Chapter -1

GENERAL INTRODUCTION
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1.1 INTRODUCTION:

Bioengineering is the field which has been develop recently by cross fertilization of the engineering and Biological Sciences so that both are utilized for the benefitting marking.

Biomechanics is an important branch of Bioengineering Bio fluid, mechanism, is an important branch of Biomechanics in this branch the principles of fluid mechanics are applied to analysis the Problem biology.

Red blood cells are roughly disc-shaped with diameter of a about ‘8’lim and a thickness of about ‘2’lim. Capillary diameters range from about 4 to ‘10’lim, so in the capillary and generally travel in single file, further, the red blood cells are highly deformable and rarely maintain disc-shape during normal flow there blood cell consists on an elastic membrane filled with
hemoglobin solution with is a newtonian fluid with a viscosity of about –6 centipoises the suspending fluid of the red blood cells is the plasma which is October Newtonian fluid with a viscosity of about 1-2 Centipoise at body temperature the densities of the blood cells and the plasma are very nearly equal so that the flow may be regarded as a suspension of neutrally buoyant particles.

The Reynolds number of capillary blood flow is generally about ‘0.01’ so that inertial terms for both the motion of the fluid and of the suspended particles may be neglected. Hence the motion of the blood plasma may be treated as a problem of an incompressible viscous flow at low Reynolds number.

The Stokes equation of motions are then

\[ \nabla p = \nabla^2 u \]  \hspace{1cm} (1.1)

\[ \nabla \cdot u = 0 \]  \hspace{1cm} (1.2)
where $u$ is the velocity and $P$ is the pressure, the equation of motion of the particle again at zero acceleration must be satisfied together with the constitutive equations of the particle and continuity of the surface of the particle.

In general the particle will be deformed and moved by the stress applied to it by the fluid; there are also integral conditions for each the must be satisfied which arise because the inertial and buoyant force are considered negligible. These conditions are that the sum of the force exerted on the particle and the sum of their moment must be equal to zero; this is usually known as the zero drag condition in the case of a uniform speed of translation of the particles which a fixed deformed shape. The internal motion of the particle may be disregarded since the fluid internally will then exert at most a uniform pressure. If the particle is elastic it is then in a state of quasi-equilibrium and the zero drag and zero moment conditions suffice to determine its motion.
The particle differential equation on the stream functions of fourth order where as the governing equation (1.1) and (1.2) are only of Second order in terms of velocity and pressure an example of the use the stream function approach is given by tong and fung (1). It can be seen that the boundary condition including the zero drag condition are some what more different to apply than when velocity and pressure and used as primary variables.

Another study of capillary flow using the stream function approach has been given by Gupta and Seshadri (2).

1.2 Historical Background of Microcirculation –

The historical development of ideas and knowledge of the circulation of blood has been elegantly recorded at length in a special volume edited by Fishman and Richards [3]. It appears that the ancient Greeks were quite familiar with the anatomy of the heart and the major blood vessels, but did not realize that the blood
circulates in a closed circuit. They did know that the heart valves permit flow in only one direction. Galen (131-201 A.D.) taught that the heart was the source of heat and blood. There is some motion of the flow through the lungs in Galen’s writing but Michael Servatus is given credit for first properly describing the flow of blood from the arteries to veins in the pulmonary circulation. The idea that the systemic circulation also "in a circle" is the famous discovery of William Harvey (1578-1657).

The first direct observation of capillary vessels was reported by Marcello Malpighi [4], shortly after Malpighi, Leeuwenhoek (1688) reported seeing individual red blood cells flowing in clear plasma in the tail of a line tadpole. Landis [5] gives this summary which could distinguish single globules following each other, compressed and in single file through the narrowest channels. Sometimes the corpuscles change into long ovals as the vessel narrowed.
The assessment of the resistance offered by the microcirculation was begun by Hales [6] who first measured blood pressure in a "mare tied down to agate". Hales realized that the principal resistance in the circulation is due to the "capillary arteries" and "capillary veins". He estimated the force driving the flow in a capillary as the product of the cross-sectional area and the difference of arterial and venous pressure which he measured as '80' inches of blood. He remarks "whence we see both from experiment and calculation, that the force of the blood in these fine capillaries can be put very little and the longer such capillaries are the slower will the motion of the blood be in them".

A more sophisticated estimate of microvascular resistance was made by Young [7]. He realized that in small tubes the pressure drop was proportional to the velocity, but attempted to write a single formula for water for the full range of sizes, from
rivers to fine tubes and came fairly close to the limited data available. Using this formula, he estimates the pressure drop in a mathematical model of the entire circulation made up of thirty segments from the aorta of '3/4' inch diameter and '9' inches long bifurcating twenty nine times with a diameter ratio of 1/1.26 and a length ratio 1/1.961. Finally, he estimates the relative viscosity of blood and water and comes remarkably close to the typical value of 3.5. Hence there can be no doubt that the resistance of the inertial surface of the arteries to the motion of the blood must be much greater than would be found in the case of water. Young also gives his own estimate of red cell diameters as 1/3000 to 1/3600 inches or about '7.1 - 8.5μ'.

