Chapter - 9

WEAKLY πg-CLOSED SETS

9.1 INTRODUCTION

Recently many authors have studied various classes of generalized closed sets, in general topology. Dontchev and Noiri [24] have introduced the concept of πg-closed sets and studied their most fundamental properties in topological spaces. Also, recently, Ekici and Noiri [33] have introduced a generalization of πg-closed sets and πg-open sets. In this chapter, we study a new class of generalized closed sets in topological spaces. We introduce the notions of weakly πg-closed sets and weakly πg-open sets, which are weaker forms of πg-closed sets and πg-open sets, respectively. Also, the relationships among related generalized closed sets are investigated.

9.2 PRELIMINARIES

Definition 9.2.1

A subset A of a space (X, τ) is said to be b-open set [6] if $A \subset \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$.

The complement of b-open set is called b-closed set.

Definition 9.2.2

A subset A of a space (X, τ) is called a generalized α-closed (briefly, gα-closed) set [66] if $\alpha \text{cl}(A) \subset U$ whenever $A \subset U$ and U is
α-open in \((X, \tau)\). The complement of \(g\alpha\)-closed set is called \(g\alpha\)-open set.

**Definition 9.2.3**

A subset \(A\) of a topological space \((X, \tau)\) is called:

(i) a weakly \(g\)-closed (briefly, \(wg\)-closed) set [119] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

(ii) a weakly \(\bar{g}\)-closed (briefly, \(w\bar{g}\)-closed) set [100] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \#gs-open in \((X, \tau)\).

(iii) a weakly \(\omega\)-closed set [103] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \((X, \tau)\).

(iv) a regular weakly generalized closed (briefly, \(rwg\)-closed) set [73] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \((X, \tau)\).

**Definition 9.2.4**

Let \(X\) and \(Y\) be topological spaces. A function \(f : X \to Y\) is called

(i) completely continuous [8] if \(f^{-1}(V)\) is regular open in \(X\) for each open set in \(Y\).

(ii) irresolute [20] if \(f^{-1}(V)\) is semi-open in \(X\) for every semi-open subset \(V\) in \(Y\).

**Definition 9.2.5 [44]**

A space \((X, \tau)\) is called \(\pi g\)-\(T_{1/2}\) if every \(\pi g\)-closed set is closed.

**Definition 9.2.6 [38]**
A topological space \((X, \tau)\) is said to be locally \(\pi g\)-indiscrete if every \(\pi g\)-open set of \(X\) is closed in \(X\).

**Definition 9.2.7 [94]**

A subset \(A\) of \((X, \tau)\) is said to be \(\pi gp\)-closed if \(pcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\pi\)-open in \(X\).

**Definition 9.2.8 [94]**

A space \((X, \tau)\) is called \(\pi gp\)-T\(_{1/2}\) if every \(\pi gp\)-closed set is preclosed.

**Definition 9.2.9 [45]**

A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be almost \(\pi gp\)-continuous if \(f^{-1}(\text{int}(\text{cl}(V)))\) is a \(\pi gp\)-open set in \(X\) for every \(V \in \sigma\).

**Definition 9.2.10 [37]**

A function \(f : X \rightarrow Y\) is called \((\pi g, s)\)-continuous if the inverse image of each regular open set of \(Y\) is \(\pi g\)-closed in \(X\).

**Definition 9.2.11 [102]**

A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(w\tilde{g}\)-continuous if the inverse image of every open set in \((Y, \sigma)\) is \(w\tilde{g}\)-open in \(X\).

**Definition 9.2.12 [99]**

A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\tilde{g}\)-continuous if the inverse image of every open set in \((Y, \sigma)\) is \(\tilde{g}\)-open in \(X\).
**Definition 9.2.13 [30]**

A topological space $X$ is called $\pi g$-compact if every cover of $X$ by $\pi g$-open sets has finite subcover.

**Definition 9.2.14 [41]**

A space $X$ is said to be almost connected if $X$ cannot be written as a disjoint union of two non-empty regular open sets.

**Definition 9.2.15 [37, 38]**

A space $X$ is called $\pi g$-connected if $X$ is not the union of two disjoint nonempty $\pi g$-open sets.

