CHAPTER 2
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New Classes of Continuity

Continuity of functions is one of the interesting concept in Mathematics. Many mathematical concepts are represented by functions. The study of characterisations and generalisations of continuity has been done by means of several generalised closed sets. Topologists have introduced and studied strong and weak forms of continuous functions. Levine [42] and Noiri [56] introduced strongly and perfectly continuous functions which are stronger forms of continuous functions. Slightly continuity was introduced by Jain[35] and it has been developed by many others. Irresolute function is a strong form of continuous functions.

This chapter defines and investigates the properties of different continuous functions in terms of $\tilde{g}_\alpha$-sets. It also deals with the strong form of continuous functions namely slightly $\tilde{g}_\alpha$-continuous functions, $\tilde{g}_\alpha$-irresolute functions, strongly $\tilde{g}_\alpha$-continuous functions and perfectly $\tilde{g}_\alpha$-continuous functions and establishes the relationship between them. The behaviour of $\tilde{g}_\alpha$-compactness and $\tilde{g}_\alpha$-connectedness are discussed in terms of the different forms of continuous functions using $\tilde{g}_\alpha$-sets.
2.1 $\tilde{g}_\alpha$-continuous functions

In this section $\tilde{g}_\alpha$-continuous function is defined and its properties are discussed by comprehensively analysing its relationship with other continuous functions. Composition of two $\tilde{g}_\alpha$-continuous functions need not be $\tilde{g}_\alpha$-continuous. The requirements for the composition of two $\tilde{g}_\alpha$-continuous functions is taken care of.

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Definition 2.1.1 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\tilde{g}_\alpha$-continuous if $f^{-1}(V)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.

Example 2.1.2 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $Y = \{p, q\}$, $\sigma = \{\phi, Y, \{p\}\}$ $\tilde{G}_\alpha C(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. The function $f$ is defined as $f(a) = f(b) = p, f(c) = q$. The function $f$ is $\tilde{g}_\alpha$-continuous.

Theorem 2.1.3 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}_\alpha$-continuous if and only if $f^{-1}(U)$ is $\tilde{g}_\alpha$-open in $(X, \tau)$ for every open set $U$ in $(Y, \sigma)$.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $\tilde{g}_\alpha$-continuous and $U$ be an open set in $(Y, \sigma)$ then $f^{-1}(U^c)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$. Also $f^{-1}(U^c) = (f^{-1}(U))^c$ and so $f^{-1}(U)$ is $\tilde{g}_\alpha$-open in $(X, \tau)$. Conversely let $U$ be a closed set in $Y$ then $U^c$ is open in $(Y, \sigma)$. By hypothesis $f^{-1}(U^c)$ is $\tilde{g}_\alpha$-open in $(X, \tau)$. Again $f^{-1}(U^c) = (f^{-1}(U))^c$ Thus $f^{-1}(U)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$. Therefore $f$ is $\tilde{g}_\alpha$-continuous.

Theorem 2.1.4 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}_\alpha$-continuous if and only if $f : (X, \tau^{\tilde{g}_\alpha}) \rightarrow (Y, \sigma)$ is continuous where $\tau^{\tilde{g}_\alpha}$ represents the class of all $\tilde{g}_\alpha$-open sets.
Proof. Let \( f : (X, \tau) \to (Y, \sigma) \) be \( \tilde{g}_\alpha \)-continuous. i.e For any open set \( U \) in \( (Y, \sigma) \)
\( f^{-1}(U) \in \tau^{\tilde{g}_\alpha} \). Therefore \( f : (X, \tau^{\tilde{g}_\alpha}) \to (Y, \sigma) \) is continuous. Conversely assume that \( f : (X, \tau^{\tilde{g}_\alpha}) \to (Y, \sigma) \) is continuous. i.e For any open set \( U \) in \( (Y, \sigma) \), \( f^{-1}(U) \in \tau^{\tilde{g}_\alpha} \). i.e \( f^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \( (X, \tau) \). Thus \( f \) is \( \tilde{g}_\alpha \)-continuous.

Remark 2.1.5 The composition of two \( \tilde{g}_\alpha \)-continuous functions need not be \( \tilde{g}_\alpha \)-continuous as seen from the following example. To overcome the hurdle if composition of a \( \tilde{g}_\alpha \)-continuous function and a continuous function is considered then their composition is \( \tilde{g}_\alpha \)-continuous.

Example 2.1.6 Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}, \{b\}, \{a, c\}, \{a, b\}\} \), \( \sigma = \{\phi, Y, \{a, b\}\} \), \( \eta = \{\phi, Z, \{b\}\} \), \( \tilde{G}_\alpha C(X) = \{\phi, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\} \), \( \tilde{G}_\alpha C(Y) = \{\phi, Y, \{c\}, \{a, c\}, \{b, c\}\} \). The function \( f \) is defined as \( f(a) = a, f(b) = c, f(c) = b \) and \( g \) is the identity function. \( f : (X, \tau) \to (Y, \sigma) \), \( g : (Y, \sigma) \to (Z, \eta) \). Both \( f \) and \( g \) are \( \tilde{g}_\alpha \)-continuous. For the closed set \( \{a, c\} \) in \( (Z, \eta) \), \( (gof)^{-1}(\{a, c\}) = f^{-1}g^{-1}(\{a, c\}) = \{a, b\} \) which is not \( \tilde{g}_\alpha \)-closed in \( (X, \tau) \).
Hence \( gof \) is not \( \tilde{g}_\alpha \)-continuous.

Proposition 2.1.7 If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) are \( \tilde{g}_\alpha \)-continuous and \( (Y, \sigma) \) is a \#T_{\tilde{g}_\alpha} \)-space then their composition \( gof : (X, \tau) \to (Z, \eta) \) is \( \tilde{g}_\alpha \)-continuous.

Proof. Let \( G \) be any closed set in \( Z \) then \( g^{-1}(G) \) is closed in \( Y \). Since \( f \) is \( \tilde{g}_\alpha \)-continuous \( (gof)^{-1}(G) = f^{-1}g^{-1}(G) \) is \( \tilde{g}_\alpha \)-closed in \( (X, \tau) \). Thus \( gof \) is \( \tilde{g}_\alpha \)-continuous.

Proposition 2.1.8 If \( f : (X, \tau) \to (Y, \sigma) \) is \( \tilde{g}_\alpha \)-continuous and \( g : (Y, \sigma) \to (Z, \eta) \) is continuous then their composition \( gof : (X, \tau) \to (Z, \eta) \) is \( \tilde{g}_\alpha \)-continuous.

Proof. Let \( G \) be any closed set in \( Z \) then \( g^{-1}(G) \) is closed in \( Y \). Since \( f \) is \( \tilde{g}_\alpha \)-continuous \( (gof)^{-1}(G) = f^{-1}g^{-1}(G) \) is \( \tilde{g}_\alpha \)-closed in \( (X, \tau) \). Thus \( gof \) is \( \tilde{g}_\alpha \)-continuous.
Proposition 2.1.9 Let \((X, \tau), (Y, \sigma)\) and \((Z, \eta)\) be any three topological spaces. If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(\alpha\)-irresolute and \(g : (Y, \sigma) \rightarrow (Z, \eta)\) is \(\alpha\)-continuous, then their composition \(gof : (X, \tau) \rightarrow (Z, \sigma)\) is \(\tilde{g}_{\alpha}\)-continuous.

Proof. Let \(F\) be any closed set in \((Z, \eta)\). Since \(g : (Y, \sigma) \rightarrow (Z, \eta)\) is \(\alpha\)-continuous, \(g^{-1}(F)\) is \(\alpha\)-closed in \((Y, \sigma)\). Since \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(\alpha\)-irresolute \((gof)^{-1}(F) = f^{-1}(g^{-1}(F))\) is \(\alpha\)-closed which is \(\tilde{g}_{\alpha}\)-closed in \((X, \tau)\) and so \(gof\) is \(\tilde{g}_{\alpha}\)-continuous.

Remark 2.1.10 The following figure represents the composition of different functions. 1. \(\tilde{g}_{\alpha}\)-continuous 2. continuous 3. \(\alpha\)-continuous

Figure 3

Proposition 2.1.11 Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be \(\tilde{g}_{\alpha}\)-continuous and \(A\) be a \(\tilde{g}_{\alpha}\)-closed subset of \((X, \tau)\) then the restriction \(f/A : (A, \tau_A) \rightarrow (Y, \sigma)\) is \(\tilde{g}_{\alpha}\)-continuous.