The clear defilation of the capillaries is attributed (by Landis, [8]) to Marshall Hall (1831) who noted those vessels which "do not become smaller by subdivision, nor larger by conjunction; but they are characterized by continual and successive
union and division or an astomoses whilst they retain a nearly uniform diameter".

Many of the aspects of the mechanics of blood flow in the microcirculation which are currently being studied were previously noted in a qualitative or rough quantitative manner. The book by Krogh [9] which represents the foremost statement of knowledge of the microcirculation as of some seventy five years ago, mentions many topics on which quantitative information is being developed at present by experimental and mathematical methods. A few quotations from Krogh's first chapter illustrate the point. "In the capillaries the pulse does not as a rule cause any variation in diameter of the vessels but the velocity variation can often be very distinctly seen. "A special reduction in the number of corpuscles, which may even amount to their complete washing away from a certain capillary field, is sometimes brought about by the process which I have termed plasma skinning".
"When examined under a fairly high power, so that the capillary walls can be distinctly seen, some are found which allow the corpuscles to pass in a continuous current and these generally exhibit a definite axial stream surrounded by a plasma zone through which a white corpuscle will occasionally come rolling along. Other is so narrow that the corpuscles have to pass in single file and come continuously in contact with the wall. Others again are even narrower and the corpuscles can pass only in a deformed state. The simplest deformation is observed in capillaries down to about '4-5µ' diameters where the edges of the flat disk like corpuscles are bent in while the length of the corpuscle measured during the passage does not exceed its diameter in the free state. In still narrower capillaries the red corpuscles are greatly deformed and compressed into a shape like sausages, the length of which may be double the normal diameter".
"In single capillaries the flow may become retarded or accelerated from no visible cause; in capillary anastomoses the direction of flow may change from time to time".

It is generally agreed by historians of medicine that the book by Harvey Published in 1628 was the first to enunciate clearly that blood flows in circulatory manner, thus requiring the existence of porosities of the flesh through which the blood passes enroute from arteries to veins.

It is remained for Poiseuille [10] by careful and accurate experiments to establish the law governing flow of water, alcohol and mercury in fine glass capillaries in the form

\[ Q = K \frac{PD^4}{L} \]  \hspace{1cm} (1.3)

Where Q is the discharge due to the pressure drop P over the length L, through a capillary of diameter. The coefficient K was found to be dependent on the liquid flowing and the temperature T

\[ K = k (1 + AT + AT^2) \]  \hspace{1cm} (1.4)
where \( k \), \( A \) and \( A' \) are constants depending on the fluid flowing.

The form of equation (1.4) which is more often called Poiseuille's law is

\[
Q = \frac{128 PD^4}{\pi \mu L}
\]  

(1.5)

where \( \mu \) is the viscosity of the fluid.

The well-known Fahreus-Lindqvist effect is the reduction in apparent viscosity of blood with reduction in diameter of tubes below about 500 \( \mu \) down to about 10 \( \mu \) in diameter. In capillaries with diameters below 10 \( \mu \), the apparent viscosity increases as the tube diameter decreases. This is sometimes called the reverse Fahraeus - Lindqvist effect.

The experiment of Prothero and Burton [11] utilized a model in which air bubbles separated by short regions of liquid estimulated red blood cells in capillary. The first theoretical model of capillary blood flow was the axial-train or stacked-coins model
in which a solid cylindrical core represents a line of axisymmetric blood cells flowing in a cylindrical tube. In terms of the diameter ratio \( \lambda'' \), (core diameter divided by tube diameter) the ratio of the core velocity \( U \), to the mean velocity \( V \), of entire suspension is

\[
\frac{U}{V} = \frac{2}{1 + \lambda''^2} \quad (1.6)
\]

Further, the relative apparent viscosity \( \eta \) of the suspension (ratio of apparent viscosity of the suspension to suspending fluid viscosity) is

\[
\eta = \frac{I}{1 - \lambda''^4} \quad (1.7)
\]

These results indicate that this model produces a Fahraeus effect \( (U > V) \), equation (1.5) and an inverse Fahraeus-Lindqvist effect, equation (1.6). If cell diameter is held fixed and tube diameter decreases, then \( \lambda \) increases and \( \eta \) increases.
The stacked coins model is a first approximation to discrete cellular flow in capillaries. The effects of deformability of red blood cells are large and important. The red blood cell is easily deformed by the stresses exerted on it by the suspending plasma in capillary flow. White blood cells are much less deformable than red blood cells and this is an important factor in the role they play in capillary flow.

White blood cells are known to be viscoelastic and hence their entry into capillary and subsequent behaviour will be timedependent. The higher stiffness and greater volume of white blood cells as compared to red blood cells results in a lower cell velocity for the white blood cells in many capillaries. The red blood cells then tend to accumulate in a high hematocrit region behind the white blood cells; this is called train formation and has been analyzed to some extent both on a discrete cell basis and by
continuum theory. Another aspect of capillary flow that has been analyzed to some extent is the effect of non-axis symmetric forms that red blood cells often take on in capillaries.

