Chapter - 6

HERSCHEL-BULKLEY MODEL OF VISCOSOUS FLUIDS IN CIRCULAR CYLINDERS OF SMALL RADIUS
A theoretical study of blood flow in narrow circular vessels have been done in this paper. Two layer blood flow model, the central core layer and peripheral layer, both satisfying Herschel-Bulkey constitutive equation.

\[ \tau = \mu e^n + \tau_y \]

has been considered for the analysis. Apparent fluidity of the blood and flow rate have been discussed with respect to peripheral layer thickness, yield stress and parameter n.

6.1 INTRODUCTION: A study of blood flow through narrow circular tubes is very important from physiological point of view. From several studies it has been established that for blood flowing in narrow vessels, a cells free plasma layer exists near the wall.
Bugliarell and Sevilla (1970) have proposed two fluid models, in which either both layers (peripheral plasma layer and core hematocrit layer) are of Newtonian fluid with different viscosities or both are of Casson fluid with different yield stresses and viscosities. Scott Blair and Spanner (1974) reported that in normal range of shear rates (except at very high and very low shear rates there is no difference in Casson models and Herschel-Bulkley models of experimental Data and in some cases, the latter model gives better results also. Herschel-Bulkley equation has one more additional parameter n than Casson equation. Therefore we can obtain more informations by using this equation.

Many authors have the view that the slip at the solid surface has no evidence. Therefore in our analysis we have considered two layer flow model of blood, both satisfying the Herschel-Bulkley constitutive equation of motion with no slip at the wall. Velocity filed and apparent fluidity have been obtained. Results are discussed
for different values of \( n \) and compared when peripheral plasma layer follows the Newtonian equation.

6.2 MATHEMTICAL ANALYSIS— In the analysis, a two layer fluid model of blood with a central core region of \( R_0 \) radius and a peripheral layer of thickness \( \delta = R - R_0 \) both satisfying Herschel Bulkley equation.

\[
\tau = \mu e^n + \tau_y \quad (1)
\]

is considered

Where \( \tau \) represents shear stress, \( \tau_y \) the yield stress, \( \mu \) the coefficient of viscosity, \( e \) the shear strain rate and \( n \) (positive) the parameter representing the Non-Newtonian effect.

The equations of motion and continuity for fully developed steady viscous incompressible laminar flow in the cylindrical co-ordinate system \( (r, \theta, z) \) whose origin lies on the axis of the vessel, are

\[
\rho = -\frac{\partial P}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) \quad (2)
\]
\[ O = \frac{\partial p}{\partial r} \]  

(3)

and

\[ \frac{\partial v}{\partial z} = 0 \]  

(4)

where \( p \) is pressure, \( \tau_{rz} \) the shear normal to \( r \) in \( z \) direction, \( v \)-the axial velocity Integration of equation (2) gives

\[ r \tau_{rz} = \frac{r^2}{2} - \frac{\partial p}{\partial z} + c \]

or

\[ \tau = \frac{r}{2} \frac{\partial p}{\partial z} + \frac{c}{r} \]

where \( C \) is a constant.

Assuming that the stress be finite on the axis, (at \( r = 0 \))

We find that \( c = 0 \), hence

\[ \tau_{rz} = \frac{r}{2} \frac{\partial p}{\partial z} \]  

...(5)
If \( \tau_{r_1}, \tau_{r_2}, V_1, \mu_1 \) represent values of the corresponding quantities in peripheral region and \( \tau_{r_2}, \tau_{r_y}, V_2, \mu_2 \) represent the values of corresponding quantities in core region.