Proof. Let \(G\) be any closed set in \((Y, \sigma)\). Since \(f\) is \(\tilde{g}_{\alpha}\)-continuous \(f^{-1}(G)\) is \(\tilde{g}_{\alpha}\)-closed in \((X, \tau)\). Let \(f^{-1}(G) \cap A = B\). We claim that \(B\) is \(\tilde{g}_{\alpha}\)-closed in \((A, \tau_A)\). Let \(U\) be any \#gs-open set of \((A, \tau_A)\) such that \(B \subseteq U\). Since \(U\) is \#gs-open set in \((A, \tau_A)\), \(U = F \cap A\) for some \#gs-open set \(F\) in \((X, \tau)\). Now \(B \subseteq F \cap A\) and hence \(B \subseteq F\). Since \(B\) is \(\tilde{g}_{\alpha}\)-closed in \((X, \tau)\), \(\alpha Cl(B) \subseteq F\), \(\alpha CL_A(B) = \alpha Cl(B) \cap A \subseteq F \cap A = U\) and thus \(B\) is \(\tilde{g}_{\alpha}\) closed in \((A, \tau_A)\) and hence \(f/A : (A, \tau_A) \rightarrow (Y, \sigma)\) is \(\tilde{g}_{\alpha}\)-continuous. (since \((f/A)^{-1}(G) = f^{-1}(G) \cap A = B\)
Theorem 2.1.12 Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function and arbitrary union of \( \tilde{g}_\alpha \)-open set is also \( \tilde{g}_\alpha \)-open in \( X \). Then the following statements are equivalent.

(i) The function \( f \) is \( \tilde{g}_\alpha \)-continuous.

(ii) The inverse of each open set in \((Y, \sigma)\) is \( \tilde{g}_\alpha \)-open in \((X, \tau)\).

(iii) For each point \( x \) in \((X, \tau)\) and each open set \( V \) in \((Y, \sigma)\) with \( f(x) \in V \), there is a \( \tilde{g}_\alpha \)-open set \( U \) in \((X, \tau)\) such that \( x \in U \), \( f(U) \subseteq V \).

(iv) For each \( x \) in \((X, \tau)\), the inverse of every neighbourhood of \( f(x) \) is a \( \tilde{g}_\alpha \)-neighbourhood of \( x \).

(v) For each \( x \) in \((X, \tau)\) and each neighbourhood \( N \) for \( f(x) \), there is a \( \tilde{g}_\alpha \)-neighbourhood \( V \) of \( x \) such that \( f(V) \subseteq N \).

(vi) For each subset \( A \) of \((X, \tau)\), \( f(\tilde{g}_\alpha \text{-Cl}(A)) \subseteq \text{Cl}(f(A)) \).

(vii) For each subset \( B \) of \((Y, \sigma)\), \( \tilde{g}_\alpha \text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\text{Cl}(B)) \).

Proof.

(i) \( \Leftrightarrow \) (ii) This follows from Theorem 2.1.3.

(i) \( \Leftrightarrow \) (iii) Suppose that (iii) holds and let \( V \) be an open set in \((Y, \sigma)\) and let \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and thus there exists a \( \tilde{g}_\alpha \)-open set \( U_x \) such that \( x \in U_x \) and \( f(U_x) \subseteq V \). Now, \( x \in U_x \subseteq f^{-1}(V) \) and \( f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x \). Since arbitrary union of \( \tilde{g}_\alpha \)-open sets is \( \tilde{g}_\alpha \)-open, \( f^{-1}(V) \) is \( \tilde{g}_\alpha \)-open in \((X, \tau)\) and therefore \( f \) is \( \tilde{g}_\alpha \)-continuous. Conversely, let \( f(x) \in V \) where \( V \) is open in \((Y, \sigma)\). Since \( f \) is \( \tilde{g}_\alpha \)-continuous, we have \( x \in f^{-1}(V) \in \tau^{\tilde{g}_\alpha} \).

Let \( U = f^{-1}(V) \). Then \( x \in U \) and \( f(U) \subseteq V \).

(ii) \( \Rightarrow \) (iv) For \( x \) in \((X, \tau)\), let \( N \) be a neighbourhood of \( f(x) \). Then there exists an open set \( U \) in \((Y, \sigma)\) such that \( f(x) \in U \subseteq N \). Consequently, \( f^{-1}(U) \) is a \( \tilde{g}_\alpha \)-open set in \((X, \tau)\) and \( x \in f^{-1}(U) \subseteq f^{-1}(N) \). Thus, \( f^{-1}(N) \) is a \( \tilde{g}_\alpha \)-neighbourhood of \( x \).
Proof. Conversely.

Example 2.1.14 Let $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a, b\}\}, \sigma = \{\phi, Y, \{a\}\}$
\[
\widetilde{G}_\alpha C(X) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}. \quad \text{The function } f \text{ is the identity function, } f \text{ is}
\]
\(\tilde{g}_\alpha\)-continuous but not continuous. Since \(f^{-1}(\{b, c\}) = \{b, c\}\) which is not closed in \((X, \tau)\).

**Proposition 2.1.15** Every \(\alpha\)-continuous function is \(\tilde{g}_\alpha\)-continuous but not conversely.

**Proof.** Let \(V\) be any closed set in \((Y, \sigma)\). Then \(f^{-1}(V)\) is \(\alpha\)-closed in \((X, \tau)\). Since every \(\alpha\)-closed set is \(\tilde{g}_\alpha\)-closed, \(f\) is \(\tilde{g}_\alpha\)-continuous.

**Example 2.1.16** Let \(X = Y = \{a, b, c\}\), \(\tau = \{\phi, X, \{a, b\}\}\), \(\sigma = \{\phi, Y, \{a\}\}\)
\(\tilde{G}_\alpha C(X) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}\). The function \(f\) is the identity function, \(f\) is \(\tilde{g}_\alpha\)-continuous but not \(\alpha\)-continuous. Since \(f^{-1}(\{b, c\}) = \{b, c\}\) which is not \(\alpha\)-closed in \((X, \tau)\).

**Proposition 2.1.17** Every \(\tilde{g}_\alpha\)-continuous function is \(\tilde{g}_\alpha\)-LC-continuous but not conversely.

**Proof.** Let \(V\) be any closed set in \((Y, \sigma)\). Then \(f^{-1}(V)\) is \(\tilde{g}_\alpha\)-closed in \((X, \tau)\). Since every \(\tilde{g}_\alpha\)-closed set is \(\tilde{g}_\alpha\)-LC-closed, \(f\) is \(\tilde{g}_\alpha\)-LC-continuous.

**Example 2.1.18** Let \(X = \{a, b, c\}\), \(\tau = \{\phi, X, \{a\}\}\), \(Y = \{p, q, r\}\), \(\sigma = \{\phi, Y, \{a\}\}, \{q\}, \{p, q\}, \{p, r\}\}\)
\(\tilde{G}_\alpha C(X) = \{\phi, X, \{c\}, \{b\}, \{b, c\}\}\). The function \(f\) is defined as \(f(a) = p, f(b) = q, f(c) = r\). Then \(f\) is \(\tilde{g}_\alpha\)-LC-continuous but not \(\tilde{g}_\alpha\)-continuous. Since \(f^{-1}(\{p, q\}) = \{a, b\}\) which is not \(\tilde{g}_\alpha\)-closed in \((X, \tau)\).

**Proposition 2.1.19** Every \(\tilde{g}_\alpha\)-continuous function is \(g\alpha\)-continuous but not conversely.

**Proof.** Let \(V\) be any closed set in \((Y, \sigma)\). Then \(f^{-1}(V)\) is \(\tilde{g}_\alpha\)-closed in \((X, \tau)\). Since every \(\tilde{g}_\alpha\)-closed set is \(g\alpha\)-closed, \(f\) is \(g\alpha\)-continuous.

**Example 2.1.20** Let \(X = Y = \{a, b, c\}\), \(\tau = \{\phi, X, \{a\}, \{b, c\}\}\), \(\sigma = \{\phi, Y, \{a\}\}\)
\(\tilde{G}_\alpha C(X) = \{\phi, X, \{a\}, \{b, c\}\}\). The function \(f\) is defined as \(f(a) = b, f(b) = c, f(c) = a\) \(f\) is \(g\alpha\)-continuous but not \(\tilde{g}_\alpha\)-continuous. Since \(f^{-1}(\{b, c\}) = \{a, b\}\) which is not \(\tilde{g}_\alpha\)-closed in \((X, \tau)\).
Proposition 2.1.21 Every $\tilde{g}_\alpha$-continuous function is $\alpha g$, $sg$ and $gs$-continuous but not conversely.

