u.b.}\{a, b\}, a \land b = \text{g.l.b.}\{a, b\}\) and given a lattice \((L, \lor, \land)\) the partial order \(\leq\) on \(L\) is defined by \(a \leq b\) iff \(a \lor b = b\) or equivalently \(a \land b = a\).

The dual of the lattice \(L\), denoted by \(L^\partial\) is obtained by swapping \(\lor\) and \(\land\) in \(L\).

Definition 1.10. Let \((L_1, \lor_1, \land_1)\) and \((L_2, \lor_2, \land_2)\) be any two lattices. A map \(f : L_1 \to L_2\) is said to be
i) a homomorphism if it satisfies $f(a \lor b) = f(a) \lor f(b)$ and $f(a \land b) = f(a) \land f(b)$ for all $a, b \in L_1$.

ii) a dual homomorphism if it satisfies $f(a \lor b) = f(a) \land f(b)$ and $f(a \land b) = f(a) \lor f(b)$ for all $a, b \in L_1$.

iii) an isomorphism if $f$ is a bijective homomorphism.

iv) a dual isomorphism if $f$ is a bijective dual homomorphism.

Two lattices $L_1$ and $L_2$ are said to be dually isomorphic written as $L_1 \cong L_2$ if, and only if, there exists dual isomorphisms $\phi: L_1 \to L_2$ and $\psi: L_2 \to L_1$ such that $\phi \circ \psi = id_{L_2}$ and $\psi \circ \phi = id_{L_1}$ where $id_S$ denotes an identity map on $S$ given by $id_S(x) = x, \forall x \in S$.

**Definition 1.11.** Let $(L_1, \lor, \land)$ and $(L_2, \lor, \land)$ be any two lattices. In the set $L_1 \times L_2$ of all ordered pairs $(a, b)$ with $a \in L_1, b \in L_2$ define the join $\lor$, the meet $\land$ componentwise.

That is, $(a, b) \lor (c, d) = (a_1 \lor c_1, c_2 \lor d_2), (a, b) \land (c, d) = (a_1 \land c_1, c_2 \land d_2)$. Then $(L_1 \times L_2, \lor, \land)$ is a lattice called the direct product of the two lattices $L_1$ and $L_2$.

If $L_1 = L_2 = L$, then we write $L^2$ for $L_1 \times L_2$.

**Definition 1.12.** Let $(L, \lor, \land)$ be a lattice. Let $A$ be a subset of $L$. Then $A$ is said to be a sublattice of $L$, if $a_1, a_2 \in A$ implies $a_1 \lor a_2, a_1 \land a_2 \in A$. In symbol,
A \leq L. A nonempty subset S of L is called a complete sublattice of L if \( \vee S \in L, \wedge S \in L \).

**Definition 1.13.** Let \((L, \vee, \wedge)\) be a lattice. Let I be a nonempty subset of L. Then I is said to be an ideal of L if it satisfies the following conditions:

i) \(a, b \in I\) implies \(a \vee b \in I\),

ii) \(a \in I, x \in L\) and \(x \leq a \implies x \in I\).

The dual concept of an ideal is called a filter.

Let L be a lattice and \(a \in L\). Then \([a] = \{x \in L / x \leq a\}\) and \([a] = \{x \in L / a \leq x\}\) are the principal ideal and principal filter generated by a respectively.

**Definition 1.14.** A lattice \((L, \vee, \wedge)\) is said to be modular if for all \(a, b, c \in L\) with \(a \leq c\), implies \(a \vee (b \wedge c) = (a \vee b) \wedge c\).

Every sublattice, every homomorphic image and direct product of distributive lattices are distributive lattices.

**Theorem 1.15.** A lattice L is modular if, and only if, L has no sublattice isomorphic to \(N_3\).

**Definition 1.16.** A lattice \((L, \vee, \wedge)\) is said to be distributive if for all \(a, b, c \in L\), one of the following distributive laws hold.

i) \(a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\)

ii) \(a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\).
Theorem 1.17. A lattice $L$ is distributive if, and only if, $L$ has no sublattice isomorphic to $N_5$ or $M_3$.

The lattices $N_5$ and $M_3$ are given in figure 1.1.

Definition 1.18. An element $u \in L$ is said to be the greatest element (or all element) of $L$ if $a \leq u$ for all $a \in L$, it is usually denoted by 1.

Definition 1.19. An element $u \in L$ is said to be the least element (or null element) of $L$ if $u \leq a$ for all $a \in L$, it is usually denoted by 0.

