Preface

This thesis embodies the work done by me under the guidance of Dr. E.K.R. Nagarajan.

In this thesis, we consider only finite nontrivial lattices.

In Chapter 1, we present some basic definitions and theorems on lattices which are used for the subsequent chapters. For Lattice theoretic terminology, we follow [2, 7, 9, 31].

The congruence lattice, Con(L) of any lattice L is a finite distributive lattice according to a result of N. Funayama and T. Nakayama [6]. The converse is a result of R.P. Dilworth from 1944, ‘Every finite distributive lattice D can be represented as the congruence lattice Con(L) of a finite lattice L. This result was first published in 1962 in the paper [19] of G. Gratzer and E.T. Schmidt. They have been publishing papers on this for the past 51 years and also in congruence preserving extension.

Let K be a lattice. A lattice L is said to be a congruence preserving extension of K if L is an extension of K (ie) $K \subseteq L$ and for every congruence $\theta$ of K there exist one congruence $\overline{\theta}$ of L.
satisfying $\bar{\theta} / K = \theta$. The map $\theta \mapsto \bar{\theta}$ is an isomorphism between $\text{Con}(K)$ and $\text{Con}(L)$.

In the second chapter, we study the congruence preserving extensions of all distributive lattices viz. chains and Boolean algebras. In 1968, E.T.Schmidt introduced $M_3[D]$ construction for finite distributive lattice $D$. $M_3[D]$ is a modular non distributive congruence preserving extension of a finite distributive lattice $D$. In the Boolean triple construction the elements of $M_3[D]$ belonging to $D^3$. As the number of elements in $M_3[D]$ are more it is difficult to draw the lattice $M_3[D]$ even for a small non trivial lattice $D$. In this chapter we give a new constructions for a bounded complemented lattice $D$, which are denoted by $\Delta_{B_n}$ for a Boolean algebra with $n$ atoms.

In 1998, G.Gratzer and F.Wehrung proved that $M_3(L)$ a Boolean triple construction is modular, every finite lattice $L$, $M_3(L) \subset M_3[L]$ is a congruence preserving extension of $L$. Also in [29], G. Gratzer and F.Wehrung proved that if $A$ is a simple lattice and $B$ is any finite lattice then the lattice tensor product $A \otimes B$ is a congruence preserving extension of the lattice $B$. 
From the above results, we observe that congruence preserving extension of a lattice is not unique. Also for every lattice $L$, the number of elements in $A \otimes B$ is very high. This motivated us to find a congruence preserving extension of a lattice which has less number of elements and which is easy to draw. First we consider a chain.

In section 2.2, we give different constructions for finite chains, which are all distributive and are congruence preserving extension of finite chains. Also we compare the number of elements in the construction.

Next we introduce a new concept called relative separator and using this concept we obtain a congruence preserving extension of a chain. This construction has minimum number of elements when we compare it to the previous constructions. We will denote the extension of a chain with $n$ elements as $C_{R_n}$.

Consider a chain with $n$ elements. For every chain with 3 elements in $C_n$, we introduce a separator and we call the resultant lattice as $C_{R_n}$. The lattice $C_{R_n}$ construction is isomorphic to the lattice $C_{R_n'}$. Also we discuss how this concept is related to the direct product.
In section 2.3, we give different constructions for finite distributive lattices in which some are modular and some are nonmodular congruence preserving extension of distributive lattices. Also we compare the number of elements in the construction. A finite chain (lattice) with one relative separator is the minimal distributive congruence preserving extension of $C_n$. In section 2.4 we construct a modular congruence preserving extension of all distributive lattices.

In chapter three, we study about congruence preserving extension of lattices by gluing. The gluing concept was introduced by of M. Hall and R.P. Dilworth [30] to prove that there exists a modular lattice that cannot be embedded in any complemented modular lattice. Followed by Hall and Dilworth, in [9] G.Gratzer studied about congruence preserving extension of lattices by gluing.