Proof. Let $V$ be any closed set in $(Y, \sigma)$. Then $f^{-1}(V)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$. Since every $\tilde{g}_\alpha$-closed set is $\alpha g$, $sg$ and $gs$ closed, $f$ is $\alpha g$, $sg$ and $gs$-continuous.

Example 2.1.22 In Example 2.1.20 $f^{-1}(\{b, c\}) = \{a, b\}$ which is not $\tilde{g}_\alpha$-closed but it is $\alpha g, gs$ and $sg$ closed.

Remark 2.1.23 $\tilde{g}_\alpha$-continuity is independent of $\omega$ and $g$-continuity.

Example 2.1.24 In Example 2.1.20 $f^{-1}(\{b, c\}) = \{a, b\}$ which is not $\tilde{g}_\alpha$-closed but it is $g$ and $\omega$-closed. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{p\}\}$ $\tilde{g}_\alpha C(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$, $GC(X) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$. The function $f$ is defined as $f(a) = f(c) = p, f(b) = q$ and $f^{-1}(\{q\}) = \{b\}$ is $\tilde{g}_\alpha$-closed and not $g, \omega$-closed in $(X, \tau)$.

Proposition 2.1.25 Any $\tilde{g}_\alpha$-continuous function is $(\Lambda, \tilde{g}_\alpha)$-continuous but not conversely.

Proof. Let $U$ be a closed set in $(Y, \sigma)$ then $f^{-1}(U)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$ which is $(\Lambda, \tilde{g}_\alpha)$-closed in $(X, \tau)$. Hence $f$ is $(\Lambda, \tilde{g}_\alpha)$-continuous.

Example 2.1.26 Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$, $\tau = \{\phi, X, \{a\}\}$, $\sigma = \{\phi, Y, \{c\}\}$. The function $f$ is the identity function. $(\Lambda, \tilde{g}_\alpha)O(X) = P(X), here f is (\Lambda, \tilde{g}_\alpha)$-continuous but it is not $\tilde{g}_\alpha$-continuous. Since $f^{-1}(\{c\}) = \{c\}$ is not $\tilde{g}_\alpha$-open in $(X, \tau)$.

Remark 2.1.27 From the above result we have the following diagram which gives the relationship between the different classes of continuous functions where $A \rightarrow B$ represents $A$ implies B, $A \longleftarrow B$ represents $A$ does not imply $B$ and $A \leftrightarrow B$ represents $A$ and $B$ are independent.

(1) Continuity (2) $\alpha$-continuity (3) $g\alpha$-continuity (4) $g\alpha$-continuity (5) $g\alpha$-continuity (6) $g\alpha$-continuity (7) $g\alpha$-continuity (8) $g\alpha$-continuity (9) $g\alpha$-continuity.
2.2 $\tilde{g}_\alpha$-irresolute functions

This section introduces strong forms of $\tilde{g}_\alpha$-continuous functions namely $\tilde{g}_\alpha$-irresolute function, strongly $\tilde{g}_\alpha$-continuous function and perfectly $\tilde{g}_\alpha$-continuous function and establishes the relationship between them. It also derives the properties of the functions. It is proved that composition of two $\tilde{g}_\alpha$-irresolute functions is also $\tilde{g}_\alpha$-irresolute.

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Definition 2.2.1 A function $f : (X, \tau) \to (Y, \sigma)$ is called a $\tilde{g}_\alpha$-irresolute function if the inverse image of every $\tilde{g}_\alpha$-closed set in $(Y, \sigma)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$.

Remark 2.2.2 The following examples show that the notion of irresolute functions and $\tilde{g}_\alpha$-irresolute functions are independent.

Example 2.2.3 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{c\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a, b\}\}$, $\tilde{G}_\alpha C(X) = \{\phi, X, \{c\}, \{a, b\}\}$, $\tilde{G}_\alpha C(Y) = \{\phi, Y, \{c\}, \{a, c\}, \{b, c\}\}$. The function $f$ is the identity function. Then $f$ is irresolute but not $\tilde{g}_\alpha$-irresolute. Since for the $\tilde{g}_\alpha$-closed set $\{a, c\}$ in $(Y, \sigma)$, $f^{-1}(\{a, c\}) = \{a, c\}$ which is not $\tilde{g}_\alpha$-closed in $(X, \tau)$.
Example 2.2.4 Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a, b\}\} \), 
\( \sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\} \), \( \tilde{G}_\alpha C(X) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\} \), 
\( \tilde{G}_\alpha C(Y) = \{\phi, Y, \{c\}, \{a, c\}, \{b, c\}\} \). The function \( f \) is the identity function. Then \( f \) is \( \tilde{g}_\alpha \)-irresolute but not irresolute. Since for the semi-closed set \( \{b\} \) in \( (Y, \sigma) \), 
\( f^{-1}(\{a, c\}) = \{a, c\} \) which is not semi-closed in \( (X, \tau) \).

Proposition 2.2.5 A map \( f : (X, \tau) \to (Y, \sigma) \) is \( \tilde{g}_\alpha \)-irresolute if and only if the inverse image of every \( \tilde{g}_\alpha \)-open set in \( (Y, \sigma) \) is \( \tilde{g}_\alpha \)-open in \( (X, \tau) \).

Proof. Let \( f \) be \( \tilde{g}_\alpha \)-irresolute. Let \( U \) be any \( \tilde{g}_\alpha \)-open set in \( (Y, \sigma) \) then \( f^{-1}(U^c) \) is \( \tilde{g}_\alpha \)-closed in \( (X, \tau) \). Since \( f^{-1}(U^c) = (f^{-1}(U))^c \). So \( f^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \( (X, \tau) \). Conversely let \( U \) be a \( \tilde{g}_\alpha \)-closed set in \( (Y, \sigma) \) then \( f^{-1}(U^c) \) is \( \tilde{g}_\alpha \)-open in \( (X, \tau) \). Since \( f^{-1}(U^c) = (f^{-1}(U))^c \). So \( f \) is \( \tilde{g}_\alpha \)-irresolute.

Proposition 2.2.6 If the map \( f : (X, \tau) \to (Y, \sigma) \) is \( \tilde{g}_\alpha \)-irresolute, then it is \( \tilde{g}_\alpha \)-continuous.

Proof. Let \( U \) be any open set in \( (Y, \sigma) \). Since any open set is \( \tilde{g}_\alpha \)-open and \( f \) is \( \tilde{g}_\alpha \)-irresolute \( f^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \( (X, \tau) \). Therefore \( f \) is \( \tilde{g}_\alpha \)-continuous.

Remark 2.2.7 The converse of Proposition 2.2.6 need not be true as seen from the following example.

Example 2.2.8 Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a, b\}\} \), \( \sigma = \{\phi, Y, \{a, b\}\} \) 
\( \tilde{G}_\alpha C(X) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\} \), \( \tilde{G}_\alpha C(Y) = \{\phi, Y, \{c\}, \{a, c\}, \{b, c\}\} \). The function \( f \) is the identity function. Then \( f \) is \( \tilde{g}_\alpha \)-continuous but not \( \tilde{g}_\alpha \)-irresolute. Since for the \( \tilde{g}_\alpha \)-closed set \( \{a, c\} \) in \( (Y, \sigma) \), 
\( f^{-1}(\{a, c\}) = \{a, c\} \) which is not \( \tilde{g}_\alpha \)-closed in \( (X, \tau) \).

Proposition 2.2.9 If \( f : (X, \tau) \to (Y, \sigma) \) is bijective, pre-\# gs-open and \( \alpha \)-irresolute, then \( f \) is \( \tilde{g}_\alpha \)-irresolute.
Proof. Let $A$ be a $\tilde{g}_\alpha$-closed set in $(Y, \sigma)$. Let $U$ be any $\#gs$-open set in $(X, \tau)$ such that $f^{-1}(A) \subseteq U$. Then $A \subseteq f(U)$. Since $A$ is $\tilde{g}_\alpha$-closed and $f(U)$ is $\#gs$-open in $(Y, \sigma)$, $\alpha Cl(A) \subseteq f(U)$ holds and hence $f^{-1}(\alpha Cl(A)) \subseteq U$. Since $f$ is $\alpha$-irresolute and $\alpha Cl(A)$ is $\alpha$-closed in $(Y, \sigma)$, $\alpha Cl(f^{-1}(\alpha Cl(A))) \subseteq U$ and so $\alpha Cl(f^{-1}(A)) \subseteq U$ (since $\alpha Cl(f^{-1}(\alpha Cl(A))) = f^{-1}(\alpha Cl(A))$). Therefore, $f^{-1}(A)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$ and hence $f$ is $\tilde{g}_\alpha$-irresolute.