**Definition 9.2.16 [38]**

A function $f : X \to Y$ is called $\pi g$-open if the image of each $\pi g$-open set is $\pi g$-open.

**Definition 9.2.17 [30]**

A function $f : X \to Y$ is said to be $\pi g$-irresolute if $f^{-1}(V)$ is $\pi g$-closed in $(X, \tau)$ for every $\pi g$-closed set $V$ of $(Y, \sigma)$.

**Lemma 9.2.18 [94]**

Let $Y$ be open in $X$. Then

(a) If $A$ is $\pi$-open in $Y$, then there exists a $\pi$-open set $B$ in $X$ such that $A = B \cap Y$.

(b) If $A$ is $\pi$-open in $X$, then $A \cap Y$ is $\pi$-open in $Y$. 
9.3 WEAKLY $\pi g$-CLOSED SETS

We introduce the definition of weakly $\pi g$-closed sets in a topological space and study the relationships of such sets.

**Definition 9.3.1**

A subset $A$ of a topological space $(X, \tau)$ is called a weakly $\pi g$-closed (briefly, $w\pi g$-closed) set if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open in $(X, \tau)$.

**Proposition 9.3.2**

Every $\pi g$-closed set is $w\pi g$-closed. But the converse of this implication is not true in general.

**Example 9.3.3**

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the set $\{c\}$ is $w\pi g$-closed but not $\pi g$-closed in $(X, \tau)$.

**Theorem 9.3.4**

Every $w\pi g$-closed set is $r\pi g$-closed but not conversely.

**Proof**

Let $A$ be any $w\pi g$-closed set and let $U$ be regular open set containing $A$. Then $U$ is a $\pi$-open set containing $A$. We have $\text{cl}(\text{int}(A)) \subseteq U$. Thus, $A$ is $r\pi g$-closed.

**Example 9.3.5**
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the set $\{a, b\}$ is rwg-closed but it is not a wπg-closed.

**Theorem 9.3.6**

Every wg-closed set is wπg-closed but not conversely.

**Proof**

Let $A$ be any wg-closed set and let $U$ be $\pi$-open set containing $A$. Then $U$ is an open set containing $A$. We have $\text{cl}(\text{int}(A)) \subseteq U$. Thus, $A$ is wπg-closed.

**Example 9.3.7**

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the set $\{a, c\}$ is wπg-closed but it is not a wg-closed.

**Theorem 9.3.8**

If a subset $A$ of a topological space $(X, \tau)$ is both closed and $\alpha$g-closed, then it is wπg-closed in $(X, \tau)$.

**Proof**

Let $A$ be a $\alpha$g-closed set in $(X, \tau)$ and $U$ be an $\pi$-open set containing $A$. Then $U$ is open containing $A$ and so $U \supseteq \alpha\text{cl}(A) = A \cup \text{cl}(\text{int}(\text{cl}(A)))$. Since $A$ is closed, $U \supseteq \text{cl}(\text{int}(A))$ and hence $A$ is wπg-closed in $(X, \tau)$.

**Theorem 9.3.9**
If a subset $A$ of a topological space $(X, \tau)$ is both $\pi$-open and $w\pi g$-closed, then it is closed.

**Proof**

Since $A$ is both $\pi$-open and $w\pi g$-closed, $A \supset cl(int(A)) = cl(A)$ and hence $A$ is closed in $(X, \tau)$.

**Corollary 9.3.10**

If a subset $A$ of a topological space $(X, \tau)$ is both $\pi$-open and $w\pi g$-closed, then it is both regular open and regular closed in $(X, \tau)$.

**Theorem 9.3.11**

Let $(X, \tau)$ be a $\pi g$-$T_{1/2}$ space and $A \subset X$ be $\pi$-open. Then, $A$ is $w\pi g$-closed if and only if $A$ is $\pi g$-closed.

**Proof**

Let $A$ be $\pi g$-closed. By Proposition 9.3.2, it is $w\pi g$-closed.

Conversely, let $A$ be $w\pi g$-closed. Since $A$ is $\pi$-open, by Theorem 9.3.9, $A$ is closed. Since $X$ is $\pi g$-$T_{1/2}$, $A$ is $\pi g$-closed.

**Theorem 9.3.12**

A set $A$ is $w\pi g$-closed if and only if $cl(int(A)) - A$ contains no non-empty $\pi$-closed set.

