Chapter 5

Dual congruence Preserving Ortho Extension of 2-Distributive Lattices

5.1 Introduction

Let $L$ be any bounded lattice with least element $0$ and greatest element 1. Let $L^\circ$ be the dual of $L$. Form the set $G$ by pasting the least element of $L$ with the least element of $L^\circ$ and the greatest element of $L$ with greatest element of $L^\circ$. This construction we call as the $\{0,1\}$ – pasting of $L$ and $L^\circ$ which is denoted by $P_{\{0,1\}}(L,L^\circ)$. In section 5.2, we give examples for standard lattices. Also in section 5.3, we find the congruence $s$ of ortholattices $P_{\{0,1\}}(L,L^\circ)$ and characterize the lattices $L$ for which congruence lattice of $L$ is isomorphic to congruence lattices of $P_{\{0,1\}}(L,L^\circ)$. In section 5.4, we observe that all irreducible modular ortholattice $K = MO_n$, $n > 1$ have an infinite number of proper congruence preserving extensions and also we observe that all reducible modular ortholattices have an infinite number of congruence preserving extensions.

In [9] Gratzer has raised the question “Does every lattice have proper modular congruence preserving extension? In [33] we have answered the question only for a finite lattices.

The above question gave rise to “Does every lattice have an ortholattice which is a proper congruence preserving extension of $L$?". 
In this chapter we construct an ortholattice \( P_{(0,1)}(L, L^\partial) \) for every lattice \( L \) and also we compute the congruences of \( P_{(0,1)}(L, L^\partial) \). Through out this construction we use the notation \( G \) for \( P_{(0,1)}(L, L^\partial) \).

**Definition 5.1.1 [9]:** Let \( n \geq 1 \) be an integer. A lattice \( L \) is \( n \)-distributive if for all \( x, y_1, y_2, \ldots, y_{n+1} \in L \),
\[
\bigwedge_{i=1}^{n+1} y_i = \bigwedge_{i=1}^{n+1} \big( x \lor \bigvee_{j \neq i} y_j \big) = \bigvee_{i=1}^{n+1} \big( x \land \bigwedge_{j \neq i} y_j \big).
\]

**Definition 5.1.2 [9]:** A lattice \( L \) is 1–distributive if and only if it is distributive and 2–distributive if and only if it satisfies the identity
\[
x \land (y_1 \lor y_2 \lor y_3) = (x \land (y_1 \lor y_2)) \lor (x \land (y_1 \lor y_3)) \lor (x \land (y_2 \lor y_3)).
\]
A lattice \( L \) is doubly 2-distributive if it satisfies the 2-distributive identity and its dual.

**Note 5.1.3 [9]:** The lattices \( M_3 \) and \( N_3 \) are doubly two distributive lattices.

**5.2 Construction of Ortholattices**

Let \( L \) be any bounded lattice with least element 0 and greatest element 1. Let \( L^\partial \) be the dual of \( L \). Form the set \( G \) by pasting the least element of \( L \) with the least element of \( L^\partial \) and the greatest element of \( L \) with greatest element of \( L^\partial \). This construction we call as the \( \{0,1\} \)–pasting of \( L \) and \( L^\partial \).

We define the partial order on \( G \) as the disjoint union of partial orders of \( L \) and \( L^\partial \). That is we define a relation \( \leq \) on \( G \) by \( a \leq b \) in \( G \) if, and only if, \( a, b \in L \) and \( a \leq b \) in \( L \) or \( a, b \in L^\partial \) and \( a \leq b \) in \( L^\partial \).
If \( a \in L \) and \( b \in L^\partial \), then \( a \) and \( b \) are not comparable. Then \( G \) is a partially ordered set with respect to \( \leq \).

**Lemma 5.2.1:** The partially ordered set \((G, \leq)\) is a lattice. The join in \( G \) is described by

\[
 a \lor_G b = \begin{cases} 
 a \lor_L b & \text{if } a, b \in L \\
 a \lor_{L^\partial} b & \text{if } a, b \in L^\partial \\
 1 & \text{if } a \in L, b \in L^\partial 
\end{cases}
\]

and the meet is defined as

\[
 a \land_G b = \begin{cases} 
 a \land_L b & \text{if } a, b \in L \\
 a \land_{L^\partial} b & \text{if } a, b \in L^\partial \\
 0 & \text{if } a \in L, b \in L^\partial 
\end{cases}
\]

**Proof:** Since \( \leq \) is a partial order on \( L \) and \( L^\partial, \leq \) is also a partial order on \( G \). By the join and meet formulae defined above \( G \) is a lattice.