**Proposition 2.2.10** If $f : (X, \tau) \to (Y, \sigma)$ is bijective, pre $\alpha$-closed and $\#gs$-irresolute, then the inverse map $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\tilde{g}_\alpha$-irresolute.

Proof. Let $A$ be a $\tilde{g}_\alpha$-closed in $(X, \tau)$. Let $(f^{-1})^{-1}(A) = f(A) \subseteq U$ where $U$ is $\#gs$-open in $(Y, \sigma)$. Then $A \subseteq f^{-1}(U)$ holds. Since $f^{-1}(U)$ is $\#gs$-open in $(X, \tau)$ and $A$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$, $\alpha Cl(A) \subseteq f^{-1}(U)$ and hence $f(\alpha Cl(A)) \subseteq U$. Since $f$ is pre-$\alpha$-closed and $\alpha Cl(A)$ is $\alpha$-closed in $(X, \tau)$, $f(\alpha Cl(A))$ is $\alpha$-closed in $(Y, \sigma)$, we have $\alpha Cl(f(\alpha Cl(A))) \subseteq U$ and hence $\alpha Cl(f(A)) \subseteq U$. Thus, $f(A)$ is $\tilde{g}_\alpha$-closed in $(Y, \sigma)$ and so $f^{-1}$ is $\tilde{g}_\alpha$-irresolute.

**Proposition 2.2.11** Let $(X, \tau)$ be any topological space and $(Y, \sigma)$ be a $\#T_{\tilde{g}_\alpha}$ space and $f : (X, \tau) \to (Y, \sigma)$ be a map. Then the following are equivalent:

(i) $f$ is $\tilde{g}_\alpha$-irresolute.

(ii) $f$ is $\tilde{g}_\alpha$-continuous.

Proof.

(i) $\Rightarrow$ (ii). Let $f$ be $\tilde{g}_\alpha$-irresolute. Let $U$ be a closed set in $(Y, \sigma)$. Since $Y$ is a $\#T_{\tilde{g}_\alpha}$-space $U$ is $\tilde{g}_\alpha$-closed in $Y$. Since $f$ is $\tilde{g}_\alpha$-irresolute $f^{-1}(U)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$. Thus $f$ is $\tilde{g}_\alpha$-continuous.

(ii) $\Rightarrow$ (ii) Let $F$ be a $\tilde{g}_\alpha$-closed set in $(Y, \sigma)$. Since $Y$ is a $\#T_{\tilde{g}_\alpha}$-space $F$ is closed in $Y$. By hypothesis $f^{-1}(F)$ is $\tilde{g}_\alpha$-closed in $(X, \tau)$. Therefore $f$ is $\tilde{g}_\alpha$-irresolute.
Definition 2.2.12 A map \( f : (X, \tau) \to (Y, \sigma) \) is called strongly \( \tilde{g}_\alpha \)-continuous if the inverse image of every \( \tilde{g}_\alpha \)-open set in \( (Y, \sigma) \) is open in \( (X, \tau) \).

Proposition 2.2.13 If \( f : (X, \tau) \to (Y, \sigma) \) is strongly \( \tilde{g}_\alpha \)-continuous then it is continuous.

Proof. Let \( U \) be any open set in \( (Y, \sigma) \). Since every open set is \( \tilde{g}_\alpha \)-open \( U \) is \( \tilde{g}_\alpha \)-open in \( (Y, \sigma) \). Then \( f^{-1}(U) \) is open in \( (X, \tau) \). Hence \( f \) is continuous.

Remark 2.2.14 The converse of Proposition 2.2.13 need not be true as seen from the following example.

Example 2.2.15 Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}\}, \sigma = \{\phi, Y, \{c\}\} \). The function \( f \) is defined as \( f(a) = c, f(b) = a, f(c) = b \). The function \( f \) is continuous but \( f \) is not strongly \( \tilde{g}_\alpha \)-continuous. Since for the \( \tilde{g}_\alpha \)-open set \( \{b, c\} \) in \( (Y, \sigma) \), \( f^{-1}(\{b, c\}) = \{a, c\} \) which is not open in \( (X, \tau) \).

Proposition 2.2.16 Let \( (X, \tau) \) be any topological space and \( (Y, \sigma) \) be a \#T_{\tilde{g}_\alpha} \)-space and \( f : (X, \tau) \to (Y, \sigma) \) be a map. Then the following are equivalent:

\( (i) \) \( f \) is strongly \( \tilde{g}_\alpha \)-continuous.

\( (ii) \) \( f \) is continuous.

Proof. \( (i) \Rightarrow (ii) \) Follows from Proposition 2.2.13.

\( (ii) \Rightarrow (i) \) Let \( U \) be any \( \tilde{g}_\alpha \)-open set in \( (Y, \sigma) \). Since \( (Y, \sigma) \) is a \#T_{\tilde{g}_\alpha} \)-space, \( U \) is open in \( (Y, \sigma) \) and since \( f \) is continuous, we have \( f^{-1}(U) \) is open in \( (X, \tau) \). Therefore \( f \) is strongly \( \tilde{g}_\alpha \)-continuous.

Proposition 2.2.17 If \( f : (X, \tau) \to (Y, \sigma) \) is strongly continuous, then it is strongly \( \tilde{g}_\alpha \)-continuous.

Proof. Let \( U \) be a \( \tilde{g}_\alpha \)-open set in \( (Y, \sigma) \). Then \( f^{-1}(U) \) is both open and closed in \( (X, \tau) \). Hence \( f \) is strongly \( \tilde{g}_\alpha \)-continuous.
Remark 2.2.18  The converse of Proposition 2.2.17 need not be true as seen from the following example.

Example 2.2.19 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a, b\}\}$ $\tilde{G}_aO(Y) = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$. The function $f$ is the identity function. The function $f$ is strongly $\tilde{g}_a$-continuous but not strongly continuous. Since for the subset $\{a, b\}$ in $(Y, \sigma)$, $f^{-1}(\{a, b\}) = \{a, b\}$ which is not closed in $(X, \tau)$

Proposition 2.2.20 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map and both $(X, \tau)$ and $(Y, \sigma)$ are $T_{\tilde{g}_a}$-spaces. Then the following are equivalent:

(i) $f$ is strongly $\tilde{g}_a$-continuous.

(ii) $f$ is continuous.

(iii) $f$ is $\tilde{g}_a$-irresolute.

(iv) $f$ is $\tilde{g}_a$-continuous.

Proof. Follows from Propositions 2.2.11 and 2.2.16.

Proposition 2.2.21 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is strongly $\tilde{g}_a$-continuous if and only if the inverse image of every $\tilde{g}_a$-closed set in $(Y, \sigma)$ is closed in $(X, \sigma)$.

Proof. Since $f^{-1}(F^c) = (F^{-1}(F))^c$ where F is a $\tilde{g}_a$-closed set in $(Y, \sigma)$, proof is similar to Proposition 2.2.5.

Proposition 2.2.22 If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are strongly $\tilde{g}_a$-continuous, then their composition $gof : (X, \tau) \rightarrow (Z, \eta)$ is also strongly $\tilde{g}_a$-continuous.

Proof. Let $U$ be a $\tilde{g}_a$-open set in $(Z, \eta)$. Since $g$ is strongly $\tilde{g}_a$-continuous, $g^{-1}(U)$ is open in $(Y, \sigma)$. Since $g^{-1}(U)$ is open, it is $\tilde{g}_a$-open in $(Y, \sigma)$. As $f$ is also strongly $\tilde{g}_a$-continuous, $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)$ is open in $(X, \tau)$ and so $gof$ is strongly $\tilde{g}_a$-continuous.
Proposition 2.2.23  Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two functions. Then their composition \( gof : (X, \tau) \to (Z, \eta) \) is

(i) strongly \( \tilde{g}_\alpha \)-continuous if \( g \) is strongly \( \tilde{g}_\alpha \)-continuous and \( f \) is continuous.

(ii) \( \tilde{g}_\alpha \)-irresolute if \( g \) is strongly \( \tilde{g}_\alpha \)-continuous and \( f \) is \( \tilde{g}_\alpha \)-continuous. (or \( f \) is \( \tilde{g}_\alpha \)-irresolute)

(iii) continuous if \( g \) is \( \tilde{g}_\alpha \)-continuous and \( f \) is strongly \( \tilde{g}_\alpha \)-continuous.

Proof.

(i) Let \( U \) be a \( \tilde{g}_\alpha \)-open set in \((Z, \eta)\). Since \( g \) is strongly \( \tilde{g}_\alpha \)-continuous, \( g^{-1}(U) \) is open in \((Y, \sigma)\). Since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is open in \((X, \tau)\) and so \( gof \) is strongly \( \tilde{g}_\alpha \)-continuous.