**Proof**

Necessity. Let $F$ be a $\pi$-closed set such that $F \subset cl(int(A)) - A$. Since $F^c$ is $\pi$-open and $A \subset F^c$, from the definition of $w\pi g$-closed set it follows that
cl(int(A)) \subseteq F^c. i.e. F \subseteq (cl(int(A)))^c. This implies that F \subseteq (cl(int(A))) \cap (cl(int(A)))^c = \phi.

Sufficiency. Let A \subseteq G, where G is \(\pi\)-open set in X. If cl(int(A)) is not contained in G, then cl(int(A)) \cap G^c is a non-empty \(\pi\)-closed subset of cl(int(A)) – A, we obtain a contradiction. This proves the sufficiency and hence the theorem.

**Corollary 9.3.13**

A w\(\pi\)g-closed set A is regular closed if and only if cl(int(A)) – A is \(\pi\)-closed and cl(int(A)) \supseteq A.

**Proof**

Necessity. Since the set A is regular closed, cl(int(A)) – A = \(\phi\) is regular closed and hence \(\pi\)-closed.

Sufficiency. By Theorem 9.3.12, cl(int(A)) – A contains no non-empty \(\pi\)-closed set. That is cl(int(A)) – A = \(\phi\). Therefore, A is regular closed.

**Theorem 9.3.14**

Let \((X, \tau)\) be a topological space and B \(\subset A \subset X\). If B is w\(\pi\)g-closed set relative to A and A is both open and w\(\pi\)g-closed subset of X then B is w\(\pi\)g-closed set relative to X.

**Proof**

Let B \(\subset U\) and U be a \(\pi\)-open in \((X, \tau)\). Then B \(\subset A \cap U\). Since B is w\(\pi\)g-closed relative to A, cl\(_A\)(int\(_A\)(B)) \(\subset A \cap U\). That is A \(\cap cl(int(B)) \subset A \cap U\). We have A \(\cap cl(int(B)) \subset U\) and then \([A \cap cl(int(B))] \cup \)
(\text{cl}(\text{int}(B)))^c \subset U \cup (\text{cl}(\text{int}(B)))^c. \text{ Since } A \text{ is } \text{wpg-closed in } (X, \tau), \text{ we have}
\text{cl}(\text{int}(A)) \subset U \cup (\text{cl}(\text{int}(B)))^c. \text{ Therefore } \text{cl}(\text{int}(B)) \subset U \text{ since } \text{cl}(\text{int}(B)) \text{ is}
\text{not contained in } (\text{cl}(\text{int}(B)))^c. \text{ Thus, } B \text{ is } \text{wpg-closed set relative to } (X, \tau).

**Corollary 9.3.15**

If A is both open and wpg-closed and F is closed in a topological space (X, \tau), then A \cap F is wpg-closed in (X, \tau).

**Proof**

Since F is closed, we have A \cap F is closed in A. Therefore cl_A(A \cap F) = A \cap F in A. Let A \cap F \subset G, where G is \pi-open in A. Then cl_A(\text{int}_A(A \cap F)) \subset G and hence A \cap F is wpg-closed in A. By Theorem 9.3.14, A \cap F
is wpg-closed in (X, \tau).

**Theorem 9.3.16**

If A is wpg-closed and A \subset B \subset \text{cl}(\text{int}(A)), then B is wpg-closed.

**Proof**

Since A \subset B, \text{cl}(\text{int}(B)) - B \subset \text{cl}(\text{int}(A)) - A. \text{ By Theorem 9.3.12}
\text{cl}(\text{int}(A)) - A \text{ contains no non-empty } \pi\text{-closed set and so } \text{cl}(\text{int}(B)) - B.

Again by Theorem 9.3.12, B is wpg-closed.

**Theorem 9.3.17**

Let \( (X, \tau) \) be a topological space and A \subset Y \subset X and Y be open. If A is \text{wpg-closed in } X, \text{ then } A \text{ is wpg-closed relative to } Y.