Since

\[
e = -\frac{dv}{dr}
\]

Equation (1) gives

\[
\frac{dv_1}{dr} = -\left\{ \frac{1}{\mu_1} \left( \frac{\partial p}{\partial z} - \tau_{r_1} \right) \right\}^{\frac{1}{n}}_{r_0 \leq r \leq R} \quad \text{(6)}
\]

\[
\frac{dv_2}{dr} = -\left\{ \frac{1}{\mu_2} \left( \frac{r}{2} \frac{\partial p}{\partial z} - \tau_{r_y} \right) \right\}^{\frac{1}{n}}_{r_0 \leq r \leq R} \quad \text{(7)}
\]

\[
\frac{dv_y}{dr} = 0 \quad \text{for} \quad 0 \leq r \leq R_y \quad \text{(8)}
\]

Where \( V_y \) is the plug velocity. Integrating equations (6), (7), (8) and applying boundary conditions,

\[
\begin{align*}
\tau_{r_1} &= \tau_{r_2}, \quad V_2 = V_1 \text{ at } r = R_0 \\
V_1 &= 0, \text{ at } r = R \\
e &= 0, \quad \tau_{r_2} = \tau_{r_y}, \quad V_2 = V_y \text{ at } r = R_y
\end{align*}
\]

(9)
We obtain

\[ V_1 = \frac{n}{n+1} \left( \frac{\tau_R}{\mu_1} \right)^\frac{1}{n} R \left\{ (1 - \beta_1)^{\frac{n+1}{n}} - \left( \frac{r}{R} - \beta_1 \right)^{\frac{n+1}{n}} \right\} \quad (10) \]

\[ V_2 = \frac{n}{n+1} \left( \frac{\tau_R}{\mu_2} \right)^\frac{1}{n} R \left\{ (x - \beta_2)^{\frac{n+1}{2}} - \left( \frac{r}{R} - \beta_2 \right)^{\frac{n+1}{n}} \right\} \]

\[ + \frac{n}{n+1} \left( \frac{\tau_R}{\mu_1} \right)^\frac{1}{n} R \left\{ (1 - \beta_1)^{\frac{n+1}{n}} - (x - \beta_1)^{\frac{n+1}{n}} \right\} \quad (11) \]

\[ V_y = \frac{n}{n+1} \left( \frac{\tau_R}{\mu_1} \right)^\frac{1}{n} R \left\{ (1 - \beta_1)^{\frac{n+1}{n}} - (x - \beta_1)^{\frac{n+1}{n}} \right\} \]

\[ + \frac{n}{n+1} \left( \frac{\tau_R}{\mu_2} \right)^\frac{1}{n} R (x - \beta_1)^{\frac{n+1}{n}} \quad (12) \]

Where \( x = \frac{R_0}{R}, \ \beta_1 = \frac{\tau_1}{\tau_R}, \ \beta_2 = \frac{\tau_2}{\tau_R} \)

\( \tau_R \) the shear stress at the wall, Flow rate is given by

\[ Q = \int_0^R 2\pi V \cdot rdr \]

\[ = \pi R_y^2 V_y + \int_{R_y}^{R_0} 2\pi V_2 \cdot rdr + \int_{R_y}^{R_0} 2\pi V_1 \cdot rdr \]

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\[ Q = \frac{\pi n}{2(n+1)} \left\{ \frac{\tau_R}{\mu_1} \right\} \frac{1}{n} R^3 \left[ 2 \left( 1 - \beta_2 \right) \frac{n+1}{n} - 2x^2 \left( x - \beta_1 \right) \frac{n+1}{n} \right] \]

\[-\frac{4n}{2n+1} \left\{ \left( 1 - \beta_1 \right) \frac{n+1}{n} - \left( x - \beta_1 \right) \frac{2n+1}{n} \right\} + \frac{4n^2}{(2n+1)(3n+1)} \left\{ \left( 1 - \beta_1 \right) \frac{3n+1}{n} - \left( x - \beta_1 \right) \frac{3n+1}{n} \right\} \]

\[ + \left( \frac{\mu_1}{\mu_2} \right) \frac{1}{n} \left\{ 2x^2 \left( x - \beta_2 \right) \frac{n+1}{n} - \frac{4nx}{2n+1} \left( x - \beta_2 \right) \frac{2n+1}{n} \right\} + \frac{3n^2}{(2n+1)(3n+1)} \left( x - \beta_2 \right) \frac{3n+1}{n} \]