Definition 1.20. A lattice $L$ is said to be a bounded lattice if it has the greatest element 1 and the least element 0. It is usually denoted by $(L, \lor, \land, 0, 1)$.

Definition 1.21. An element $a$ of a bounded lattice $L$ is an atom if $0 \prec a$ and a dual atom if $a \prec 1$.

Definition 1.22. Let $(L, \lor, \land, 0, 1)$ be a bounded lattice and $a \in L$, an element $x \in L$ is said to be a complement of $a$ if $a \lor x = 1$ and $a \land x = 0$. The complement of $a$ is denoted by $a'$.
Definition 1.23. A bounded lattice $L$ is said to be a complemented lattice if every element in the lattice has a complement.

In a lattice, an element can have more than one complement.

Definition 1.24. A distributive complemented lattice is said to be a Boolean algebra. It is usually denoted by $(B, \lor, \land, \neg, 0, 1)$. Boolean algebra with $n$ atoms is denoted by $B_n$.

Definition 1.25. Let $L$ be a lattice and $a, b \in L$. Suppose $a \leq b$ and $[a, b] = \{x \in L / a \leq x \leq b\}$. $[a, b]$ is said to be complemented if for every $x \in [a, b]$, there exists $t \in [a, b]$ such that $x \land t = a$ and $x \lor t = b$. $L$ is said to be relatively complemented if every interval of the form $[a, b]$ (with $a \leq b$) is a complemented lattice.

Equivalently, a relatively complemented lattice is a lattice in which every element has a relative complement in any interval containing it.

Definition 1.26. Let $L$ be a lattice with $0$ and $a \in L$. Then $L$ is said to be sectionally complemented if every interval of the form $[0, a]$ is complemented. That is, if $0 \leq x \leq a$, then there exist $t \in [0, a]$ such that $x \land t = 0$ and $x \lor t = a$.

Lemma 1.27. In a bounded distributive lattice, an element can have only one complement.

Lemma 1.28. If a bounded distributive lattice has a complement, then it also has a relative complement in any interval containing it.
Example 1.29. The lattice $N_6$ shown in figure 1.2 is a sectionally complemented lattice.

Definition 1.30. A congruence relation on a lattice $L$ is an equivalence relation $\theta$ which is compatible with the join and the meet.

Theorem 1.31. A reflexive binary relation $\theta$ on a lattice $L$ is said to be a congruence relation on $L$ if, and only if, the following three properties are satisfied, for all $a,b,c,d \in L$:

i) $a \equiv b \pmod{\theta}$ if, and only if, $a \land b \equiv a \lor b(\theta)$.

ii) $a \leq b \leq c$, $a \equiv b \pmod{\theta}$, and $b \equiv c \pmod{\theta}$ imply that $a \equiv c \pmod{\theta}$

iii) $a \leq b$ and $a \equiv b(\theta)$ imply that $a \land t \equiv b \land t \pmod{\theta}$ and $a \lor t \equiv b \lor t \pmod{\theta}$.

Definition 1.32. Let $L$ be a lattice. Let $\omega$ be defined on $L$ by $x \equiv y \pmod{\omega}$ if and only if $x = y$ in $L$. Then $\omega$ is a congruence relation and is called the null
congruence. Let $\tau$ be defined on $L$ by $x \equiv y \pmod{\tau}$ for all $x, y \in L$. Then $\tau$ is a congruence relation on $L$ and is called the all congruence.

The lattice of all congruence relations on a lattice $L$ is denoted by $\text{Con}(L)$. If $L$ is non trivial, then $\text{Con}(L)$ contains the two element sublattice $\{\omega, \tau\}$.

**Definition 1.33.** A lattice $L$ is said to be simple if $\text{Con}(L) = \{\omega, \tau\}$.

**Definition 1.34.** Let $L$ be a lattice and $\theta$ be a congruence relation on $L$. Let $L/\theta = \{[a] \theta / a \in L\}$. Let $[a] \theta \lor [b] \theta = [a \lor b] \theta$ and $[a] \theta \land [b] \theta = [a \land b] \theta$. Then with respect to $\lor$ and $\land$, $L/\theta$ becomes a lattice and is called the quotient lattice of $L$ modulo $\theta$.