**Proof**
Let $A \subseteq Y \cap G$ where $G$ is $\pi$-open in $(X, \tau)$. Since $A$ is $w\pi g$-closed in $(X, \tau)$, $A \subseteq G$ implies $\text{cl}(\text{int}(A)) \subseteq G$. That is $Y \cap (\text{cl}(\text{int}(A))) \subseteq Y \cap G$ where $Y \cap \text{cl}(\text{int}(A))$ is closure of interior of $A$ in $(Y, \sigma)$. Thus, $A$ is $w\pi g$-closed relative to $(Y, \sigma)$.

**Theorem 9.3.18**

If a subset $A$ of a topological space $(X, \tau)$ is nowhere dense, then it is $w\pi g$-closed.

**Proof**

Since $\text{int}(A) \subseteq \text{int}(\text{cl}(A))$ and $A$ is nowhere dense, $\text{int}(A) = \emptyset$. Therefore $\text{cl}(\text{int}(A)) = \emptyset$ and hence $A$ is $w\pi g$-closed in $(X, \tau)$.

The converse of Theorem 9.3.18 need not be true as seen in the following example.

**Example 9.3.19**

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then the set $\{a\}$ is $w\pi g$-closed in $(X, \tau)$ but not nowhere dense in $(X, \tau)$.

**Remark 9.3.20**

If any subsets $A$ and $B$ of topological space $X$ are $w\pi g$-closed, then their intersection need not be $w\pi g$-closed.

**Example 9.3.21**

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. In this topological space the subsets $\{a, c\}$ and $\{a, d\}$ are $w\pi g$-closed but their intersection $\{a\}$ is not $w\pi g$-closed in $(X, \tau)$.
Proposition 9.3.22

Every $g\alpha$-closed set is $w\pi g$-closed but not conversely.

Proof

Let $A$ be any $g\alpha$-closed subset of $(X, \tau)$ and let $U$ be an $\pi$-open set containing $A$. Then $U$ is $\alpha$-open set containing $A$. Now $G \supset \alpha \text{cl}(A) \supset \text{cl}(\text{int}(\text{cl}(A))) \supset \text{cl}(\text{int}(A))$. Thus, $A$ is $w\pi g$-closed in $(X, \tau)$.

The converse of Proposition 9.3.22 need not be true as seen in the following example.

Example 9.3.23

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then the set $\{a, b, c\}$ is $w\pi g$-closed but not $g\alpha$-closed in $(X, \tau)$.

Remark 9.3.24

$w\pi g$-closedness is independent of semi-closedness, $\beta$-closedness, b-closedness, sg-closedness and $\tilde{g}$ $s$-closedness in $(X, \tau)$.

Example 9.3.25

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the set $\{a, b, c\}$ is $w\pi g$-closed in $(X, \tau)$ but not semi-closed, $\beta$-closed, b-closed, sg-closed and $\tilde{g}$ $s$-closed in $(X, \tau)$.

Example 9.3.26
Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \).

Then the set \( \{a, c\} \) is semi-closed, \( \beta \)-closed, \( b \)-closed, sg-closed and \( \tilde{g} \) s-closed in \((X, \tau)\) but not \( w\pi g \)-closed in \((X, \tau)\).

**Remark 9.3.27**

The following diagrams show the relationships established between \( w\pi g \)-closed sets and some other sets where \( A \rightarrow B \) (resp. \( A \leftrightarrow B \)) represents \( A \) implies \( B \) but not conversely (resp. \( A \) and \( B \) are independent of each other).

**Diagram I**

\[
\text{closed} \rightarrow w\tilde{g} \text{-closed} \rightarrow \text{weakly } \omega \text{-closed} \rightarrow wg \text{-closed} \rightarrow w\pi g \text{-closed} \\
\downarrow \\
\text{rwg-closed}
\]

**Diagram II**

semi-closed \( \rightarrow \) \( \beta \)-closed \( \rightarrow \beta \)-closed \( \rightarrow \) b-closed \( \rightarrow \) \( w\pi g \)-closed

wr - w\pi g-closed

\[
\begin{array}{c}
\text{sg-closed} \\
\tilde{g} \text{-s-closed}
\end{array}
\]

**Definition 9.3.28**

A subset \( A \) of a topological space \( X \) is called \( w\pi g \)-open set if \( A^c \) is \( w\pi g \)-closed in \( X \).
Proposition 9.3.29

(i) Every \( \pi g \)-open set is \( w \pi g \)-open;

(ii) Every \( g \)-open set is \( w \pi g \)-open.