In [9], G.Gratzer has posed the problem,“Does every modular lattice have a proper, modular, congruence-preserving extension?” In [34], we give a positive solution to the above problem for finite cases.
G. Gratzer and et al. have studied about the congruence of gluing of lattices and congruence preserving extension of lattices. In [28], G. Gratzer and Wehrung have proved that $A \otimes B, A \langle B \rangle$ are congruence preserving extensions of $B$ if $A$ is a simple lattice. In section 3.1 of this chapter we have given a construction for a smaller proper modular congruence preserving extension of $A$ if $B$ is a simple lattice. In section 3.2 of this chapter we have given a construction for a smaller modular congruence preserving extension of $A$ if $B$ is not a simple lattice. In section 3.3 of this chapter we study non modular congruence preserving extensions of $A$ by gluing of a simple (non simple) lattice $B$. In section 3.4 we study the congruence of gluing of $A$ and $B$ where $B$ is a Boolean algebra, and give the characterization for $\text{Con}(G(A,B))$. In [29], G. Gratzer and F. Wehrung introduced the lattice tensor product and they proved that for a finite bounded lattice $A$ and $B$, the isomorphism $\text{Con} A \langle B \rangle \cong \left( \text{Con} A \right) \langle \text{Con} B \rangle$ which is a special case of a result of G. Gratzer and F. Wehrung and a generalization of a 1981 result of G. Gratzer, H. Lasker and R. W. Quackenbush. The result is “For finite lattices $A$ and $B$ the tensor product of congruence lattices is isomorphic to congruence lattice of the
tensor product, in formula, \( \text{Con}(A \otimes B) \cong (\text{Con}A) \otimes \text{Con}B \).

Followed by G. Gratzer and F. Wehrung [11,12,13], G. Gratzer and M. Greenberg studied about lattice tensor product of infinite lattices.

In chapter four, we study about an isoform lattices and isoform ortholattices. Also we discuss an isoform congruence preserving extension of some of the lattices. The congruence lattice of a finite lattice was characterized by a classical result of R.P. Dilworth as a finite distributive lattice \( D \). Many Papers were published improving this result by representing a finite distributive lattice \( D \) as the congruence lattice of a finite lattice \( L \) with additional properties.

In [24], Isoform lattices have been defined. A nontrivial congruence \( \theta \) of a lattice \( L \) is Isoform if any two congruence classes \( A \) and \( B \) of \( \theta \) are isomorphic and are of the same size. A lattice \( L \) is Isoform if all of its congruence are Isoform. In [24], it has been proved the theorem Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite Isoform lattice \( L \).

In [18] a much stronger result was proved by G. Gratzer, R.W. Quackenbush and E.T. Schmidt in 2004. “Every finite lattice
has a congruence preserving extension to a finite isoform lattice. They raised the problem whether this result can be extended to congruence - finite lattices, that is, to lattices with finitely many congruences. In section 4.2.1, we introduce the concept of n – Isoform lattices. A congruence \( \theta \neq \omega, \tau \) of a lattice \( L \) is said to be n – Isoform if any two congruence classes A and B of \( \theta \) are isomorphic and are of size n. That is \( A \cong B \) and \( |A| = |B| = n \). A lattice \( L \) is said to be n – Isoform, if all of its congruences \( \theta \neq \omega, \tau \) are n – Isoform. We observe that the lattice \( B_2 \) is 2–isoform and prove that for every \( n > 3 \), there exists a n – Isoform lattice.

In chapter five, we introduced the \{0,1\} pasting of lattice \( L \) and its dual lattice. For any lattice \( L \) we obtain an ortholattice extension of a lattice \( L \). In section 5.1 of this chapter we give examples only for some standard lattices. In section 5.2 of this chapter we find the congruence of pasting of lattices. Also we observe that every doubly 2-distributive lattice \( L \) has dual congruence preserving ortho extension of \( L \). In section 5.4, we discuss the congruence preserving extension of modular ortholattices.