(ii) Let \( U \) be a \( \tilde{g}_\alpha \)-open set in \((Z, \eta)\). Since \( g \) is strongly \( \tilde{g}_\alpha \)-continuous, \( g^{-1}(U) \) is open in \((Y, \sigma)\). As \( f \) is \( \tilde{g}_\alpha \)-continuous, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \((X, \tau)\) and so \( gof \) is \( \tilde{g}_\alpha \)-irresolute.

(iii) Let \( U \) be an open set in \((Z, \eta)\). Since \( g \) is \( \tilde{g}_\alpha \)-continuous, \( g^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \((Y, \sigma)\). As \( f \) is also strongly \( \tilde{g}_\alpha \)-continuous, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is open in \((X, \tau)\) and so \( gof \) is continuous.

Definition 2.2.24  A map \( f : (X, \tau) \to (Y, \sigma) \) is called perfectly \( \tilde{g}_\alpha \)-continuous if the inverse image of every \( \tilde{g}_\alpha \)-open set in \((Y, \sigma)\) is both open and closed in \((X, \tau)\).

Proposition 2.2.25  If \( f : (X, \tau) \to (Y, \sigma) \) is perfectly \( \tilde{g}_\alpha \)-continuous, then it is strongly \( \tilde{g}_\alpha \)-continuous.

Proof. Since \( f : (X, \tau) \to (Y, \sigma) \) is perfectly \( \tilde{g}_\alpha \)-continuous, \( f^{-1}(U) \) is both open and closed in \((X, \tau)\) for every \( \tilde{g}_\alpha \)-open set \( U \) in \((Y, \sigma)\). Therefore, \( f \) is strongly \( \tilde{g}_\alpha \)-continuous.

Remark 2.2.26  The converse of Proposition 2.2.25 need not be true as seen from the following example.
Example 2.2.27 Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{\phi, Y, \{a, b\}\}\). \( \bar{G}_aO(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, \bar{G}_aO(Y) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}\). The function \( f \) is the identity function. The function \( f \) is strongly \( \bar{g}_a \)-continuous but not perfectly \( \bar{g}_a \)-continuous. Since for the \( \bar{g}_a \)-open set \( \{a\} \) in \((Y, \sigma)\), \( f^{-1}\{\{a\}\} = \{a\} \) which is not closed in \((X, \tau)\).

Proposition 2.2.28 If \( f : (X, \tau) \to (Y, \sigma) \) is strongly continuous, then it is perfectly \( \bar{g}_a \)-continuous.

Proof. Since \( f : (X, \tau) \to (Y, \sigma) \) is strongly continuous, \( f^{-1}(U) \) is both open and closed in \((X, \tau)\), for every \( \bar{g}_a \)-open set \( U \) in \((Y, \sigma)\). Therefore, \( f \) is perfectly \( \bar{g}_a \)-continuous.

Remark 2.2.29 The converse of Proposition 2.2.28 need not be true as seen from the following example.

Example 2.2.30 Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}\) = \( \sigma \), \( \bar{G}_aO(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} = \bar{G}_aO(Y)\). The function \( f \) is the identity function then \( f \) is perfectly \( \bar{g}_a \)-continuous but not strongly-continuous. Since for the subset \( \{a, b\} \) in \((Y, \sigma)\), \( f^{-1}\{\{a, b\}\} = \{a, b\} \) which is not closed or open in \((X, \tau)\).

Proposition 2.2.31 Let \( (X, \tau) \) be a discrete topological space, \((Y, \sigma)\) be any topological space and \( f : (X, \tau) \to (Y, \sigma) \) be a map. Then the following statements are equivalent:

(i) \( f \) is perfectly \( \bar{g}_a \)-continuous.

(ii) \( f \) is strongly \( \bar{g}_a \)-continuous.

Proof. (i) \( \Rightarrow \) (ii) Let \( U \) be any \( \bar{g}_a \)-open set in \((Y, \sigma)\) then \( f^{-1}(U) \) is both open and closed in \((X, \tau)\).

(ii) \( \Rightarrow \) (i) Let \( U \) be any \( \bar{g}_a \)-open set in \((Y, \sigma)\). By hypothesis \( f^{-1}(U) \) is open in \((X, \tau)\). Since \((X, \tau)\) is a discrete space, \( f^{-1}(U) \) is also closed in \((X, \tau)\). \( f^{-1}(U) \) is both open and closed in \((X, \tau)\) and so \( f \) is perfectly \( \bar{g}_a \)-continuous.
Proposition 2.2.32 A map \( f : (X, \tau) \to (Y, \sigma) \) is perfectly \( \tilde{g}_\alpha \)-continuous if and only if the inverse image of every \( \tilde{g}_\alpha \)-closed set in \( (Y, \sigma) \) is both open and closed in \( (X, \tau) \).

Proof. Let \( U \) be any \( \tilde{g}_\alpha \)-closed set in \( (Y, \sigma) \), since \( f \) is perfectly \( \tilde{g}_\alpha \)-continuous then \( f^{-1}(U^c) \) is both open and closed in \( (Y, \sigma) \). Since \( f^{-1}(U^c) = (f^{-1}(U))^c \), \( f^{-1}(U) \) is both open and closed in \( (X, \tau) \).

Conversely let \( U \) be any \( \tilde{g}_\alpha \)-open set in \( (Y, \sigma) \). Since \( g^{-1}(U^c) = (g^{-1}(U))^c \), \( g^{-1}(U) \) is both and closed in \( (X, \tau) \). Hence \( f \) is perfectly \( \tilde{g}_\alpha \)-continuous.

Proposition 2.2.33 If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) are perfectly \( \tilde{g}_\alpha \)-continuous, then their composition \( gof : (X, \tau) \to (Z, \eta) \) is also perfectly \( \tilde{g}_\alpha \)-continuous.

Proof. Let \( U \) be any \( \tilde{g}_\alpha \)-open set in \( (Z, \eta) \) then \( g^{-1}(U) \) is both open and closed in \( (Y, \sigma) \). Since any open (closed) set is \( \tilde{g}_\alpha \)-open(\( \tilde{g}_\alpha \)-closed) set in \( (X, \tau) \) and \( f \) is perfectly \( \tilde{g}_\alpha \)-continuous. \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is both and closed in \( (X, \tau) \).

Hence \( gof \) is perfectly continuous.

Proposition 2.2.34 Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two functions. Then their composition \( gof : (X, \tau) \to (Z, \eta) \) is

(i) \( \tilde{g}_\alpha \)-continuous if \( g \) is strongly continuous and \( f \) is \( \tilde{g}_\alpha \)-continuous.

(ii) \( \tilde{g}_\alpha \)-irresolute if \( g \) is perfectly \( \tilde{g}_\alpha \)-continuous and \( f \) is \( \tilde{g}_\alpha \)-continuous. (or \( f \) is \( \tilde{g}_\alpha \)-irresolute)

(iii) strongly \( \tilde{g}_\alpha \)-continuous if \( g \) is perfectly \( \tilde{g}_\alpha \)-continuous and \( f \) is continuous (or \( f \) is strongly-continuous).

(iv) perfectly \( \tilde{g}_\alpha \)-continuous if \( g \) is strongly continuous and \( f \) is perfectly \( \tilde{g}_\alpha \)-continuous.
Proof.

(i) Let $U$ be any open set in $(Z, \eta)$ then $g^{-1}(U)$ is both open and closed in $(Y, \sigma)$. Hence $f^{-1}(g^{-1}(U))$ is $\tilde{g}_\alpha$-open in $(X, \tau)$. Thus $gof$ is $\tilde{g}_\alpha$-continuous.

(ii) Let $U$ be any $\tilde{g}_\alpha$-open set in $(Z, \eta)$ then $g^{-1}(U)$ is open or closed in $(Y, \sigma)$. Since $f$ is $\tilde{g}_\alpha$-continuous $f^{-1}(g^{-1}(U))$ is $\tilde{g}_\alpha$-closed or $\tilde{g}_\alpha$-open in $(X, \tau)$. Thus $gof$ is $\tilde{g}_\alpha$-irresolute.

(iii) Let $U$ be any $\tilde{g}_\alpha$-open set in $(Z, \eta)$. Then $g^{-1}(U)$ is both open and closed in $(Y, \sigma)$ and hence $f^{-1}(g^{-1}(U))$ is open and closed in $(X, \tau)$ and hence $gof$ is strongly $\tilde{g}_\alpha$-continuous.