Another theoretical extension under current development concerns the behaviour of blood in capillaries of larger diameters so that two rows of cells form, both non-axisymmetric. It has been shown experimentally that such double file formation take place more readily as the diameter of the capillary increases and as the hematocrit increases.

A number of interrelated issues still requiring elucidation concern the distribution of blood cells in capillaries. It is well known that capillary hematocrit is lower than that in the large arteries or veins. The location of particles in a capillary and their position at the entrance of a bifurcation will be affected by gravity. There are also controversial issues to be resolved, such as the
reason for discharge hematocrits measured by withdrawing blood from microvessels appears to yield systemic rather than the lower capillary hematocrit.

From the above items it can be seen that theoretical analysis of capillary blood flow is currently far from resolving many details and biochemical aspects of physiologic interest. But by the beginning of the 21st Century, accurate computer models including realistic 3D geometry of the cells and the tube walls, as well as the biochemistry of adhesion and activation were developed along with appropriate statistical methods of representing interactions of red cells, leukocytes and platelets in large networks of capillaries.

1.3 Newtonian and non-Newtonian Fluids:

A Newtonian fluid is that for which the shearing stress between any two adjacent layers is linearly proportional to the shear rate, that is
\[ \tau = \mu \frac{du}{dy} \] (1.8)

The above equation is also known as Newton's law of friction, where \( \mu \) is defined to be the measure of the viscosity of fluid and depend to a great extent on its temperature.

Non-Newtonian fluids are usually considered to be those for which the shearing stress between two adjacent layers is not linearly proportional to the shear rate, that is the viscosity of a non-Newtonian fluid is not constant at a given temperature and pressure but do depends on the rate of shear, more generally, on the previous kinematic history of the fluid. Non-linear fluids in shear flow may be classified into two ways

1.4 **Time-independent fluids**

In these fluids the rate of shear at any point is some function of the shear stress at that point and depends on nothing-else,
thus

\[ \frac{du}{dy} = \bar{\gamma} = f(\tau) \]  \hspace{1cm} (1.9)

The above equation (1.9) shows that the rate of shear \( \gamma \) at any point in the fluid is the function of \( f(\tau) \).

These types are -

- **Bingham Plastic Fluids**
- **Pseudo Plastic fluids**
- **Dilatant fluids**

1.5 **Bingham Plastic fluids** -

Such type of fluid is characterized by a flow curve, which is a straight line having an intercept \( \tau_0 \) (which is also known as yield stress) on the shear stress axis. The constitutive equation for such type of fluid is given by.

\[ \tau = \tau_0 + \eta_p \gamma \hspace{1cm} ; \hspace{1cm} \tau > \tau_0 \]

and \( \bar{\gamma} = 0 \hspace{1cm} ; \text{ when } |\tau| > \tau_0 \)
where $\eta$ is the plastic viscosity, the slope of the flow curve. Common examples of such fluids are slurries, drilling muds, greases, oil paints, toothpaste and sludges. The behaviour of the fluid is that its structural rigidity resists any stress less than yield stress. But when the applied stress exceeded the yield stress then the system behaves as a Newtonian fluid under a shear stress $\tau - \tau_0$ and when the shear stress falls below $\tau_0$, the structure is reformed.

1.6 **Pseudo plastic fluids**

This category of fluids show no yield value and the flow curve indicates that the ratio of shear stress to the rate of shear, "which is termed as viscosity" falls increasingly with shear rate and further the flow curve becomes linear at very high rates of shear. The geological relation is

$$\tau = k |\gamma|^{n-1} \gamma$$ (1.10)
where \( k \) is the measure of viscosity and \( n \) is a measure of the degree of Non-Newtonian behaviour, greater its departure from unity the more pronounced are the Non-Newtonian properties of the fluid. The viscosity for a power law fluid in terms of \( k \) and \( n \) can be expressed as

\[
\eta = \frac{\tau}{\gamma} = k |\gamma|^{n-1} \quad (1.11)
\]

For Pseudo-plastic fluids \( n < 1 \) and hence the viscosity function decreases as the rate of shear increases.

The Herschel - Bulkley equation

\[
\gamma = \frac{1}{k} (\tau - \tau_0)^n \quad \text{if} \quad \tau > \tau_0
\]

\[
= 0 \quad \text{if} \quad |\tau| \leq \tau_0 \quad (1.13)
\]