Theorem 9.3.30

A subset \( A \) of a topological space \( X \) is \( w \pi g \)-open if \( G \subseteq \text{int}(\text{cl}(A)) \) whenever \( G \subseteq A \) and \( G \) is \( \pi \)-closed.

Proof

Let \( A \) be any \( w \pi g \)-open. Then \( A^c \) is \( w \pi g \)-closed. Let \( G \) be a \( \pi \)-closed set contained in \( A \). Then \( G^c \) is a \( \pi \)-open set in \( X \) containing \( A^c \). Since \( A^c \) is \( w \pi g \)-closed we have \( \text{cl}(\text{int}(A^c)) \subseteq G^c \). Therefore \( G \subseteq \text{int}(\text{cl}(A)) \).

Conversely, we suppose that \( G \subseteq \text{int}(\text{cl}(A)) \) whenever \( G \subseteq A \) and \( G \) is \( \pi \)-closed. Then \( G^c \) is a \( \pi \)-open set containing \( A^c \) and \( G^c \supseteq (\text{int}(\text{cl}(A)))^c \). It follows that \( G^c \supseteq \text{cl}(\text{int}(A^c)) \). Hence \( A^c \) is \( w \pi g \)-closed and so \( A \) is \( w \pi g \)-open.

9.4 WEAKLY \( \pi g \)-CONTINUOUS FUNCTIONS

Definition 9.4.1

Let \( X \) and \( Y \) be topological spaces. A function \( f : X \to Y \) is called weakly \( \pi g \)-continuous (briefly, \( w \pi g \)-continuous) if \( f^{-1}(U) \) is a \( w \pi g \)-open set in \( X \), for each open set \( U \) in \( Y \).

Example 9.4.2
Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, Y\} \). The function \( f : (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(a) = b \), \( f(b) = c \) and \( f(c) = a \) is w\( \pi \)-g-continuous, because every subset of \( X \) is w\( \pi \)-g-closed.

**Proposition 9.4.3**

Every \( \pi \)-g-continuous function is w\( \pi \)-g-continuous.

**Proof**

It follows from Proposition 9.3.29 (i).

The converse of Proposition 9.4.3 need not be true as per the following example.

**Example 9.4.4**

Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, Y\} \). Let the function \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function. Then \( f \) is w\( \pi \)-g-continuous but it is not \( \pi \)-g-continuous.

**Theorem 9.4.5**

A function \( f : X \rightarrow Y \) is called w\( \pi \)-g-continuous if and only if \( f^{-1}(U) \) is a w\( \pi \)-g-closed set in \( X \) for each closed set \( U \) in \( Y \).

**Proof**

Let \( U \) be any closed set in \( Y \). According to the assumption \( f^{-1}((U^c)) = X \setminus f^{-1}(U) \) is w\( \pi \)-g-open in \( X \), so \( f^{-1}(U) \) is w\( \pi \)-g-closed in \( X \).

The converse can be proved in a similar manner.

**Theorem 9.4.6**
Suppose that X and Y are spaces and the family of $\pi g$-open sets of X is closed under arbitrary unions. If a function $f : X \to Y$ is contra $\pi g$-continuous and Y is regular, then f is $w\pi g$-continuous.

**Proof**

Let $f : X \to Y$ be contra $\pi g$-continuous and Y be regular. By Theorem 16 of [38], f is $\pi g$-continuous. Hence, f is $w\pi g$-continuous.

**Theorem 9.4.7**

Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f is contra $\pi g$-continuous and $(X, \tau)$ is locally $\pi g$-indiscrete, then f is $w\pi g$-continuous.

**Proof**

Let $f : X \to Y$ be contra $\pi g$-continuous and $(X, \tau)$ be locally $\pi g$-indiscrete. By Theorem 21 of [38], f is continuous. Hence, f is $w\pi g$-continuous.

**Theorem 9.4.8**

Suppose that a topological space $(X, \tau)$ is $\pi gp$-$T_{1/2}$ and submaximal and Y is regular. If f is almost $\pi gp$-continuous, then f is $w\pi g$-continuous.