(iv) Let $U$ be any $\tilde{g}_\alpha$-open set in $(Z, \eta)$ then $g^{-1}(U)$ is open and closed in $(Y, \sigma)$ which is $\tilde{g}_\alpha$-open in $(Y, \sigma)$ then $f^{-1}(g^{-1}(U))$ is both open and closed in $(X, \tau)$. Hence $gof$ is perfectly $\tilde{g}_\alpha$-continuous.

**Remark 2.2.35** From the above discussions we have the following table which gives the relationship between different types of continuous functions. The symbol “1” in a cell means that a function on the corresponding row implies a function on the corresponding column. The symbol “0” means that a function on the corresponding row does not imply a function on the corresponding column.

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Table 1
2.3 Slightly $\tilde{g}_{\alpha}$-continuous functions

Slightly continuous function concept introduced by Jain[35] in 1980 helped to investigate various properties of topological spaces. In this section slightly $\tilde{g}_{\alpha}$-continuous function which is a weaker form of slightly continuous function is defined, the relationship between other existing functions and its properties are derived.

This section in its entirety has been published in International Journal of Mathematical Sciences and Applications, Vol1, No.3, September (2011),1455,1461.

Definition 2.3.1 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called slightly $\tilde{g}_{\alpha}$-continuous if the inverse image of every clopen set in $(Y, \sigma)$ is $\tilde{g}_{\alpha}$-open in $(X, \tau)$.

Example 2.3.2 Let $X = \{a, b, c\} = Y, \tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b\}\}$
$\sigma = \{\phi, Y, \{a\}, \{b, c\}\}$ $\tilde{G}_{\alpha}O(X) = \{\phi, X, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$. The function $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined as $f(a) = c, f(b) = a, f(c) = b$. The function $f$ is slightly $\tilde{g}_{\alpha}$-continuous.

Theorem 2.3.3 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function then the following are equivalent.

(i) $f$ is slightly $\tilde{g}_{\alpha}$-continuous.

(ii) the inverse image of every clopen set $V$ of $(Y, \sigma)$ is $\tilde{g}_{\alpha}$-closed in $(X, \tau)$.

(iii) the inverse image of every clopen set $V$ of $(Y, \sigma)$ is $\tilde{g}_{\alpha}$-clopen in $(X, \tau)$.

Proof.

(i) $\Rightarrow$ (ii) Let $V$ be clopen in $Y \Rightarrow V^c$ is clopen in $Y$. By (i) $f^{-1}(V^c)$ is $\tilde{g}_{\alpha}$-open in $X$. Since $(f^{-1}(V))^c = f^{-1}(V^c)$, it follows that $f^{-1}(V)$ is $\tilde{g}_{\alpha}$-closed.

(ii) $\Rightarrow$ (iii) By (i) and (ii) $f^{-1}(V)$ is $\tilde{g}_{\alpha}$-clopen in $(X, \tau)$.
(iii) ⇒ (i) Let \( V \) be a clopen subset of \((Y, \sigma)\). By (iii) \( f^{-1}(V) \) is \( \tilde{g}_\alpha \)-clopen in \((X, \tau)\). Hence \( f \) is slightly \( \tilde{g}_\alpha \)-continuous.

**Proposition 2.3.4** Every slightly continuous function is slightly \( \tilde{g}_\alpha \)-continuous.

**Proof.** Let \( U \) be a clopen set in \((Y, \sigma)\) then \( f^{-1}(U) \) is open in \((X, \tau)\). Since every open set is \( \tilde{g}_\alpha \)-open, \( f^{-1}(U) \) is \( \tilde{g}_\alpha \)-open. Hence \( f \) is slightly \( \tilde{g}_\alpha \)-continuous.

**Proposition 2.3.5** Every \( \tilde{g}_\alpha \)-continuous function is slightly \( \tilde{g}_\alpha \)-continuous.

**Proof.** Let \( U \) be a clopen set in \((Y, \sigma)\) then \( f^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \((X, \tau)\). Hence \( f \) is slightly \( \tilde{g}_\alpha \)-continuous.

**Remark 2.3.6** The converse of the Proposition 2.3.5 need not be true as shown in the following Example.

**Example 2.3.7** Let \( X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \)
\( Y = \{p, q\}, \sigma = \{\phi, Y, \{p\}\} \)
\( \tilde{G}_\alpha O(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}. \)

The function \( f : (X, \tau) \to (Y, \sigma) \) be defined as \( f(a) = q, f(b) = f(c) = p \). The function \( f \) is slightly \( \tilde{g}_\alpha \)-continuous but not \( \tilde{g}_\alpha \)-continuous. Since \( f^{-1}(\{p\}) = \{b, c\} \) is not \( \tilde{g}_\alpha \)-open in \((X, \tau)\).

**Theorem 2.3.8** If the function \( f : (X, \tau) \to (Y, \sigma) \) is slightly \( \tilde{g}_\alpha \)-continuous and \((Y, \sigma)\) is a locally indiscrete space then \( f \) is \( \tilde{g}_\alpha \)-continuous.

**Proof.** Let \( U \) be an open subset of \((Y, \sigma)\). Since \( Y \) is locally indiscrete \( U \) is closed in \( Y \). Since \( f \) is slightly \( \tilde{g}_\alpha \)-continuous \( f^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \( X \). Hence \( f \) is \( \tilde{g}_\alpha \)-continuous.

**Theorem 2.3.9** If the function \( f : (X, \tau) \to (Y, \sigma) \) is slightly \( \tilde{g}_\alpha \)-continuous and \((X, \tau)\) is a \( \#T_{\tilde{g}_\alpha} \)-space then \( f \) is slightly continuous.

**Proof.** Let \( U \) be a clopen subset of \((Y, \sigma)\). Since \( f \) is slightly \( \tilde{g}_\alpha \)-continuous \( f^{-1}(U) \) is \( \tilde{g}_\alpha \)-open in \( X \). Since \( X \) is a \( \#T_{\tilde{g}_\alpha} \)-space \( f^{-1}(U) \) is open in \( X \). Hence \( f \) is slightly continuous.
Theorem 2.3.10 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be functions.

(i) If $f$ is $\tilde{g}_\alpha$-irresolute and $g$ is slightly $\tilde{g}_\alpha$-continuous then $gof : (X, \tau) \to (Z, \eta)$ is slightly $\tilde{g}_\alpha$-continuous.

(ii) If $f$ is $\tilde{g}_\alpha$-irresolute and $g$ is $\tilde{g}_\alpha$-continuous then $gof : (X, \tau) \to (Z, \eta)$ is slightly $\tilde{g}_\alpha$-continuous.

(iii) If $f$ is $\tilde{g}_\alpha$-irresolute and $g$ is slightly continuous then $gof : (X, \tau) \to (Z, \eta)$ is slightly $\tilde{g}_\alpha$-continuous.

Proof.

(i) Let $U$ be a clopen set in $(Z, \eta)$. Then $g^{-1}(U)$ is $\tilde{g}_\alpha$-open in $(Y, \sigma)$. Since $f$ is $\tilde{g}_\alpha$-irresolute $f^{-1}(g^{-1}(U))$ is $\tilde{g}_\alpha$-open in $(X, \tau)$. Since $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$, $gof$ is slightly $\tilde{g}_\alpha$-continuous.

(ii) Let $U$ be a clopen set in $(Z, \eta)$. Then $g^{-1}(U)$ is $\tilde{g}_\alpha$-open in $(Y, \sigma)$. Since $f$ is $\tilde{g}_\alpha$-irresolute $f^{-1}(g^{-1}(U))$ is $\tilde{g}_\alpha$-open in $(X, \tau)$. Since $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$, $gof$ is slightly $\tilde{g}_\alpha$-continuous.

(iii) Let $U$ be a clopen set in $(Z, \eta)$. Then $g^{-1}(U)$ is open in $(Y, \sigma)$ and any open set is $\tilde{g}_\alpha$-open. Since $f$ is $\tilde{g}_\alpha$-irresolute $f^{-1}(g^{-1}(U))$ is $\tilde{g}_\alpha$-open in $(X, \tau)$. Since $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$, $gof$ is slightly $\tilde{g}_\alpha$-continuous.

Theorem 2.3.11 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be functions. If $f$ is surjective and strongly $\tilde{g}_\alpha$-open and $g$ is slightly $\tilde{g}_\alpha$-continuous then $g$ is slightly $\tilde{g}_\alpha$-continuous.

Proof. Let $U$ be a clopen set in $(Z, \eta)$ then $f^{-1}(g^{-1}(U))$ is $\tilde{g}_\alpha$-open in $(X, \tau)$. Since $f$ is strongly $\tilde{g}_\alpha$-open and surjective we have $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is $\tilde{g}_\alpha$-open in $(Y, \sigma)$. Hence $g$ is slightly $\tilde{g}_\alpha$-continuous.