**Theorem 1.35.** Let $L$ and $K$ be lattices. Let $\theta$ be a congruence relation of $L$ and let $\phi$ be a congruence relation of $K$. Define the relation $\theta \times \phi$ on $L \times K$ by

$$\langle a_1, b_1 \rangle \equiv \langle a_2, b_2 \rangle (\theta \times \phi) \text{ iff } a_1 \equiv a_2(\theta) \text{ and } b_1 \equiv b_2(\phi).$$

Then $\theta \times \phi$ is a congruence relation on $L \times K$. Conversely, every congruence relation of $L \times K$ is of this form.

**Theorem 1.36.** The lattice $\text{Con}(L)$ is distributive, for any lattice $L$.

**Corollary 1.37.** The congruence lattice of a finite modular lattice is Boolean.

**Note 1.38.** $\text{Con}(B_n) \cong B_n$ and $\text{Con}(C_n) \cong B_{n-1}$

**Definition 1.39.** Let $K$ be a lattice. A lattice $L$ is said to be a congruence preserving extension of $K$ if $L$ is an extension of $K$ that is $K \subset L$, and every congruence $\theta$ of $K$ has exactly one extension $\overline{\theta}$ of $L$ satisfying $\overline{\theta}/K = \theta$. 
The map $\theta \to \overline{\theta}$ is an isomorphism between $\text{Con}(K)$ and $\text{Con}(L)$.

**Definition 1.40.** Let $K$ and $L$ be any two lattices. If $\theta_K$ is a binary relation on $K$ and $\theta_L$ is a binary relation on $L$, the reflexive product $\theta_K \circ \theta_L$ is defined as $\theta_K \cup \theta_L \cup (\theta_K \circ \theta_L)$.

**Definition 1.41.** Let $K$ and $L$ be any two lattices and $K \subset L$. Then every congruence $\theta$ of $L$ reflects (or restricts) to $K$: the relation $\theta \cap K^2 = \theta \cap K$ on $K$ is a congruence of $K$. So, we get the reflection map (also called restriction map) $\text{re}: \text{Con}(L) \to \text{Con}(K)$ that maps a congruence $\theta$ of $L$ to $\theta \cap K$.

**Definition 1.42.** Let $L$ be a lattice, we call a congruence relation $\theta$ of $L$ regular if any congruence class of $\theta$ determines the congruence. A lattice $L$ is said to be regular, if all the congruences of $L$ are regular.

**Definition 1.43.** A congruence relation on a lattice is said to be isoform, if all the congruence classes are isomorphic sublattices. A lattice is said to be isoform if all of its congruences are isoform.

**Definition 1.44.** Let $L$ be bounded distributive lattice. The lattice $M_3[L]$ is defined as the set of all balanced triples, that is triples $(x, y, z) \in L^3$ satisfying the condition $x \land y = y \land z = z \land x$.

**Note 1.45.** If $L$ is not distributive, then $M_3[L]$ fails to be a lattice.

**Definition 1.46.** For every lattice $L$, call an element $(a_1, a_2, a_3) \in L^3$ a Boolean triple if there exists $(u, v, w) \in L^3, (a_1, a_2, a_3) = (v \land w, u \land w, u \land v)$. This is equivalent
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\[ (x, y, z) = \langle (x \lor y) \land (x \lor z), (y \lor x) \land (y \lor z), (z \lor x) \land (z \lor y) \rangle. \]

Let \( L \) be a lattice \( M_3(L) = \{ (x, y, z) \in L^3 / x = (x \lor y) \land (x \lor z), y = (y \lor x) \land (y \lor z), z = (z \lor x) \land (z \lor y) \} \)

be a lattice \( M_3(L) = \{ (x, y, z) \in L^3 / x = (x \lor y) \land (x \lor z), y = (y \lor x) \land (y \lor z), z = (z \lor x) \land (z \lor y) \} \)

is the set of all Boolean triples of \( L \).

**Definition 1.47.** An ortholattice is a lattice \( \langle L, \lor, \land, ', 0, 1 \rangle \) where \( \langle L, \lor, \land, 0, 1 \rangle \)

is a bounded lattice and \( ': L \to L \) is an unary operation called orthocomplementation satisfying the conditions (i) \( a \lor a' = 1 \) and \( a \land a' = 0 \),

(ii) if \( a \leq b \), then \( b' \leq a' \) and (iii) \( a'' = a \) for all \( a, b \in L \).

**Definition 1.48.** An orthomodular lattice is an ortholattice \( \langle L, \lor, \land, ', 0, 1 \rangle \)

satisfying the orthomodular law, \( x \leq y \) implies \( x \lor (x' \land y) = y \), where \( x, y \in L \).