**Proof**

Let f be almost $\pi gp$-continuous. By Theorem 30 of [45], f is almost $\pi g$-continuous. Also, by Theorem 38 of [45], f is $\pi g$-continuous. Hence, f is $w\pi g$-continuous.

**Theorem 9.4.9**
Let $Y$ be a regular space and $f : X \to Y$ be a function. Suppose that the collection of $\pi g$-closed sets of $X$ is closed under arbitrary intersections. Then if $f$ is ($\pi g$, $s$)-continuous, $f$ is $w\pi g$-continuous.

**Proof**

Let $f$ be ($\pi g$, $s$)-continuous. By Theorem 24 of [37], $f$ is $\pi g$-continuous. Thus, $f$ is $w\pi g$-continuous.

**Remark 9.4.10**

Every $\tilde{g}$-continuous function is $w\tilde{g}$-continuous and every $w\tilde{g}$-continuous function is $w\pi g$-continuous but not conversely as shown in [102] and in the below example.

**Example 9.4.11**

Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $w\pi g$-continuous but not $w\tilde{g}$-continuous.

**Proposition 9.4.12**

If $f : X \to Y$ is perfectly continuous and $\pi$-irresolute, then it is $R$-map.

**Proof**

Let $V$ be any regular open subset of $Y$. According to the assumption, $f^{-1}(V)$ is both $\pi$-open and closed in $X$. Since $f^{-1}(V)$ is closed it is $w\pi g$-closed. Then $f^{-1}(V)$ is both $\pi$-open and $w\pi g$-closed. Hence by Corollary 9.3.10 it is regular open in $X$, so $f$ is $R$-map.

**Definition 9.4.13**
A topological space $X$ is weakly $\pi g$-compact (briefly, $w\pi g$-compact) if every $w\pi g$-open cover of $X$ has a finite subcover.

**Remark 9.4.14**

Every $w\pi g$-compact space is $\pi g$-compact.

**Theorem 9.4.15**

Let $f : X \rightarrow Y$ be a surjective $w\pi g$-continuous function. If $X$ is $w\pi g$-compact, then $Y$ is compact.

**Proof**

Let $\{A_i : i \in I\}$ be an open cover of $Y$. Then $\{f^{-1}(A_i) : i \in I\}$ is a $w\pi g$-open cover of $X$. Since $X$ is $w\pi g$-compact, it has a finite subcover, say $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$. Since $f$ is surjective $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $Y$ and hence $Y$ is compact.

**Definition 9.4.16**

A topological space $X$ is weakly $\pi g$-connected (briefly, $w\pi g$-connected) if $X$ cannot be written as the disjoint union of two non-empty $w\pi g$-open sets.

**Theorem 9.4.17**

If a topological space $X$ is $w\pi g$-connected, then $X$ is almost connected and $\pi g$-connected.

**Proof**

It follows from the fact that each regular open set and each $\pi g$-open set is $w\pi g$-open.
**Theorem 9.4.18**

For a topological space $X$ the following statements are equivalent:

(i) $X$ is $w\pi g$-connected.

(ii) The empty set $\emptyset$ and $X$ are only subsets which are both $w\pi g$-open and $w\pi g$-closed.

(iii) Each $w\pi g$-continuous function from $X$ into a discrete space $Y$ which has at least two points is a constant function.

**Proof**

(i) $\Rightarrow$ (ii). Let $S \subset X$ be any proper subset, which is both $w\pi g$-open and $w\pi g$-closed. Its complement $X \setminus S$ is also $w\pi g$-open and $w\pi g$-closed. Then $X = S \cup (X \setminus S)$ is a disjoint union of two non-empty $w\pi g$-open sets which is a contradiction with the fact that $X$ is $w\pi g$-connected. Hence, $S = \emptyset$ or $X$.

(ii) $\Rightarrow$ (i). Let $X = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and $A$, $B$ are $w\pi g$-open. Since $A = X \setminus B$, $A$ is $w\pi g$-closed. According to the assumption $A = \emptyset$, which is a contradiction.