Theorem 2.3.12 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be functions. If $f$ is surjective and strongly $\tilde{g}_\alpha$-open and $\tilde{g}_\alpha$-irresolute then $gof : (X, \tau) \to (Z, \eta)$ is slightly $\tilde{g}_\alpha$-continuous if and only if $g$ is slightly $\tilde{g}_\alpha$-continuous.
Proof. Let $U$ be a clopen set in $(Z, \eta)$ then $f^{-1}(g^{-1}(U))$ is $\tilde{g}_{\alpha}$-open in $(X, \tau)$. Since $f$ is strongly $\tilde{g}_{\alpha}$-open and surjective we have $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is $\tilde{g}_{\alpha}$-open in $(Y, \sigma)$. Hence $g$ is slightly $\tilde{g}_{\alpha}$-continuous.

Conversely let $g$ be slightly $\tilde{g}_{\alpha}$-continuous and $U$ be a clopen set in $(Z, \eta)$ then $g^{-1}(U)$ is $\tilde{g}_{\alpha}$-open in $(Y, \sigma)$. Since $f$ is $\tilde{g}_{\alpha}$- irresolute $f^{-1}(g^{-1}(U))$ is $\tilde{g}_{\alpha}$-open in $(X, \tau)$. Hence $gof$ is slightly $\tilde{g}_{\alpha}$-continuous.

Remark 2.3.13 From the above discussions we have the following table which gives the relationship between the following classes of continuous functions. The symbol “1” in a cell means that a function on the corresponding row implies a function on the corresponding column. Th symbol “0” means that a function on the corresponding row does not imply a function on the corresponding column.

1. Slightly continuous functions. 2. $\tilde{g}_{\alpha}$-continuous functions 3. slightly $\tilde{g}_{\alpha}$-continuous functions.

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Table 2

2.4 Applications

Applying the $\tilde{g}_{\alpha}$-sets, this section defines $\tilde{g}_{\alpha}$-$T_2$ spaces, $\tilde{g}_{\alpha}$-compact and $\tilde{g}_{\alpha}$-connected spaces. The different forms of continuous functions defined in the first three sections of this chapter help to investigate the properties of $\tilde{g}_{\alpha}$-$T_2$ spaces, $\tilde{g}_{\alpha}$-compact and $\tilde{g}_{\alpha}$-connected spaces.

Definition 2.4.1 A topological space $(X, \tau)$ is called a $\tilde{g}_{\alpha}$-Hausdorff space (or $\tilde{g}_{\alpha}$-$T_2$) if for each pair of distinct points $x,y$ of $X$ there exist disjoint $\tilde{g}_{\alpha}$-open sets $U,V$ containing $x$ and $y$ respectively. 74
Theorem 2.4.2 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an injective $\tilde{g}_{\alpha}$-continuous function and $Y$ is $T_2$ then $X$ is $\tilde{g}_{\alpha}$-$T_2$.

Proof. Suppose that $x, y$ are distinct points of $X$, since $f$ is injective $f(x) \neq f(y)$ and $Y$ is $T_2$, there exist disjoint open sets $G, H$ in $Y$ such that $f(x) \in G, f(y) \in H, G \cap H = \phi$. Let $U = f^{-1}(G), V = f^{-1}(H)$. By hypothesis $U, V$ are $\tilde{g}_{\alpha}$-open sets in $X$. Also $x \in f^{-1}(G) = U, y \in f^{-1}(H) = V, U \cap V = \phi$. Hence $X$ is $\tilde{g}_{\alpha}$-$T_2$.

Theorem 2.4.3 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\tilde{g}_{\alpha}$-continuous map

(i) If $X$ is a $\# T_{\tilde{g}_{\alpha}}$-space then $f$ is a continuous map.

(ii) If $X$ is a $T_{\tilde{g}_{\alpha}}$-space then $f$ is an $\alpha$-continuous map.

Proof.

(i) Let $U$ be a closed set in $Y$. Since $f$ is $\tilde{g}_{\alpha}$-continuous $f^{-1}(U)$ is a $\tilde{g}_{\alpha}$-closed set in $X$. Since $X$ is a $\# T_{\tilde{g}_{\alpha}}$-space it is closed in $X$. Hence $f$ is a continuous map.

(ii) Let $U$ be a closed set in $Y$. Since $f$ is $\tilde{g}_{\alpha}$-continuous $f^{-1}(U)$ is a $\tilde{g}_{\alpha}$-closed set in $X$. Since $X$ is a $T_{\tilde{g}_{\alpha}}$-space it is $\alpha$-closed in $X$. Hence $f$ is an $\alpha$-continuous map.

Definition 2.4.4 A space $(X, \tau)$ is $\tilde{g}_{\alpha}$-connected if $X$ can not be written as the union of two non empty disjoint $\tilde{g}_{\alpha}$-open sets.

Theorem 2.4.5 For a topological space $(X, \tau)$, the following are equivalent:

(i) $(X, \tau)$ is $\tilde{g}_{\alpha}$-connected.

(ii) The only subsets of $(X, \tau)$ which are both $\tilde{g}_{\alpha}$-open and $\tilde{g}_{\alpha}$-closed are the empty set and $X$.

Proof. $(i) \Rightarrow (ii)$ Let $U$ be a $\tilde{g}_{\alpha}$-open and $\tilde{g}_{\alpha}$-closed subset of $(X, \tau)$. Then $U^c$ is both $\tilde{g}_{\alpha}$-open and $\tilde{g}_{\alpha}$-closed in $(X, \tau)$. Since $(X, \tau)$ is the disjoint union of the
\[ \tilde{g}_\alpha \text{-open sets } U \text{ and } U^c, \text{ by assumption one of these must be empty. i.e., } U = \phi \text{ or } U = X. \]

(ii) \Rightarrow (i) Suppose that \( X = A \cup B \) where \( A \) and \( B \) are disjoint non-empty \( \tilde{g}_\alpha \)-open subsets of \( (X, \tau) \). Then \( A \) is both \( \tilde{g}_\alpha \)-open and \( \tilde{g}_\alpha \)-closed subset of \( (X, \tau) \) and therefore by assumption, \( A = \phi \) or \( A = X \). Thus, \( (X, \tau) \) is \( \tilde{g}_\alpha \)-connected.

**Proposition 2.4.6** Every \( \tilde{g}_\alpha \)-connected space is connected.

**Proof.** Let \((X, \tau)\) be a \( \tilde{g}_\alpha \)-connected space. Suppose that \((X, \tau)\) is not connected. Then \( X = A \cup B \) where \( A \) and \( B \) are disjoint nonempty open sets in \((X, \tau)\). Then \( A \) and \( B \) are \( \tilde{g}_\alpha \)-open and \( X = A \cup B \), where \( A \) and \( B \) are disjoint nonempty and \( \tilde{g}_\alpha \)-open sets in \((X, \tau)\). This contradicts the fact that \((X, \tau)\) is \( \tilde{g}_\alpha \)-connected and so \((X, \tau)\) is connected.

**Proposition 2.4.7** If \((X, \tau)\) is a \#T\( \tilde{g}_\alpha \)-space and connected, then \((X, \tau)\) is \( \tilde{g}_\alpha \)-connected.

**Proof.** Suppose \( X \) is not \( \tilde{g}_\alpha \)-connected let \( A \) and \( B \) are two non empty disjoint \( \tilde{g}_\alpha \)-open subsets of \( X \) such that \( X = A \cup B \). Since \( X \) is a \#T\( \tilde{g}_\alpha \)-space \( A \) and \( B \) are open which is a contradiction to our assumption that \( X \) is connected. Hence \( X \) is \( \tilde{g}_\alpha \)-connected.

**Proposition 2.4.8** If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \tilde{g}_\alpha \)-continuous surjection and \( (X, \tau) \) is \( \tilde{g}_\alpha \)-connected, then \((Y, \sigma)\) is connected.

**Proof.** Suppose that \( Y = A \cup B \), where \( A \) and \( B \) are disjoint nonempty open sets of \((Y, \sigma)\). Since \( f \) is a \( \tilde{g}_\alpha \)-continuous and onto, \( X = f^{-1}(A) \cup f^{-1}(B) \) where, \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint nonempty \( \tilde{g}_\alpha \)-open sets in \((X, \tau)\). This contradicts the fact that \((X, \tau)\) is \( \tilde{g}_\alpha \)-connected and so \((Y, \sigma)\) is connected.