The equation is reduced to the Bingham fluids when \( n = 1 \), to that for Power law fluid when \( \tau_0 = 0 \) and that for Newtonian fluid when \( n = 1 \) and \( \tau_0 = 0 \).
Other fluid describing Pseudo - Plastic behaviours are

\[ \tau = A \sin^{-1}\left( \frac{\gamma}{C} \right) \]  \hspace{1cm} (1.14)

\[ \tau = \frac{\gamma}{B} + C \sin\left( \frac{\tau}{A} \right) \]  \hspace{1cm} (1.15)

\[ \tau = A \gamma + \sin^{-1}(C\gamma) \]  \hspace{1cm} (1.16)

\[ \frac{1}{\eta} = \frac{1}{\eta_0} + m^{-\frac{1}{n}} (\tau^2)^{\frac{1-n}{2n}} \]  \hspace{1cm} (1.17)

\[ \tau = \frac{A\gamma}{(B + |\gamma|)} + \eta \]  \hspace{1cm} (1.18)

\[ \tau^{1/2} = \tau_0^{1/2} + \eta^{1/2} \gamma^{1/2} \]  \hspace{1cm} (1.19)

1.7 Dilitant fluids-

These fluids are similar to Pseudo Plastic type showing no yield stress but the viscosity increase with increasing rate of shear. Power law relation, for the index \( n \) greater than unity, is applicable for these fluids.
Shear stress vs shear rate relation for Bingham fluids

(a) Pseudo - Plastic fluids

(b) Dilatants fluids

(c) The dashed line show Newtonian Behaviour.
1.8 Remark on the Rheological Model of Non-Newtonian fluid

At present the rheology of Non-Newtonian fluids is more less inductive, since up to now it seems impossible to derive realistic constitutive-equations directly from a set of first principle of continuum mechanics or/and kinetic theory. According to Bird [12], the essential idea of model building is to develop constitutive equations containing a small number of measurable and easily interpretable constants which are useful in making fluid dynamical calculations. The model proposed by Oldroyd [13, 14] has become popular because of its physical interpretations showing that this model has a fundamental meaning for non-Newtonian fluids, especially if these fluids consist of suspensions of particles or solutions of micromolecules in Newtonian solvents. From statistical calculations made by Bird [12] for dilute suspensions of rigid dumbels with Brownion motion in a Newtonian solvent, they
derived relations between the constituents occurring in the constitutive equations of the fluids.

In the early days of rheology, constitutive equations of elastico viscous fluids were primarily formulated for simple shear flows or pure strain flows. Jeffreys [15] had first coined the term elastico-viscosity for fluid exhibiting not only viscosity but also elastic properties. Jeffreys-Oldroyd model has been modified by Bird [12] in such a way that to a order of approximation the model is consistent with statistical results.

For simple shear flow defined by

\[ v_1 = v(z_2, t), \quad v_2 = 0, \quad v_3 = 0, \quad (1.20) \]

the shear rate of flow is given by \( q = \frac{\partial v}{\partial z_2} \) (1.2) and the corresponding shear stress by \( \tau_{12} \).
The theoretical constitutive equation of the Jefreys model is

\[ \tau_{12} + \lambda_1 \tau_{12} = \eta_0(\lambda_2 q) \quad (1.22) \]

Where \( \lambda_1 \) and \( \lambda_2 \) are constant with the dimension of time and \( \eta_0 \) has the significance of dynamic viscosity.

An interpretation of equation has been given by Frohlich and Sack [16] who calculated the rheological constitutive equation of a dilute suspension of small elastic spheres in a Newtonian liquid.

1.9 Visco-elastic Fluids-

Those fluid which possess a certain degree of elasticity in addition to viscosity are called visco-elastic fluid. Now we will discuss about various visco-elastic fluids.

(a) Rivlin Ericksen fluids-

The constitutive equation for the above fluid has been proposed Rivlin - Ericksen [17] as

\[ S = -pI + \phi_1 A_1 + \phi_2 A_2 + \phi_3 A_1^2 + \phi_4 A_2^2 + \phi_5 (A_1 A_2 + A_2 A_1) + \phi_6 \]
\[
(A_1^2A_2 + A_2A_1^2) + \phi_7 + (A_1A_2^2 + A_2^2A_1) + \phi_8 + (A_1^2A_2^2 + A_2^2A_1^2)
\]

(1.23)

where \( p \) is an arbitrary hydrostatic pressure and \( \phi \)'s are polynomial functions, matrices \( A_1 \) and \( A_2 \) are defined by

\[
A_y^{(1)} = (v_{y} + v_{y,i})
\]

\[
A_y^{(2)} = \frac{\partial A_y^{(1)}}{\partial t} + v_p A_{y,p}^{(1)} + A_{y,p}^{(1)} V_{p,j} + A_{y,p}^{(1)} V_{p,i}
\]

\( v_p \) being velocity vector.

On neglecting the square and products and \( A_2 \), we have

\[
S = -pI + \phi_1 A_1 + \phi_2 A_2 + \phi_3 A_1^2
\]

(1.21)

here \( \phi_1 \) is called coefficient of ordinary viscosity, \( \phi_2 \) the coefficient of viscoelasticity and \( \phi_3 \) the coefficient of cross-viscosity.

Caro et al. [18] have adopted a different approach to obtain the constitutive equation (1.24). In this case, the constitutive equation is
\[ S = - p I + \phi_1 E^{(1)} + \phi_2 E^{(2)} + \phi_3 E^{(1)^2} \]  

(1.25)

where,

\[ E^{(2)}_{ij} = A_{i,j} + A_{j,i} + 2 V_{m,i} V_{m,j} \]

In the above equation, S is the stress tensor, \( v_i \) and \( A_i \) are the components of velocity and acceleration in the direction of the \( i^{th} \) coordinate \( x_i \), \( p \) is an indeterminate hydrostatic pressure.