We obtain the apparent fluidity as

\[ \phi_a = \frac{1}{\mu_1} \left[ 2 \left( \frac{1+2n-2n^2}{n(2n+1)} \beta_1 \right) + 2 \left( \frac{\mu_1}{\mu_2} \right) \frac{1}{n} \left\{ \frac{n+1}{3n+1} - \frac{n+1}{2n+1} \beta_2 \right\} \right] \]

\[ + \frac{2\delta}{R} \left\{ M_1 - \left( \frac{\mu_1}{\mu_2} \right) \frac{1}{n} M_2 \right\} \]

(14)

Where

\[ M_1 = \frac{4n^2 + 3n + 1}{n(2n+1)} - \frac{2n^2 + n + 1}{n^2} \beta_1 \]

(15)

\[ M_2 = \frac{3n^2 + 1}{n(2n+1)} + \frac{4n^3 - 2n^2 - 3n - 1}{n^2(2n+1)} \beta_2 \]

(16)

From equation (14) we obtain that apparent fluidity increases when \( \delta \) increases.
Which follow the observed trend of the blood rheology in narrow tubes.

6.3 DISCUSSION: Many blood flow models through narrow vessels with different boundary conditions have been proposed in the blood rheology. In present paper we discuss for two different cases.

Case 1: When $\beta_1 = 0$, i.e. when peripheral layer is Newtonian and

Case 2: When $\beta_1 \neq 0$, i.e. when peripheral layer is Non Newtonian

By plotting graphs in $r/R$ and velocity $V$, we find that velocity profile is blunted near the axis of the tube and decreases slowly with respect to $r/R$ i.e. with respect to $r$ in the core region. In the peripheral region velocity decreases fastly. The trends of the curves are same in both cases ($\beta = 0, \beta_1 \neq 0$). The values obtained for $\beta_1 = 0$ are greater than those obtained for $\beta_1 \neq 0$. 

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Also by plotting graphs for apparent fluidity $\phi_a$ with parameters $n$, peripheral layer thickness $\delta$ and yield stress $\beta_2$. We observe that $\phi_a = \text{increases with } n \text{ up to } n = 1.6 \text{ and decreases for higher values of } n (n > 1.6)$. $\phi_a$ increases with $\delta$ and decreases with respect to $\beta_2$. From graphs we also observe that values obtained for $\beta_1 = 0$ are greater than those obtained for $\beta_1 \neq 0$. This shows that yield stress of the peripheral fluid causes a loss in apparent fluidity of the blood.
6.4 REFERENCES:


$r = 0.2$

$0.10 = \frac{R}{R}$

$I = u$ $\% = H$

FIG. 6.1: Velocity distribution
Fig. 6.2: Variation of apparent fluidity with parameter \( n \).

\[ \mu_1 = 1.2 \text{ centipoise} \]
\[ \beta_2 = 2.2 \text{ centipoise} \]
\[ \beta_2 = 0.2, \delta R = 0.10 \]
Figure 6.3: Variation of apparent mobility with peripheral layer thickness.

\[ \frac{\rho}{R} \]

\[ \phi = 0.20 \]

\[ \tau = 0.01 \]

\[ \eta = 1.2 \text{ centipoise} \]

\[ \eta = 1.2 \text{ centipoise} \]

\[ \rho = 0.00 \]

\[ \phi = 0.00 \]
Fig. 6.4: Variation of apparent density with yield stress.

\[ p^2 \]

- \[ p^1 = 0.10 \]
- \[ p^1 = 0.00 \]

\[ \phi = 0.15 \]
\[ \phi = 0.12 \] centrifuged
\[ \phi = 0.11 \] centrifuged
\[ \phi = \]