**Proposition 2.4.9** If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \tilde{g}_\alpha \)-irresolute surjection and \((X, \tau)\) is \( \tilde{g}_\alpha \)-connected, then \((Y, \sigma)\) is \( \tilde{g}_\alpha \)-connected.
Proof. Since any \( g_\alpha \)-irresolute function is \( g_\alpha \)-continuous, proof is similar to Proposition 2.4.8.

**Proposition 2.4.10** If \( f : (X, \tau) \to (Y, \sigma) \) is strongly \( g_\alpha \)-continuous onto map, where \( (X, \tau) \) is a connected space, then \( (Y, \sigma) \) is \( g_\alpha \)-connected.

Proof. Since any strongly \( g_\alpha \)-continuous function is \( g_\alpha \)-continuous, proof is similar to Proposition 2.4.8.

**Theorem 2.4.11** If \( f : (X, \tau) \to (Y, \sigma) \) is a slightly \( g_\alpha \)-continuous surjective function and \( X \) is \( g_\alpha \)-connected then \( Y \) is connected.

Proof. Suppose \( Y \) is not connected. Then there exists non-empty disjoint open sets \( U \) and \( V \) such that \( Y = U \cup V \). Therefore \( U \) and \( V \) are clopen sets in \( Y \). Since \( f \) is slightly \( g_\alpha \)-continuous \( f^{-1}(U), f^{-1}(V) \) are non empty disjoint \( g_\alpha \)-open sets in \( X \). Also \( f^{-1}(Y) = X = f^{-1}(U) \cup f^{-1}(V) \). This shows that \( X \) is not \( g_\alpha \)-connected, a contradiction and hence \( Y \) is connected.

**Theorem 2.4.12** If \( f \) is a slightly \( g_\alpha \)-continuous function from a \( g_\alpha \)-connected space \( (X, \tau) \) onto a space \( (Y, \sigma) \) then \( Y \) is not a discrete space.

Proof. Suppose that \( Y \) is a discrete space. Let \( A \) be a proper nonempty open subset of \( Y \). Then \( f^{-1}(A) \) is a proper nonempty \( g_\alpha \)-clopen subset of \( X \), which is a contradiction to the assumption that \( X \) is \( g_\alpha \)-connected.

**Theorem 2.4.13** A space \( X \) is \( g_\alpha \)-connected if every slightly \( g_\alpha \)-continuous function from \( X \) into any \( T_0 \) space \( Y \) is constant.

Proof. Let every slightly \( g_\alpha \)-continuous function from a space \( X \) into \( Y \) be constant. If \( X \) is not \( g_\alpha \)-connected there exists a proper nonempty \( g_\alpha \)-clopen subset \( A \) of \( X \). Let \( (Y, \tau) \) be such that \( Y = \{a, b\}, \tau = \{\emptyset, Y, \{a\}, \{b\}\} \) be a topology. Let \( f : X \to Y \) be any function such that \( f(A) = \{a\} \) and \( f(X - A) = \{b\} \). Then \( f \) is a non-constant and slightly \( g_\alpha \)-continuous function which is a contradiction. Hence \( X \) is \( g_\alpha \)-connected.
Definition 2.4.14 A space \((X, \tau)\) is said to be \(\tilde{g}_\alpha\)-compact if every \(\tilde{g}_\alpha\)-open cover of \(X\) has a finite sub cover where \(\tilde{g}_\alpha\)-open cover is the collection of \(\tilde{g}_\alpha\)-open sets in \(X\) whose union is equal to \(X\).

Proposition 2.4.15 A \(\tilde{g}_\alpha\)-continuous image of a \(\tilde{g}_\alpha\)-compact space is compact.

Proof. Let \(f : (X, \tau) \to (Y, \sigma)\) be a \(\tilde{g}_\alpha\)-continuous onto map, where \((X, \tau)\) is a \(\tilde{g}_\alpha\)-compact space. Let \(\{A_i : i \in \Lambda\}\) be an open cover of \((Y, \sigma)\). Then \(\{f^{-1}(A_i) : i \in \Lambda\}\) is a \(\tilde{g}_\alpha\)-open cover of \((X, \tau)\). Since \((X, \tau)\) is \(\tilde{g}_\alpha\)-compact, it has a finite sub cover, say \(\{f^{-1}(A_1), f^{-1}(A_2), \ldots f^{-1}(A_n)\}\). Since \(f\) is onto, \(\{A_1, A_2, \ldots A_n\}\) is an open cover of \((Y, \sigma)\) and so \((Y, \sigma)\) is compact.

Proposition 2.4.16 If \(f : (X, \tau) \to (Y, \sigma)\) is a strongly \(\tilde{g}_\alpha\)-continuous onto map where \((X, \tau)\) is a compact space, then \((Y, \sigma)\) is \(\tilde{g}_\alpha\)-compact.

Proof. Let \(\{A_i : i \in \Lambda\}\) be a \(\tilde{g}_\alpha\)-continuous cover of \((Y, \sigma)\). Since \(f\) is strongly \(\tilde{g}_\alpha\)-continuous, \(\{f^{-1}(A_i) : i \in \Lambda\}\) is an open cover of \((X, \tau)\). Since \((X, \tau)\) is compact, it has a finite sub cover say, \(\{f^{-1}(A_1), f^{-1}(A_2), \ldots f^{-1}(A_n)\}\) and since \(f\) is onto, \(\{A_1, A_2, \ldots A_n\}\) is a finite sub cover of \((Y, \sigma)\) and therefore \((Y, \sigma)\) is \(\tilde{g}_\alpha\)-compact.

Proposition 2.4.17 If a map \(f : (X, \tau) \to (Y, \sigma)\) is a perfectly \(\tilde{g}_\alpha\)-continuous onto map, where \((X, \tau)\) is compact, then \((Y, \sigma)\) is \(\tilde{g}_\alpha\)-compact.

Proof. Since every perfectly \(\tilde{g}_\alpha\)-continuous function is strongly \(\tilde{g}_\alpha\)-continuous and therefore the result follows from Proposition 2.4.16.

Theorem 2.4.18 If \(f : (X, \tau) \to (Y, \sigma)\) is a slightly \(\tilde{g}_\alpha\)-continuous injection and \(Y\) is clopen \(T_1\) then \(X\) is \(\tilde{g}_\alpha\)-\(T_1\).

Proof. Suppose that \(Y\) is clopen \(T_1\). For any distinct points \(x\) and \(y\) in \(X\), there exists \(V, W \in CO(Y)\) such that \(f(x) \in V, f(y) \notin V, f(x) \notin W, f(y) \in W\). Since \(f\) is slightly \(\tilde{g}_\alpha\)-continuous \(f^{-1}(V), f^{-1}(W)\) are \(\tilde{g}_\alpha\)-open subsets of \((X, \tau)\) such that \(x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W), y \in f^{-1}(W)\). This shows that \(X\) is \(\tilde{g}_\alpha\)-\(T_1\).
Theorem 2.4.19 If \( f : (X, \tau) \to (Y, \sigma) \) is a slightly \( \tilde{g}_\alpha \)-continuous injection and \( Y \) is clopen \( T_2 \) then \( X \) is \( \tilde{g}_\alpha \)-\( T_2 \).

Proof. Suppose that \( Y \) is clopen \( T_2 \). For any distinct points \( x \) and \( y \) in \( X \), there exist disjoint clopen sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U, f(y) \in V \). Since \( f \) is slightly \( \tilde{g}_\alpha \)-continuous \( f^{-1}(U), f^{-1}(V) \) are \( \tilde{g}_\alpha \)-open subsets of \((X, \tau)\) containing \( x \) and \( y \) respectively. Also \( f^{-1}(U) \cap f^{-1}(V) = \phi \) (since \( U \cap V = \phi \)). Hence \( X \) is \( \tilde{g}_\alpha \)-\( T_2 \).

Theorem 2.4.20 Let \( f : (X, \tau) \to (Y, \sigma) \) be a slightly \( \tilde{g}_\alpha \)-continuous injective function. If \( Y \) is ultra \( T_2 \) then \( X \) is \( \tilde{g}_\alpha \)-\( T_2 \).

Proof. Let \( x, z \) be any two points of \( X \). Then since \( f \) is injective and \( Y \) is ultra \( T_2 \), there exist disjoint clopen sets \( U, V \) in \( Y \) containing \( f(x) \) and \( f(z) \) respectively. Since \( f \) is slightly \( \tilde{g}_\alpha \)-continuous \( x \in f^{-1}(U) \in \tilde{G}_\alpha O(X), z \in f^{-1}(V) \in \tilde{G}_\alpha O(X) \) and \( f^{-1}(U) \cap f^{-1}(V) = \phi \). Hence \( X \) is \( \tilde{g}_\alpha \)-\( T_2 \).