1.20 Reiner - Rivlin Fluids-

Reiner [19] and Rivlin [20] established that for the anistropic fluid, the most general relation between the stress tensor \( \tau_{ij} \) and the rate of deformation tensor \( e_{ij} \) has the form

\[ \tau_{ij} = - p \delta_{i,j} + 2 \mu e_{i,j} + 2 \mu_c e_{ik} e_{kj} \]  

(1.26)

where,

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

where \( p \) is an arbitrary hydrostatic pressure, \( \mu \) is the coefficient of viscosity, \( \mu_c \) is the coefficient of cross viscosity.
depending on the invariants \( I_1, I_2, \) and \( I_3 \), where \( I_1 = e_{ii}, I_2 = \frac{1}{2} (e_{ij} e_{ji}), \) 
\[ I_3 = e_{ij} e_{jk} e_{ki} \]

1.21 Equation of Motion:

(a) Conservation of Mass (Equation of Continuity) -

We derive the conservation of mass within an arbitrary volume \( V \) bounded by surface \( S \). We equate the mass of fluids \( ds \) flowing out through vector surface element of the surface \( S \) to the loss of \(-\frac{\partial}{\partial t} \int_V \rho dV\) mass \( \int_V \rho v \cdot ds \) from volume \( V \).

That is
\[ -\frac{\partial}{\partial t} \int_V \rho dV + \int_S \rho v \cdot ds = 0 \quad (1.27) \]

Applying Gauss divergence theorem to the surface integral and have
\[ \int_V (\rho dV + \text{div}(\rho v)) dV = 0 \]

Since \( V \) is an arbitrary volume, hence we have
\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho v) = 0 \quad (1.28) \]
as the equation of continuity.
(b) Conservation of Momentum (Equation of Motion)-

The linear momentum of an element $dv$ is $\rho v dv$, where $\rho$ density, velocity and $dV$ the volume. Then the conservation of momentum states that the rate of change of linear momentum of the material particle is equal to the net force on the control volume due to surface and body force.

That is

$$\frac{d}{dt} \int_v \rho v dV = \text{total force} \quad (1.29)$$

If $f$ is the body force per unit mass and $\sigma_{ij}$ is the stress tensor then equation of motion can be written as

$$\rho \frac{Dv_i}{Dt} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \quad (1.30)$$

(c) Conservation of Energy (Heat Equation)-

The first law of thermodynamics states that the rate of increase of energy of the material is equal to the rate of working of the exterior forces on the material minus the net rate of heat loss from the body.
If e is the specific internal energy of a particle and \( \frac{1}{2}v^2 \) is the specific kinetic energy of that particle, then the energy equation has the form

\[
\rho \frac{De}{Dt} = -\frac{\partial q_i}{\partial x_i} + \sigma_{ij} \frac{d}{dy} + \rho S
\]  
(1.31)

where \( \sigma_{ij} = -\rho \delta_{ij} + \tau_{ij} \)

\[
q_i = -k \frac{\partial T}{\partial X_i}
\]

\[
e = C_v T = C_p T = C
\]

\[
d_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i})
\]

Where T is temperature, \( C_p \) and \( C_v \) are specific heats at constant pressure and constant volume respectively, k the thermal conductivity of the fluid.

1.22 Equation of Motion for Incompressible Flow, in Different Co-ordinate Systems:

(a) In cartesian co-ordinates \((x,y,z)\), velocity \((u_x, u_y, u_z)\)

and body force \((X,Y,Z)\) -

Continuity equation is

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0
\]  
(1.32)
operators \((u, \nabla)\) and \(\Delta\) have the forms

\[
(u, \nabla) = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}
\]  

(1.33)

\[
\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]  

(1.34)

Equation of motion in the stress form is

\[
\rho \frac{Du_x}{Dt} = \rho X + \left[ \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z} \right]
\]

\[
\rho \frac{Du_y}{Dt} = \rho Y + \left[ \frac{\partial p_{yx}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{yz}}{\partial z} \right]
\]

\[
\rho \frac{Du_z}{Dt} = \rho Z + \left[ \frac{\partial p_{zx}}{\partial x} + \frac{\partial p_{zy}}{\partial y} + \frac{\partial p_{zz}}{\partial z} \right]
\]  

(1.35)

If the acceleration terms are written fully, equation (1.35) assumes the form

\[
\rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right)
\]

\[
= X - \frac{\partial \rho}{\partial x} + \eta \left[ \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_x}{\partial y} \frac{\partial u_x}{\partial z} \right]
\]

\[
\rho \left( \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \right)
\]

\[
= Y - \frac{\partial \rho}{\partial y} + \eta \left[ \frac{\partial^2 u_y}{\partial x^2} \frac{\partial u_y}{\partial x} \frac{\partial u_y}{\partial z} \right]
\]

\[
\rho \left( \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right)
\]  