**Theorem 1.49.** Let \( \langle L, \lor, \land, ', 0, 1 \rangle \) be an ortholattice. The following statements are equivalent:

(i) \( L \) is orthomodular

(ii) If \( a \leq b \) and \( b \land a' = 0 \) then \( a = b \)

(iii) \( O_b \) is not a subalgebra of \( L \)

(iv) if \( a \leq b \), then \( \Gamma\{a, b\} \) is a Boolean subalgebra of \( L \)

(v) \( aCb \iff bCa \).
Definition 1.50. An ortholattice \((L, \lor, \land, ', 0, 1)\) is said to be modular ortholattice if for \(x, z \in L\) such that \(x \leq z\), \(x \lor (y \land z) = (x \lor y) \land z\) for all \(y \in L\).

Definition 1.51. A congruence relation on an orthomodular lattice \(L\) is an equivalence relation \(\theta\) which is compatible with the join, the meet and the orthocomplementation.

Definition 1.52. Let \(a, b \in L\) then \(a\) is said to be perspective to \(b\) written as \(a \sim b\) if \(a\) and \(b\) have a common complement.

Remark 1.53. i) Every irreducible modular ortholattice is of the form \(MO_n\), \(n > 1\) as shown in the figure 1.4 or \(C_2\).

\[ \begin{array}{c}
\text{1} \\
\text{2} \\
\text{0} \\
\end{array} \]

\[ \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
\vdots \\
a_{n-1} \\
a_n \\
\end{array} \]

Figure 1.4

ii) Every modular ortholattice is of the form \(MO_n \times (C_2)^i\) for some \(n > 1\) and for some \(i > 0\) as shown in figure 1.5. Also, Boolean algebras \(B_n\) with \(n\) atoms are modular ortholattices.
Definition 1.54. Let $A$ and $B$ be $\{\lor, 0\}$ semilattices. We denote by $A \otimes B$ the tensor product of $A$ and $B$ defined as the free $\{\lor, 0\}$ semilattice generated by the set $A^- \times B^-$ and subject to the relations.

$$\langle a, b_0 \rangle \lor \langle a, b_1 \rangle = \langle a, b_0 \lor b_1 \rangle, \text{ for } a \in A^-, b_0, b_1 \in B^-;$$

$$\langle a_0, b \rangle \lor \langle a_1, b \rangle = \langle a_0 \lor a_1, b \rangle, \text{ for } a_0, a_1 \in A^-, b \in B^-$$

Lemma 1.55. If $A$ and $B$ are finite nontrivial lattices then $A \otimes B$ is a finite lattice since it is a finite $\{\lor, 0\}$ semilattice and $\text{Con } A \otimes \text{Con } B \cong \text{Con } (A \otimes B)$

Definition 1.56. Let $A$ and $B$ be lattices. For $(a, b) \in A \times B$, we define $a \Box b = \{(x, y) \in A \times B / x \leq a \text{ or } x \leq b\}$. The box product of $A$ and $B$ denoted by $A \Box B$ is defined as the set of all finite intersection of the form $H = \bigcap_{i < n} (a_i \Box b_i)$, where $n$ is a positive integer and $(a_i, b_i) \in A \times B$, for all $i < n$. $A \Box B$ is always a lattice.
**Definition 1.57.** For arbitrary lattices $A$ and $B$, define the lattice tensor product of $A$ and $B$ as \( \perp_{A,B} = (A \times \perp_B) \cup (\perp_A \times B) \), where for any lattice $L$, 
\[
\perp_L = \begin{cases} \{0_L\}, & \text{if } L \text{ has a zero;} \\
\emptyset, & \text{otherwise}
\end{cases}
\]

For \((a,b) \in A \times B\), define \(a \odot b = \{(x, y) \in A \times B / x \leq a \text{ and } y \leq b \}\), and \(a \boxtimes b = (a \odot b) \cup \perp_{A,B}\).

An element $H$ of $A \boxtimes B$ is confined if it is contained in $a \boxtimes b$ for some $(a, b) \in A \times B$, the lattice tensor product of $A$ and $B$ denoted by $A \boxtimes B$, is the ideal of $A \boxtimes B$.

If $A$ is bounded, then $A \boxtimes B$ is always a lattice.