The lattice \((G, \lor, \land, 0, 1)\) can be made into an ortho lattice as follows.

Let \( L \) be a lattice and let \( L^\partial \) be the dual lattice of \( L \). Then there exist duality maps

\[
 \phi : L \to L^\partial \quad \text{and} \quad \psi : L^\partial \to L.
\]

Define an orthocomplementation \( \perp \) on \( G \) by

\[
 x^\perp = \begin{cases} 
 \phi(x) & \text{if } x \in L \\
 \psi(x) & \text{if } x \in L^\partial.
\end{cases}
\]

**Theorem 5.2.2:** The lattice \((G, \leq)\) formed by pasting is an ortholattice \((G, \lor, \land, ^\perp, 0, 1)\) where \( ^\perp \) is an orthocomplementation on it.

**Proof:** Let \( a, b \in L \). Then \( \phi(a) = a^\perp \) and \( \psi(a^\perp) = a \) with \( a \lor \phi(a) = 1 \) and \( a \land \phi(a) = 0 \)

Let \( a \leq b \).
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Then \( a \leq b \Rightarrow a \leq_L b \)

\[ \Rightarrow \varphi(b) \leq_{L^\perp} \varphi(a) \]

\[ \Rightarrow b^\perp \leq_{L^\perp} a^\perp \]

\[ \Rightarrow b^\perp \leq a^\perp \]

\[ \therefore a \leq b \Rightarrow a^\perp \geq b^\perp. \]

Now \( \varphi(a) = a^\perp \)

\[ \Rightarrow \varphi(a)^\perp = a^{\perp\perp} \]

\[ \Rightarrow \psi(\varphi(a)) = a^{\perp\perp} \]

\[ \Rightarrow a^{\perp\perp} = I(a) \]

\[ \Rightarrow a^{\perp\perp} = a \]

Therefore \((G, \lor, \wedge, ^\perp, 0, 1)\) is an ortholattice.

**Example 5.2.3:** Let \( L \) be chain with 3 elements \( (C_3) \), and let \( L^\perp \) be the dual of \( L \). Then \( G \) is an ortholattice. The lattices \( L, L^\perp \) and \( G \) are given in figure 5.1

![Figure 5.1](image-url)
Example 5.2.4: For \( n=4 \), a chain with 4 elements say \( L=C_4 = \{0,a,b,1\} \) and its dual lattice \( L^0 = C_4 \). Then the lattice \( G \) obtained by pasting is an ortholattice. The lattices \( L, L^0 \) and \( G \) are given in figure 5.2.

![Figure 5.2](image)

Similarly all chains \( C_n \) \( (n>2) \) form an ortholattice.

Example 5.2.5: Let \( L = B_n, n > 1 \) be a Boolean algebra, its dual lattice \( L^0 = B_n, n > 1 \). \( \{0,1\} \) pasting of two lattices \( L \) and its dual lattice \( L^0 \), we get the lattice \( G \) which form an ortholattice. For \( n=2 \), \( L = B_2, L^0 = B_2 \) then \( G \) is \( \{0,1\} \) pasting of \( L \) and \( L^0 \) form an ortholattice. It satisfies the modular law. Hence pasting of two \( B_2 \) is a modular ortholattice. The lattices \( L, L^0 \) and \( G \) are given in figure 5.3

![Figure 5.3](image)
For $n=3$, let $L=B_3$ be a Boolean algebra with 3 atoms, its dual $L^\partial=B_3$. Pasting the two lattices $L$ and $L^\partial$ we get the lattice $G$ which is an ortholattice. The lattices $L$, $L^\partial$ and $G$ are given in Figure 5.4.