(1.36)
\[ \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0 \] (1.37)

operators \((u, \nabla)\) and \(\Delta\) have the forms

\[(u, \nabla) = u_r + \frac{\partial}{\partial r} + \frac{u_\phi}{r} \frac{\partial}{\partial \phi} + u_z \frac{\partial}{\partial z} \]
\[\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \]

Equation of motion in stress form is

\[
\begin{align*}
\rho \left[ \frac{\partial u_r}{\partial t} + (u, \nabla)u_r - \frac{u_\phi^2}{r} \right] &= F_r + \left( \frac{\partial p_{rr}}{\partial r} + \frac{1}{r} \frac{\partial p_{r\phi}}{\partial \phi} + \frac{\partial p_{r\phi}}{\partial \phi} + \frac{p_r - p_\phi}{r} \right) \\
\rho \left[ \frac{\partial u_\phi}{\partial t} + (u, \nabla)u_\phi - \frac{u_r u_\phi}{r} \right] &= F_\phi + \left( \frac{\partial p_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial p_{\phi\phi}}{\partial \phi} + \frac{\partial p_{\phi\phi}}{\partial \phi} + \frac{2 p_{r\phi}}{r} \right) \\
\rho \left[ \frac{\partial u_z}{\partial t} + (u, \nabla)u_z \right] &= F_z + \left( \frac{\partial p_{zz}}{\partial r} + \frac{1}{r} \frac{\partial p_{z\phi}}{\partial \phi} + \frac{\partial p_{z\phi}}{\partial \phi} + \frac{p_z}{r} \right)
\end{align*}
\] (1.38)
where the stress tensor $p_{ij}$ ($i, j = r, \phi, z$) is given by

$$p_{rr} = -p + 2\eta \frac{\partial u_r}{\partial r}$$

$$p_{\phi\phi} = -p + n\eta \left( \frac{1}{r^2} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} \frac{\partial \phi}{\partial \phi} \right)$$

$$p_{zz} = -p + 2\eta \frac{\partial u_z}{\partial z}$$

$$p_{r\phi} = -\eta \left( \frac{1}{r} \frac{\partial u_\phi}{\partial r} + \frac{\partial u_r}{\partial r} - \frac{u_r}{r} \right)$$

$$p_{r\phi} = \eta \left( \frac{\partial u_\phi}{\partial \phi} + \frac{1}{r} \frac{\partial u_z}{\partial \phi} \right)$$

$$p_{z\phi} = \eta \left( \frac{\partial u_z}{\partial r} + \frac{1}{r} \frac{\partial u_z}{\partial z} \right)$$

Navier-Stokes equation in terms of velocity component has the form

$$p \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} - \frac{u_z}{r} \frac{\partial u_r}{\partial z} \right)$$

$$= F_r - \frac{\partial p}{\partial r} + \eta \left[ \frac{\partial^2 u_r}{\partial t^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r^2}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial z^2} \right]$$

$$p \left( \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} - \frac{u_z}{r} \frac{\partial u_\phi}{\partial z} \right)$$

$$= F_\phi - \frac{1}{r} \frac{\partial p}{\partial \phi} + \eta \left[ \frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial u_\phi}{\partial r} + \frac{u_\phi^2}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial z^2} \right]$$
\[ = F_z - \frac{\partial p}{\partial z} + \eta \left[ \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2} \right] \]

(c) In spherical polar co-ordinates \((r, \theta, \phi)\), velocity field \((u_r, u_\theta, u_\phi)\) and body force \((F_r, F_\theta, F_\phi)\)

Equation of continuity-

\[ \frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{rsin\theta} \frac{\partial (sin\theta \; u_\theta)}{\partial \theta} + \frac{1}{rsin\theta} \frac{\partial u_\phi}{\partial \phi} = 0 \]  
(1.39)

operators \((u, \nabla)\) and \(\Delta\) have the forms

\[ u \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r \sin\theta} \frac{\partial}{\partial \theta} + \frac{u_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \]

\[ \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial^2}{\partial \phi^2} \]

\[ \rho \left[ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r \sin\theta} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r \sin\theta} \frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} \right] \]

\[ = F_r \left[ \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial p}{\partial \phi} + \frac{1}{r} \left( 2p + \cot\theta \; p_{\theta \theta} - p_{\phi \phi} \right) \right] \]

\[ + \rho \left[ \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r \sin\theta} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r \sin\theta} \frac{\partial u_\theta}{\partial \phi} - \frac{u_\theta u_\phi}{r} - \frac{u_\phi^2}{r \cot\theta} \right] \]

\[ = F_\theta \left[ \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial p}{\partial \phi} + \frac{1}{r} \left( p_{\theta \theta} + \cot\theta \; 3p_{\phi \phi} - p_{\phi \phi} \cot\theta \right) \right] \]