(ii) $\Rightarrow$ (iii). Let $f : X \to Y$ be a $w\pi g$-continuous function where $Y$ is a discrete space with at least two points. Then $f^{-1}(\{y\})$ is $w\pi g$-closed and $w\pi g$-open for each $y \in Y$ and $X = \bigcup \{f^{-1}(\{y\}) \mid y \in Y\}$. According to the assumption, $f^{-1}(\{y\}) = \emptyset$ or $f^{-1}(\{y\}) = X$. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, $f$ will not be a function. Also there is no exist more than one $y \in Y$
such that \( f^{-1}(\{y\}) = X \). Hence, there exists only one \( y \in Y \) such that \( f^{-1}(\{y\}) = X \) and \( f^{-1}(\{y_1\}) = \emptyset \) where \( y \neq y_1 \in Y \). This shows that \( f \) is a constant function.

(iii) \( \Rightarrow \) (ii). Let \( S \neq \emptyset \) be both \( w\pi g \)-open and \( w\pi g \)-closed in \( X \). Let \( f : X \to Y \) be a \( w\pi g \)-continuous function defined by \( f(S) = \{a\} \) and \( f(X \setminus S) = \{b\} \) where \( a \neq b \). Since \( f \) is constant function we get \( S = X \).

**Theorem 9.4.19**

Let \( f : X \to Y \) be a \( w\pi g \)-continuous surjective function. If \( X \) is \( w\pi g \)-connected, then \( Y \) is connected.

**Proof**

We suppose that \( Y \) is not connected. Then \( Y = A \cup B \) where \( A \cap B = \emptyset \), \( A \neq \emptyset \), \( B \neq \emptyset \) and \( A, B \) are open sets in \( Y \). Since \( f \) is \( w\pi g \)-continuous surjective function \( X = f^{-1}(A) \cup f^{-1}(B) \) are disjoint union of two non-empty \( w\pi g \)-open subsets. This is contradiction with the fact that \( X \) is \( w\pi g \)-connected.

**9.5 WEAKLY \( \pi g \)-OPEN FUNCTIONS AND WEAKLY \( \pi g \)-CLOSED FUNCTIONS**

**Definition 9.5.1**
Let X and Y be topological spaces. A function \( f : X \to Y \) is called weakly \( \pi_g \)-open (briefly, \( w\pi_g \)-open) if \( f(V) \) is a \( w\pi_g \)-open set in Y for each open set \( V \) in X.

**Remark 9.5.2**

Every \( \pi_g \)-open function is \( w\pi_g \)-open but not conversely.

**Example 9.5.3**

Let \( X = Y = \{a, b, c, d\} \), \( \tau = \{\phi, \{a\}, \{a, b, d\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, Y\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( w\pi_g \)-open but not \( \pi_g \)-open.

**Definition 9.5.4**

Let X and Y be topological spaces. A function \( f : X \to Y \) is called weakly \( \pi_g \)-closed (briefly, \( w\pi_g \)-closed) if \( f(V) \) is a \( w\pi_g \)-closed set in Y for each closed set \( V \) in X.

It is clear that an open function is \( w\pi_g \)-open and a closed function is \( w\pi_g \)-closed.

**Theorem 9.5.5**

Let X and Y be topological spaces. A function \( f : X \to Y \) is \( w\pi_g \)-closed if and only if for each subset \( B \) of Y and for each open set \( G \) containing \( f^{-1}(B) \) there exists a \( w\pi_g \)-open set \( F \) of Y such that \( B \subset F \) and \( f^{-1}(F) \subset G \).

**Proof**
Let B be any subset of Y and let G be an open subset of X such that \( f^{-1}(B) \subseteq G \). Then \( F = Y \setminus f(X \setminus G) \) is \( w\pi g \)-open set containing B and \( f^{-1}(F) \subseteq G \).

Conversely, let U be any closed subset of X. Then \( f^{-1}(Y \setminus f(U)) \subseteq X \setminus U \) and \( X \setminus U \) is open. According to the assumption, there exists a \( w\pi g \)-open set F of Y such that \( Y \setminus f(U) \subseteq F \) and \( f^{-1}(F) \subseteq X \setminus U \). Then \( U \subseteq X \setminus f^{-1}(F) \). From \( Y \setminus F \subseteq f(U) \subseteq f(X \setminus f^{-1}(F)) \) \( \subseteq Y \setminus F \) follows that \( f(U) = Y \setminus F \), so \( f(U) \) is \( w\pi g \)-closed in Y. Therefore \( f \) is a \( w\pi g \)-closed function.