![Figure 5.4](image1)

**Example 5.2.6:** Let $L=N_5$ be a non modular lattice and its dual lattice $L^\partial=N_5$. Pasting the two lattices we get $G$. The resulting lattice $G$ is an ortholattice. The lattices $N_5$, dual of $N_5$ and $G$ are given in Figure 5.5.

![Figure 5.5](image2)
Example 5.2.7: Let $L = \mathbb{N}_6$, its dual lattice is $L^\partial$ and $G$ the $\{0,1\}$ pasting of $L$ and $L^\partial$ are given in figure 5.6. Then $G$ is an ortho lattice.

Example 5.2.8: Let $L = \mathbb{N}_7$ and its dual lattice $L^\partial$ are given in the figure 5.7.
Then $G$, the $\{0,1\}$ – pasting of $L$ and $L^\partial$, is an ortho lattice but neither a modular ortho lattice nor an ortho modular lattice.

**Figure 5.8**

**Example 5.2.9:** Let $L=S$ and its dual lattice $L^\partial$ are given in the figure 5.9.

**Figure 5.9**
Then $G$, the \( \{0,1\}\)-pasting of $L$ and $L^\partial$, is an ortho lattice only and is given in figure (5.10).

![Figure 5.10](image)

From the above construction we observe some results

**Lemma 5.2.10:** Let $L$ be a chain with three element. Then the \( \{0,1\}\) pasting of $L$ and its dual lattice is a Boolean algebra with two atoms.

Proof: Let $L$ be a chain with three element. That is $L=C_3$ and its dual lattice also $C_3$. Paste $L$ and its dual lattice $L^\partial$, we get the lattice $G$, which is given in figure 5.1, which is isomorphic to $B_2$. The lattice $B_2$ is a Boolean algebra with two atoms. Hence $G$ is a Boolean algebra.

**Note 5.2.11:** Every Boolean algebra is a modular ortholattice. As congruence lattice of $C_3$ is $B_2$, $G$ the \( \{0,1\}\) pasting of two copies of $C_3$ is $B_2$ and so $G$ is a congruence preserving extension of $C_3$. 

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Lemma 5.2.12: Let L be a chain with n element, n>3 then the \{0,1\} pasting of two copies of \( C_n \) is an ortho lattice.

As it has an isomorphic copy of \( O_6 \). For n=4, this is verified from figure 5.2.

Lemma 5.2.13: Let L be Boolean algebra with more than 3 atoms and its dual lattice is also Boolean algebra then the \{0,1\} pasting of L and its dual lattice is neither a modular lattice nor orthomodular lattice.

Proof: From example 5.2.5, we can easily prove that the \{0,1\} pasting of L and its dual lattice is neither a modular lattice nor orthomodular lattice.

Note 5.2.14: The \{0,1\} pasting of two distributive lattice is a distributive lattice if and only if L is a chain with three element.

Note 5.2.15: The \{0,1\} pasting of modular non distributive lattice \( L (=M_n) \) and its dual lattice \( L^\partial \) is a modular non distributive lattice \( G =MO_n \). Moreover it is a modular ortholattice.

Note 5.2.16: The \{0,1\} pasting of two distributive lattice is a modular lattice only for \( C_3, B_2 \) and \( M_n \).

Lemma 5.2.17: Let L be a modular non distributive lattice then the \{0,1\} pasting of L and its dual lattice \( L^\partial \) is a modular ortholattice.

Proof: Let L be a modular non distributive lattice say \( L=M_j (L^\partial =M_j) \)

Let G be the \{0,1\} pasting of L and \( L^\partial \). If for all \( a \leq b \) in L or \( a \leq b \) in \( L^\partial \) then \( a \leq b \) in G. For all \( c \in G \) then \( a \lor (c \land b) = (a \lor c) \land b \). Then G is a modular ortho lattice.
Note 5.2.18: Let L be a non modular lattice then G the \{0,1\} pasting of L and its dual \(L^\partial\) (non modular lattice) is an ortholattice which is not an OML / MOL. See 5.2.6, 5.2.7 and 5.2.8.

Note 5.2.19: The lattice G formed by the above construction is a modular ortho lattice if and only if L is a modular complemented lattice.

Remark 5.2.20: The number of elements of the \{0,1\} pasting of L and its dual lattice \(L^\partial\) given by \(|G| = |P_{\{0,1\}}(L, L^\partial)| = |L| + |L^\partial| - 2\).