\[ \rho \left[ \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r \sin\theta} \frac{\partial u_\phi}{\partial \theta} + u_z \frac{\partial u_\phi}{\partial z} \right] \]
\[ + \rho \left[ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi u_r}{r} - \frac{u_\theta u_r}{r} \cot \theta \right] \]

\[ + \rho \left[ \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi u_\theta}{r} - \frac{u_\theta u_\theta}{r} \cot \theta \right] \]

\[ = F_r \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{2 \partial^2 u_r}{\partial r \partial \theta} + \frac{\partial^2 u_r}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial \theta} \left( 2 p_{\phi \phi} + \cot \theta + 3 p_{\phi \phi} \right) \right] \] (1.40)

where, the stress tensor \( p_{ij} \) \((i, j = r, \theta, \phi)\) is given by

\[ p_{rr} = -\rho + 2\eta \frac{\partial u_r}{\partial r} \]

\[ p_{\theta \theta} = -\rho + 2\eta \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \]

\[ p_{\phi \phi} = -\rho + 2\eta \left( \frac{1}{r^2} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} - \frac{u_\theta}{r} \cot \theta \right) \]

\[ p_{rs} = \eta \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{r} \right) \]

\[ p_{\phi \phi} = \eta \left( \frac{1}{r^2} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi \cot \theta}{r} \right) \]

\[ p_{r\phi} = \eta \left( \frac{1}{r} \frac{\partial u_r}{\partial \phi} + \frac{u_r}{r} - \frac{u_\theta \cot \theta}{r} \right) \]
In terms of velocity components, the Navier - Stokes equation takes the form as

\[
\rho \left[ \frac{\partial u_r}{\partial t} + (u_r \nabla) u_r - \frac{u_\theta^2 + u_\phi^2}{r} \right] = F_r - \frac{\partial p}{\partial r} + \eta \left[ \Delta u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin^2 \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right]
\]

\[
\rho \left[ \frac{\partial u_\theta}{\partial t} + (u_r \nabla) u_\theta - \frac{u_r u_\theta}{r} - \frac{u_\phi^2 \cot \theta}{r} \right] = \frac{1}{r} \frac{\partial p}{\partial \theta} + \eta \left[ \Delta u_\theta - \frac{2}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} \right]
\]

\[
e \left[ \frac{\hat{\tau} u_\phi}{\hat{\tau} t} + (u_r \nabla) u_\phi + \frac{\mu_r \mu_\phi}{r} + \frac{\mu_r \mu_\phi \cot \theta}{r} \right]
\]

\[
= \frac{1}{r} \frac{\partial p}{\partial \phi} + \eta \left[ \Delta u_\phi + \frac{2}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} \right]
\]

\[
+ e \left[ \frac{\partial u_\phi}{\partial t} + (u_r \nabla) u_\phi + \frac{\mu_r u_\theta}{r} + \frac{\mu_r \mu_\phi}{r} \cot \theta \right]
\]

\[
+ \rho \left[ \frac{\partial u_\phi}{\partial t} + (u_r \nabla) u_\phi - \frac{u_r u_\phi}{r} - \frac{u_\theta u_\phi \cot \theta}{r} \right]
\]

\[
= \frac{1}{r} \frac{\partial p}{\partial \phi} + \eta \left[ \Delta u_\phi - \frac{2}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} \right]
\]
1.23 Equation in Curvilinear Co-ordinates system for compressible Fluids:

\[
\frac{\partial (h^2 u_i^2)}{\partial x^i} + \frac{p(h^2 u_i^2)}{\partial x^2} + \frac{\partial (h h_i u_i^3)}{\partial x^3} = 0
\]  

(1.41)

Equation of motion -

\[
\rho \left[ \frac{\partial u}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial u}{\partial q_k} \right]
\]

\[
= F + \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (p_1 h_2 h_3) + \frac{\partial}{\partial q_2} (p_2 h_1 h_3) + \frac{\partial}{\partial q_3} (p_3 h_1 h_2) \right]
\]

where \( u = \sum_{k=1}^{3} u_k i_k \) and \( pm = \sum_{k=1}^{3} p_{mk} i_k \)

In Cartesian co-ordinate system -

\[
\begin{align*}
q_1 &= x; & q_2 &= y; & q_3 &= z \\
h_1 &= 1; & h_2 &= 1; & h_3 &= 1
\end{align*}
\]

(1.43)

In cylindrical polar co-ordinate system -

\[
\begin{align*}
q_1 &= r; & q_2 &= \phi; & q_3 &= z \\
h_1 &= 1; & h_2 &= r; & h_3 &= 1
\end{align*}
\]