**Remark 9.5.6**

The composition of two \( w\pi g \)-closed functions need not be \( w\pi g \)-closed as we can see from the following example.

**Example 9.5.7**

Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{a, b\}, Y\} \) and \( \eta = \{\phi, \{a\}, \{b\}, \{a, b\}, Z\} \). We define \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be the identity functions. Hence both \( f \) and \( g \) are \( w\pi g \)-closed functions. For a closed set \( U = \{a\} \), \( (g \circ f)(U) = g(f(U)) = g(\{a\}) = \{a\} \) which is not \( w\pi g \)-closed in Z. Hence the composition of two \( w\pi g \)-closed functions need not be \( w\pi g \)-closed.

**Theorem 9.5.8**
Let $X$, $Y$ and $Z$ be topological spaces. If $f : X \to Y$ be a closed function and $g : Y \to Z$ be a $w\pi g$-closed function, then $g \circ f : X \to Z$ is a $w\pi g$-closed function.

**Definition 9.5.9**

A function $f : X \to Y$ is called a weakly $\pi g$-irresolute (briefly, $w\pi g$-irresolute) if $f^{-1}(U)$ is a $w\pi g$-open set in $X$ for each $w\pi g$-open set $U$ in $Y$.

**Example 9.5.10**

Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, c\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $w\pi g$-irresolute.

**Remark 9.5.11**

The following examples show that irresoluteness and $w\pi g$-irresoluteness are independent.

**Example 9.5.12**

Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, c\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is $w\pi g$-irresolute but not irresolute.

**Example 9.5.13**

Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $f$ is irresolute but not $w\pi g$-irresolute.
Remark 9.5.14

Every $\pi g$-irresolute function is $w\pi g$-continuous but not conversely. Also, the concepts of $\pi g$-irresoluteness and $w\pi g$-irresoluteness are independent of each other.

Example 9.5.15

Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b, d\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $w\pi g$-continuous but not $\pi g$-irresolute.

Example 9.5.16

Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $\pi g$-irresolute but not $w\pi g$-irresolute.

Example 9.5.17

Let $X = \{a, b, c, d\}$, $Y = \{p, q\}$, $\tau = \{\phi, \{d\}, \{b, c\}, \{b, c, d\}, X\}$ and $\sigma = \{\phi, \{p\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined as $f(a) = f(c) = f(d) = p$ and $f(b) = q$. Then $f$ is $w\pi g$-irresolute but not $\pi g$-irresolute.

Theorem 9.5.18

The composition of two $w\pi g$-irresolute functions is also $w\pi g$-irresolute.

Theorem 9.5.19

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions such that $g \circ f : X \rightarrow Z$ is $w\pi g$-closed function. Then the following statements hold:
(i) if $f$ is continuous and injective, then $g$ is $w\pi g$-closed.

(ii) if $g$ is $w\pi g$-irresolute and injective, then $f$ is $w\pi g$-closed.

**Proof**

(i) Let $F$ be a closed set of $Y$. Since $f^{-1}(F)$ is closed in $X$, we can conclude that $(g \circ f)(f^{-1}(F))$ is $w\pi g$-closed in $Z$. Hence $g(F)$ is $w\pi g$-closed in $Z$. Thus $g$ is a $w\pi g$-closed function.

(ii) It can prove in a similar manner as (i).

**Theorem 9.5.20**

If $f : X \to Y$ is a $w\pi g$-irresolute function, then it is $w\pi g$-continuous.

**Remark 9.5.21**

The converse of the above need not be true in general. Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{d\}, Y\}$. The function $f : X \to Y$ defined as $f(a) = d$, $f(b) = c$, $f(c) = b$ and $f(d) = a$. Then $f$ is $w\pi g$-continuous but not $w\pi g$-irresolute. Since $f^{-1}(\{a\}) = \{d\}$ is not $w\pi g$-open in $X$.

**Theorem 9.5.22**

If $f : X \to Y$ is a surjective $w\pi g$-irresolute function and $X$ is $w\pi g$-compact, then $Y$ is $w\pi g$-compact.

**Theorem 9.5.23**

If $f : X \to Y$ is surjective $w\pi g$-irresolute function and $X$ is $w\pi g$-connected, then $Y$ is $w\pi g$-connected.