Proof: The result is obvious from the above examples.

Theorem 5.2.21: The lattice G -\{0,1\} pasting of two modular non distributive lattice L (\(L^\partial\)) is a proper modular non distributive congruence preserving extension of L.

Proof: By Note 5.2.15, the lattice G is isomorphic to \(MO_n\). As \(MO_n \supseteq M_n\) for all n and we know that, \(\text{Con}(MO_n) \cong C_2\) and \(\text{Con}(M_n) \cong C_2\). Hence \(\text{Con}(MO_n) \cong \text{Con}(M_n)\)

Hence \(MO_n\) is a proper congruence preserving extension of \(M_n\). More over \(MO_n\) is a modular ortholattice. Hence every modular non distributive lattice L have a modular ortholattice and is a congruence preserving extension.

5.3 Congruence of \{0,1\} pasting of Lattices and their extension

Definition 5.3.1: Let \(\theta_L\) be the congruence of L and let \(\theta_L^\partial\) be the congruence of dual of L (\(L^\partial\)). Let us denote the \{(0),(1)\} separating, \{(0)\} separating \{(01)\} separating congruence of L by \(\delta_L\) and dually
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Let $\theta$ be the congruence of $\{0,1\}$ pasting of $L$ and dual of $L$, which is defined as $\theta = \delta_L \circ \delta_{\partial}$, the union of the reflexive product of every $\{(0),(1)\}$ separating congruence of $L$ with $\{(1)(0)\}$ separating congruence of dual of $L$, every $\{(0)\}$ separating congruence of $L$ with $\{(1)\}$ separating congruence of dual of $L$ and every $\{(01)\}$ separating congruence of $L$ with $\{(10)\}$ separating congruence of dual of $L$.

Example 5.3.2: Let $L$ be a chain with three elements then the $\{0,1\}$ pasting of $L$ and dual of $L$ denoted by $P_{\{0,1\}}(L,\delta^\partial)$ is $B_2$. The lattice $L$, $\delta^\partial$ and $P_{\{0,1\}}(L,\delta^\partial)$ are given in figure 5.1.

The congruences of $L$ are $\text{Con}L=\{(0)(a)(1)\}, \{(0)(a1)\}, \{(0a)(1)\}, \{(0a1)\}$ and the congruence of $L^\partial$ is $\text{Con}L^\partial=\{(0)(x)(1)\}, \{(0)(x1)\}, \{(0x)(1)\}, \{(0x1)\}$. Let $\delta_L$ be the $\{(0),(1)\}$ separating, $\{(0)\}$ separating, $\{(1)\}$ separating and $\{(01)\}$ separating congruence of $L$. Similar for $L^\partial$. Let $\delta_L=\{(0)(a)(1)\}, \{(0)(a1)\}, \{(0a)(1)\}, \{(0a1)\}=\text{Con}L$ similarly $\delta_{\partial}$. By this definition Congruence of $P_{\{0,1\}}(L,\delta^\partial)$ is obtained as

$$\begin{align*}
\omega &= \omega_L \circ \omega_{\partial} = \{(0)(a)(x)(1)\}, \\
\theta_1 &= \theta_{t_1} \circ \theta_{\partial_{t_2}} = \{(0a)(x1)\}, \\
\theta_2 &= \theta_{t_2} \circ \theta_{\partial_{t_1}} = \{(0x)(a1)\}, \\
\tau_1 &= \tau_{t_1} \circ \tau_{\partial_{t_2}} = \{(0ax1)\}.
\end{align*}$$
Therefore $\text{Con}(P_{(0,1)}(l,l^\omega)) = \{\omega, \theta_1, \theta_2, \tau\} \cong \text{Con}(B_2)$

Also $\text{Con}(B_2) \cong B_2$. Hence $\text{Con}(P_{(0,1)}(l,l^\omega)) \cong B_2 \cong \text{Con}(C_3)$.

**Example 5.3.3:** For $n=4$, $L=C_4 = \{0,a,b,1\}$ and its dual lattice $L^\omega = C_4$.

Then the lattice $P_{(0,1)}(l,l^\omega)$ is obtained by pasting the lattices $L$ and its dual lattice. The lattice $L$, $L^\omega$ and $P_{(0,1)}(l,l^\omega)$ are given in figure 5.2.