(1.44)
In spherical co-ordinate system -

\[ \begin{align*}
    q_1 &= r; & q_2 &= \theta; & q_3 &= \phi \\
    h_1 &= 1; & h_2 &= r; & h_3 &= r \sin \theta
\end{align*} \tag{1.44} \]

\[
\nabla_x u^i = \frac{1}{h_i} \frac{\partial u^j_x}{\partial x^k} h^k_i h^j_k + \delta^i_k \sum_{k=1}^{3} \frac{u^k_x}{h_i} \frac{\partial h_j}{\partial x^k}
\]

and

\[
(u^k \nabla_x) u^i = \sum_{k=1}^{3} \frac{u^k_x}{h_i h_k} \frac{\partial u^i_x}{\partial x^k} - \sum_{k=1}^{3} \frac{u^2_x}{h_i h_k^2} \frac{\partial h_k}{\partial x^i} + \sum_{k=1}^{3} \frac{u^k_x v^l_x}{h_i h_l} \frac{\partial h_l}{\partial x^k}
\] \tag{1.47}

1.24 Magnetohydrodynamic Equation:

In magnetofluid dynamics, we consider a conducting fluid. The charge density in the Maxwell equations must then be interpreted as an excess charge density which is generally not large, so we neglect the displacement current. Thus, the displacement current, excess charge density, excess body force and current due to convection of the excess charges are small. The complete sets of MHD equation are

\[ \begin{align*}
    \nabla \cdot D &= 0 & D &= \varepsilon E \\
    \nabla \cdot J &= 0 & B &= \mu_0 H \\
    \nabla \cdot B &= 0 & J &= \sigma \left[ \varepsilon u \times B \right] \\
    \nabla \times H &= J & \nabla \times E &= -\frac{\partial B}{\partial t} \\
    \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x_i} (\rho u_i) & \text{Continuity equation (1.49)}
\end{align*} \tag{1.48} \]
\[
\rho \frac{Du_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_i} + (j \times B)_i + \frac{\partial \tau_{ii}}{\partial x_i} \\
\text{Momentum equation (1.50)}
\]

\[
\rho T \frac{DS}{Dt} = \phi + \frac{J^2}{\sigma} + \nabla q + \rho R \\
\text{Energy equation (1.51)}
\]

\[
\frac{\partial H}{\partial t} = \nabla \times (u \times H) - \nabla \times [u_H (\nabla \times H)] \\
\text{Magnetic field equation (1.52)}
\]

Where, the symbols have their usual meaning.

1.25 **Rheological Properties of Blood**:

Blood is a mixture of plasma and blood cells; plasma is of water, proteins, enzymes and other substances. When plasma is tested in a viscometer, it is found that its coefficient of viscosity is constant and independent of the rate of strain. Hence it is a Newtonian fluid and has viscosity about ‘1.2’ centipoise. Inside of the red cell is a hemoglobin solution which has a viscosity of about ‘6’ centipoise. Hence it is also a Newtonian fluid. Thus blood consists of one Newtonian fluid wrapped in small packages that float in another Newtonian fluid. In large blood vessels or in
viscometers whose dimensions are much larger than the diameter of the red blood cell, the mixture appears as a homogeneous fluid. The shear stress shear strain rate relationship of the mixture can be determined by viscometry. The ratio of the shear stress to the shear strain rate is the coefficient of viscosity. Experimental results show that blood viscosity is Non-Newtonian. However, as we have discussed on large blood vessels the Non-Newtonian feature of blood is un-important and blood can be treated as a Newtonian fluid with a constant coefficient of viscosity. Infact, the modeling of blood flow through individual capillaries has been the subject of numerous theoretical studies, like Lopez and Hellum [21], Chen and Skolak [22], Lighthill [23], Wang and Skalak [24], Skalak and Cheng [25], Pries et al. [26], Quemada [27], Sugiharra and Skalak [28], Suzuki et al. [29], Whitemore [30], Singh [31, 32], Mishra et al. [33], Zakaria [34], Rahman and Sarkar [35], Damiano et al. [36].
1.26 Plasma Peripheral Layer:

It is well known that blood, consisting of red cells and plasma, undergoes a phase separation in these two constituents under steady flow in microvessels and narrow glass capillaries, Bugliarello and Sevilla [37], Pries et al. [38], Saran and Popel [39], Secomb [40, 41]. This non-uniform distribution of red cells over the vessel cross-section leads to a cell rich core flow and plasma rich region which gives rise to the so called Fahracus effect in which the instantaneous volume fraction of the red cells in the vessel or tube haematocrit, HT, is increased relative to the red cell concentration discharge from the vessel or discharge haematocrit HD. The ratio HT/HD is seen to decrease with decreasing vessel diameter in glass tube ranging between ‘20’ and ‘1000’m in diameter Cokelet [42]. Most two-phase models of blood flow in the microcirculation that invoke the continuum approximation
have started. with assumption about the viscosity and/or haematocrit distribution that led to the predictions of the velocity profile, Thamas [43], Nair et al. [44], Secomb [45], Damiano [46], Saran and Popel [47]. Analysis of these plasma peripheral layer (PPL) data has shown that for the range of varies investigated, the absolute dimensions of the layer are not significantly or consistently influenced by the flow velocity and tube it, diameter but are strongly affected by haematocrit.
1.27 REFERENCE:


(12) *Bird, R.B. (1971)*. “Macromolecular hydrodynamics” Research center University of Wisconsin U.S.A.


Ellington and T.J. Pedley) PP. 305-321 company of Biologist Cambridge.