**Definition 1.58.** Let $A$ and $B$ be finite nontrivial lattices. The tensor extension of $A$ by $B$ is defined by $A[B] = \{m(\alpha) / \alpha \in B^A\}$ where 
\[
m_a(\alpha) = \Lambda_{x \in \alpha} (A - \text{fil}(a)) \alpha = \Lambda_{x \in \alpha} x \alpha.
\]

**Lemma 1.59.** Let $A$ and $B$ be lattices. If $A$ is bounded, then the elements of $A \boxtimes B$ are exactly the finite intersection of the form $H = \cap (a_i \boxtimes b_i)$ subject to 
\[
\cap_{i < n} (a_i \boxtimes b_i) \land a_i = 0_A \quad \text{where } n > 0 \text{ and } (a_i, b_i) \in A \times B, \ i < n.
\]

For a finite simple lattice $A$, the construction $A \boxtimes B$ retains the most important property of the Boolean triple construction $M_3 \langle B \rangle$. Moreover
\[
M_3 \boxtimes B \equiv M_3 \langle B \rangle
\]
Lemma 1.60. Let \( A \) and \( B \) be lattices with 0. If either \( A \) or \( B \) is distributive, then the semilattice tensor product and the lattice tensor product coincide. That is, \( A \otimes B = A \times B \).

Remark 1.61. If \( L \) is a finite lattice and \( D \) is a finite distributive lattice, then by Schmidt \( Con L[\]D\] \( \cong (Con L)[Con D] \). In [29], it has been proved that the tensor extension \( A[B] \) of nontrivial finite lattices \( A \) and \( B \),
\[
Con A[B] \cong (Con A)[Con B]
\]

Theorem 1.62. Let \( A \) and \( B \) be nontrivial finite lattices. Then the following isomorphism holds: \( Con(A[B]) \cong (ConA)[ConB] \).

Definition 1.63. Let \( K \) and \( L \) be lattices. Let \( F \) be filter of \( K \) and let \( I \) be an ideal of \( L \). If \( F \) is isomorphic to \( I \) with \( \psi \) as the isomorphism, then we can form the gluing of \( K \) and \( L \) over \( F \) and \( I \) with respect to \( \psi \) defined as follows.

We form the disjoint union \( K \cup L \) and identify \( a \in F \) with \( a \psi \in I \) for all \( a \in F \), to obtain the set \( G \).

We order \( G \) as follows:

\[
a \leq b \text{ if and only if, } \begin{cases} a \leq_k b & \text{if } a, b \in K \\ a \leq_L b & \text{if } a, b \in L \\ a \leq_k x \text{ and } \psi(x) \leq_L b & \text{if } a \in K \text{ and } b \in L \text{ for some } x \in F \end{cases}
\]

Lemma 1.64. Let \( K, L, F, I \) and \( G \) be given as above. Then \( G \) is a lattice. The join in \( G \) is described by
and dually for the meet. If L has a zero, $0_L$, then the last clause for the join may be rephrased: $a \lor_G b = \psi(a \lor_K 0_L) \lor_L b$ if $a \in K$ and $b \in L$. G contains K and L as sublattices. In fact, K is an ideal and L is a filter of G.

**Lemma 1.65.** Let $K, L, F, I$ and $G$ be given as above. Let A be a lattice containing $K$ and $L$ as sublattices such that $F \cap L = I \cap F$. Then $K \cup L$ is a sublattice of A and it is isomorphic to G.

**Lemma 1.66.** A congruence $\theta$ of G can be uniquely written in the form $\theta = \theta_K \circ \theta_L$, where $\theta_K$ is a congruence of K and $\theta_L$ is a congruence of L satisfying the condition that $\theta_K$ restricted to F equals $\theta_L$ restricted to I (under the identification of elements by $\psi$). Conversely, if $\theta_K$ is a congruence of K and $\theta_L$ is a congruence of L satisfying the condition that $\theta_K$ restricted to F equals $\theta_L$ restricted to I, then $\theta = \theta_K \circ \theta_L$ is a congruence of G.

**Lemma 1.67.** If K and L are modular (distributive), so is gluing G of K and L.

**Lemma 1.68.** If K and L are finite semi modular lattices, then so is the gluing G of K and L.

**Lemma 1.69.** Let K and L be lattices, let $F$ be a filter of K and let $I$ be an ideal of L. Let $\psi$ be an isomorphism between F and I, let G be the gluing of K and L.
over $F$ and $I$ with respect to $\psi$. If $L$ is a congruence preserving extension of $I$, then $G$ is a congruence preserving extension of $K$.