The congruences of $L$ are $\text{Con}L=\{(0)(a)(b)(1)\}$, $\{(0)(a)(b1)\}$, $\{(0)(ab)(1)\}$, $\{(0)(a)(b1)\}$, $\{(0)(ab)(1)\}$, $\{(0)(ab1)\}$.

Similar for $L^\omega$.

The $\{(0)(1)\}$ – separating congruence of $L$ are $\omega_L = \{(0)(a)(b)(1)\}$, $\theta_3 = \{(0)(ab)(1)\}$. The $\{(0)\}$ separating congruence of $L$ are $\theta_1 = \{(0)(a)(b1)\}$, $\theta_2 = \{(0)(ab1)\}$, the $(1)$ separating congruence of $L$ are $\theta_3 = \{(0a)(b)(1)\}$, $\theta_5 = \{(0)(ab)(1)\}$ and $\{(01)\}$ separating congruence of $L$ is $\tau_L = \{(0)(a)(b1)\}$. Among the eight congruence, the congruence $\theta_4 = \{(0)(a)(b1)\}$ is not separated by any one the above separating class.

The congruences of $P_{(0,1)}(l,l^\omega)$ are obtained by the reflexive product. Hence $\text{Con}G$ isomorphic to $\text{Con}O_6$. Ie, $\text{Con}G \cong \text{Con}O_6$.

Similarly all chains $C_n$ (n$>2$) form an ortholattice.
Example 5.3.4: Let $L = B_n, n > 1$ be a Boolean algebra, its dual lattice $L^\partial = B_n, n > 1$. $\{0,1\}$ pasting of two lattices, we get the lattice $G$ which form an ortholattice.

For $n=2$, $L = B_2, L^\partial = B_2$ then $G$ is $\{0,1\}$ pasting of $L$ and $L^\partial$ form an ortholattice. It satisfies the modular law. Hence pasting of two $B_2$ is a modular ortholattice.

The congruence of $G$ is trivial. As it is isomorphic to $MO_2$. Hence $\text{Con}G \cong C_2$

Example 5.3.5: Let $L = N_5$ be a non modular lattice and its dual lattice $L^\partial = N_5$. Pasting the two lattices we get $G$. The resulting lattice $G$ is an ortholattice. The congruence of $L = N_5$ is $\text{Con}(N_5) = \{\omega, \theta_1, \theta_2, \theta_3, \tau_1\}$ where $\theta_1 = \{(0b), (ad), (c)\}, \theta_2 = \{(0a), (b), (cd)\}, \theta_3 = \{(0a), (bcd)\}, \omega$ - the null congruence and $\tau_1$ - the all congruence. The congruence of $P_{\{0,1\}}\{l,l^\partial\}$ are $\text{Con}(P_{\{0,1\}}\{l,l^\partial\}) = \{\omega, \gamma_1, \gamma_2, \gamma_3, \tau\}$ where $\gamma_1 = \{(0), (a), (b), (c), (c'), (b', a'), (l)\}, \gamma_2 = \{(0), (a), (b), (c), (c'), (b'), (a'), (a), (l)\}, \gamma_3 = \{(0), (a), (b), (c), (c'), (b', a'), (l)\}, \omega$ - the null congruence and $\tau$ - the all congruence. The congruence lattice of $P_{\{0,1\}}\{l,l^\partial\}$ is given in figure 5.11.

![Figure 5.11](image-url)
The congruence lattice of \( P_{[0,1]}(l, l^\partial) \) is the dual of the congruence of \( L = N_5 \). They have the same number of congruences and they are dually isomorphic.

**Example 5.3.6:** Let \( L = N_6 \), its dual lattice is \( L^\partial \) and \( G \) be the \{0,1\} pasting of \( L \) and \( L^\partial \) are given in figure 5.6. Then \( G \) is an ortho lattice. Here \( \text{Con}(N_6) \cong C_3 \) and the congruence of extension of \( N_6 \) isomorphic to \( C_3 \). ie, \( \text{Con}(G) \cong C_3 \).

Using the above concept, we find the congruence of \( G \), which is a congruence preserving extension of \( L \).

That is, \( \text{Con}(N_6) \cong C_3 \cong \text{Con}(G) \)

**Theorem 5.3.7:** The \{0,1\} pasting of two non modular lattice is a non modular ortholattice and has a dual congruence preserving extension of non modular lattice.

Proof: By note 5.2.18, \( G \) is non modular ortholattice. From the above examples we observe that the congruences of \{0,1\} pasting of non modular lattice \( (L) \) and its dual lattice \( (L^\partial) \) is dual congruence preserving extension of \( L \).

**5.4 Congruence preserving extension of Modular ortholattices**

**Extension for \( MO_n \)'s 5.4.1:** Consider the modular ortholattice \( K = MO_2 \), the elements of \( MO_2 \) are \( MO_2 = \{0,a,b,c,d,1\} \) and the modular ortholattice
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\( K_1 = MO_3 \), whose elements are \( MO_3 = \{0, a, b, c, d, e, f, 1\} \). Then \( K \subset K_1 \).

The congruences of \( MO_2 \) are \( \omega = \{0, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{1\}\} \) the null congruence and \( \tau = \{0, a, b, c, d, 1\} \) the all congruence. The congruence of \( MO_3 \) are \( \omega = \{0, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{1\}\} \) the null congruence, \( \tau = \{0, a, b, c, d, e, f, 1\} \) the all congruence. Since \( K \subset K_1 \), \( K_1 \) is an extension of \( K \). \( Con(K) \equiv Con(K_1) \equiv C_2 \). Therefore \( K_1 \) is the congruence preserving extension of \( K \).

\[ \begin{array}{c}
\text{Figure 5.12} \\
\end{array} \]

In General \( MO_n \subset MO_{n+i} \) and \( Con(MO_n) \equiv Con(MO_{n+i}) \equiv C_2 \). Thus we have every irreducible modular ortholattice has a congruence preserving extension.

By varying \( i = 1, 2, 3, \ldots \) we get an infinite number of congruence preserving extension for \( MO_n \)'s. From the figure 5.12 given above, we easily seen that it is modular ortho extension of \( MO_n \).
Extensions for $\text{MO}_n \times C_2$ 5.4.2:

Consider the modular ortholattice $\text{MO}_2 \times C_2$, whose elements are $\text{MO}_2 \times C_2 = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$. That is, $G = \text{MO}_2 \times C_2$.

Let $G_1 = \text{MO}_3 \times C_2$. Then $G \subset G_1$ and the congruences of $G$ are

$c_0 = \{\{0\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{j\}, \{1\}\}$ - null congruence $\omega$, congruence $\tau = \{\{0, a, b, c, d, e, f, g, h, i, j, 1\}\}$ and two nontrivial congruences are $\theta_1 = \{\{0, c\}, \{a, j\}, \{b, i\}, \{d, g\}, \{e, f\}, \{h, 1\}\}$ and $\theta_2 = \{\{0, a, b, d, e, h\}, \{c, f, g, i, j, 1\}\}$. All the congruence classes of $\theta_1$, $\theta_2$ are isomorphic and are of same size. The congruences of $G_1$ are

$\text{Con}(G_1) = \{c_0, \theta_1, \theta_2, c_1\}$

where

$\omega = \{\{0\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{j\}, \{k\}, \{l\}, \{m\}, \{n\}, \{1\}\}$ - null congruence, $\tau = \{\{0, a, b, c, d, e, f, g, h, i, j, k, l, m, n, 1\}\}$ - all congruence and the nontrivial congruences are $\theta_1 = \{\{0, d\}, \{a, m\}, \{b, l\}, \{c, k\}, \{e, j\}, \{f, i\}, \{g, h\}, \{n, 1\}\}$ and $\theta_2 = \{\{0, a, b, c, e, f, g, n\}, \{d, h, i, j, k, l, m, 1\}\}$, whose congruence classes are same and are isomorphic.