**Corollary 1.70.** Let $K, L, F, I$ and $\psi$ be given as above. If $I$ and $L$ are simple lattices, then $G$ is a congruence preserving extension of $K$.

**Lemma 1.71.** Let $K$ and $L$ be lattices such that $K \leq L$. Then $re: \text{Con } L \to \text{Con } K$ is a $\{\land, 0,1\}$ homomorphism.

**Lemma 1.72.** Let $K$ and $L$ be lattices such that $K \leq L$. If $K$ is an ideal of $L$, then $re: \text{Con}(L) \to \text{Con}(K)$ is a $\{0,1\}$ homomorphism.

**Lemma 1.73.** Let $K$ and $L$ be lattices such that $K \leq L$. Then $ext: \text{Con } K \to \text{Con } L$ is a $\{0\}$ separating join-homomorphism.

**Lemma 1.74.** Let $L$ be a finite lattice and $K \leq L$. Then the following conditions are equivalent:

(i) $K$ is a congruence-reflecting sublattice of $L$.

(ii) $L$ is a congruence-reflecting extension of $K$.

(iii) Let $p$ and $q$ be prime intervals in $K$. If $p \Rightarrow q$ in $L$, then $p \Rightarrow q$ in $K$.

**Lemma 1.75.** Let the lattice $L$ be an extension of the lattice $K$. Then $L$ is a congruence preserving extension of $K$ if, and only if, the following two conditions hold:

(i) $re(ext \ \theta) = \theta$, for any congruence $\theta$ of $K$.

(ii) $ext(re \ \theta) = \theta$, for any congruence $\theta$ of $L$. 
Lemma 1.76. Let $L$ be a finite lattice and $K \leq L$. Then $L$ is a congruence preserving extension of $K$ if, and only if, the following two conditions hold:

(i) Let $p$ and $q$ be prime intervals in $K$. If $p \Rightarrow q$ in $L$, then $p \Rightarrow q$ in $K$.

(ii) Let $p$ a prime interval of $L$. Then there exist a prime interval $q$ in $K$ such that $p \Leftrightarrow q$ in $L$.

Lemma 1.77. Let $M$ be a chopped lattice and $\theta$ be an equivalence relation on $M$ satisfying the following two conditions, for $x, y, z \in M$:

(i) If $x \equiv y(\theta)$; then $x \land z \equiv y \land z(\theta)$.

(ii) If $x \equiv y(\theta)$; and $x \lor z$ and $y \lor z$ exist then $x \lor z \equiv y \lor z(\theta)$.

Then $\theta$ is a congruence relation on $M$.

Lemma 1.78. Suppose the lattice $L$ have a direct product decomposition: $L = L_1 \times L_2$, then $M_3[L] \equiv M_3[L_1] \times M_3[L_2]$.

Theorem 1.79. Let $D$ be a bounded distributive lattice. Then $M_3[D]$ (the set of balanced triples $\langle x, y, z \rangle \in D^3$) is a modular lattice. The map $\phi : x \rightarrow \langle x, 0, 0 \rangle \in M_3[D]$ is an embedding of $D$ into $M_3[D]$, and $M_3[D]$ is a congruence preserving extension of $D$. 

### Some standard lattices and their congruence lattice

<table>
<thead>
<tr>
<th>Lattice $L$</th>
<th>$\text{Con}(L)$</th>
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<tbody>
<tr>
<td>$C_3$</td>
<td>$\text{Con}(C_3)$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$\text{Con}(C_4)$</td>
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<tr>
<td>$B_2$</td>
<td>$\text{Con}(B_2)$</td>
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<tr>
<td>Diagram</td>
<td>Description</td>
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<td><img src="image" alt="B₃" /></td>
<td>$B_3$</td>
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<tr>
<td><img src="image" alt="Con(B₃)" /></td>
<td>Con($B_3$)</td>
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<td><img src="image" alt="M₃" /></td>
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<td><img src="image" alt="Con(M₄)" /></td>
<td>Con($M_4$)</td>
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</tbody>
</table>
Preliminaries

\[ C_1 + B_2 \]

\[ B_2 + C_1 \]

\[ N_5 \]

Con\( C_1 + B_2 \)

Con\( B_2 + C_1 \)

Con\( N_5 \)
Preliminaries

\[ \text{Con}(N_6) \]

\[ N_7 \]

\[ \text{Con}(N_7) \]

\[ S_8 \]

\[ \text{Con}(S_8) \]