Since $G \subset G_1$, $G_1$ is a congruence preserving extension of $G$. Also

$\text{Con}(G = \text{MO}_2 \times C_2) \cong \text{Con}(G_1 = \text{MO}_3 \times C_2) \cong B_2$ . Since $\text{Con}(\text{MO}_2) \cong C_2$ and $\text{Con}(C_2) \cong C_2$, direct product of $C_2$ and $C_2$ are $B_2$. Hence $\text{Con}(G) \cong \text{Con}(G_1) \cong B_2$. The lattice $B_2$ is naturally an isoform lattice. The lattice $G_1$ is an isoform congruence preserving extension of $B_2$.

In general $\text{MO}_n \times (C_2)^i \subset \text{MO}_{n+k} \times (C_2)^i$ for $k > 0$ and

$\text{Con}(\text{MO}_n \times (C_2)^i) \cong \text{Con}(\text{MO}_{n+k} \times (C_2)^i)$ for some $i > 0$.

The lattice $B_n \subset \text{MO}_n \times (C_2)^i \subset \text{MO}_{n+k} \times (C_2)^i$ for $i = n-1$.
Thus we have every reducible modular ortholattice of the form $MO_n \times C_2$ has a congruence preserving extension. By varying $k = 1, 2, 3, \ldots$ we get an infinite number of congruence preserving extension for $MO_n \times C_2$.

The lattice $G$ and $G_i$ are isoform modular ortholattice. Hence every Boolean algebra has an isoform modular ortholattice congruence preserving extension.

**Extension for $MO_n \times MO_n$’s 5.4.3:** Let $MO_n$ be a modular ortholattice then
the direct product $MO_n \times MO_n$ are also Modular ortholattice and which has an extension $MO_{n+1} \times MO_n$ and is a congruence preserving extension of $MO_n \times MO_n$.

That is, $MO_{n+1} \times MO_n \supset MO_n \times MO_n$

$$\text{Con} (MO_{n+1} \times MO_n) = \text{Con} (MO_{n+1}) \times \text{Con} (MO_n)$$

$$= C_2 \times C_2$$

$$= B_2$$

$$\text{Con} (MO_n \times MO_n) = \text{Con} (MO_n) \times \text{Con} (MO_n)$$

$$= C_2 \times C_2$$

$$= B_2$$

Therefore $MO_{n+1} \times MO_n$ is a proper congruence preserving extension of $MO_n \times MO_n$.

Hence we observe that all modular ortholattices have a congruence preserving extension.

**Extension for $B_n$’s 5.4.4:** Consider the Boolean algebra $B_n$ with $n$ atoms, which is subset of the reducible modular ortholattice $MO_2 \times (C_2)^{n-1}$.

That is, $MO_2 \times (C_2)^{n-1} \supset B_n$. 

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Now \( \text{Con}(MO_2 \times (C_2)^{n-1}) \equiv \text{Con}(MO_2) \times \text{Con}(C_2^{n-1}) \)

\[ \equiv C_2 \times \text{Con}(B_{n-1}) \]

\[ \equiv C_2 \times B_{n-1} \equiv B_n. \]

Also \( \text{Con}(B_n) \equiv B_n. \) Hence \( \text{Con}(MO_2 \times C_2^{n-1}) \equiv B_n \equiv \text{Con}(B_n). \) Therefore for all \( n, \) \( MO_2 \times (C_2)^{n-1} \) is a congruence preserving extension of \( B_n \) the Boolean algebra with \( n \) atoms.

**Example 5.4.5:** Consider the Boolean algebra \( B_2 \) with two atoms is a subset of the reducible modular ortholattice \( MO_2 \times C_2, \) which is shown in figure 5.14

![Figure 5.14](image-url)

\( \text{Con}(MO_2 \times C_2) \equiv \text{Con}(MO_2) \times \text{Con}(C_2) \equiv C_2 \times C_2 \equiv B_2 \) and \( \text{Con}(B_2) \equiv B_2. \)

Hence

\( MO_2 \times C_2 \) is modular ortholattice and is a congruence preserving extension of \( B_2. \)
Conclusion: In this chapter we have constructed an ortholattice for every lattice L. We observe that some of the lattices have congruence preserving extensions of L. We study the congruence of a {0,1} pasting of lattices and their congruence preserving extensions and observe that every 2 distributive lattice L has dual congruence preserving extension.