Chapter -3

BEHAVIOR OF BLOOD FLOW IN VERY NARROW CAPILLARIES
Chapter-III

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3.1 Introduction

On microscopic scale the movement of blood through very narrow capillary must involve the passage of individual red cells in single file along it, each separated from one by a bolus of viscous fluid, Prothero and Burton (9 and 10). It is expected that the calls are deformed elastically to enable them to pass through the tube which also suffer some small elastic distension. The purpose of the study is to get some qualitative insight into the problems of flow in tubes under consideration where the concentration of lubricating film of plasma important in region to mass transfer and to hydraulic resistance as well as to the relative residence times of red
cells and plasma in capillary network. Prothero and Burton (9) pointed out that the bolus of viscous plasma between two red cells must perform relative to their motion, a toroidal circulation, toward on the tube axis and backward near the walls. Prothero and Burton estimated, from their model experiments, the pressure drop in the bolus of moving plasma between two red cells and deduced a contribution to over all capillary resistance less than that given by Poiseuille law, mainly because plasma with depleted RBC has a viscosity considerably lower than typical values measured for whole blood.

From the available experimental data it is observed that the flow resistance innanow capillaries is greater than that of plasma and that it may depend on the hematocrit, flow rate and capillary diameter. Whitemore (15) attempted the problem giving, an axial
train model. The model consists of a central core of cells (in single file) and plasma moving with a uniform speed, with a surrounding annulus of plasma in which the shearing occurs. The model is applicable in capillaries whose diameters are slightly greater than those of the RBC. Another model was developed by Bloom (1) using a rigid pill box model for the cell Lighthill (4) developed an elastohydrodynamic lubrication model for the deformation of the RBC and the flow between the cell and the capillary wall Fitz Gerald (2 and 3), Vand and Fitz–Gerald (14) refined Lighthill's model and found an analytical solution for an idealized bolus flow model. The bolus flow models generally agree that the mean resistance across the bolus increases as the bolus is decreased. Thus, the cell spacing plays an important role in the contribution to the resistance offered by the plasma in the boli. Tozeren and Skalak (13) have developed a mathematical model for the flow of
closely fitting incompressible elastic spheres in a tube under zero
drag condition (under this condition it is assumed that the resultant
force on the particle due to pressure and viscous stresses exerted
by the fluid is equal to zero. This condition of zero drag on the
neutrally buoyant particle is formulated by considering the
equilibrium of a control volume bounded by the tube wall, particle
surface and two planes tangential to the particle at the downstream
and upstream ends).

Calculations of Fitz-Gerald lubrication model for the flow of
plasma around the RBC and the deformation of cell predicted
lower resistances. A choice of membrane deformation resistance
between the value measured by Rand and Burton {11} and that
predicted by Lingård {5-8} gave better agreement between
resistance data obtained experimentally and those predicted by
Fitz-Gerald {2 and 3} and Sugihara et al. {12}. 
3.2 Mathematical Analysis:

We consider the flow of elastic incompressible sphere in a rigid tube of uniform radius. Single file flow of RBC surrounded an annulus of plasma is considered. In the case of movable buoyant particle, treated in the present study, the condition of zero-drag on the particle must be satisfied in addition to the Reynolds equation. It can be used to eliminate leak-back (which is equal to the discharge of the fluid observed relative to a reference frame fixed to the particle) leaving only pressure drop as an unknown.

Figure-3.1
The single RBC of biconcave disk shape is deformed during the flow passage in very narrow capillary, as shown in Figure-3.1. It is assumed that the inertial terms are negligible, the equation of motion in cylindrical polar co-ordinate about the axis of symmetry is

\[
\frac{\partial p}{\partial z} = \frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_r}{\partial r} \right)
\]  

(3.1)

The equation of continuity valid in the fluid is

\[
\frac{\partial}{\partial r} (r, v_r) + \frac{\partial}{\partial z} (r v_z) = 0
\]  

(3.2)

where \( v_r, v_z \) are radial and axial velocity, respectively, \( \mu \) the dynamic viscosity of fluid, \( p \) the pressure not varying with \( r \).

Boundary conditions are

when \( r = R_0; \quad v_z = -v_0; \quad v_r = 0 \)  

(3.3)

\[
r = R_C(z); \quad v_z = 0; \quad v_r = 0
\]  

(3.4)
\[ P(-b) - P(b) = \Delta P_0; \quad h = R_0 - R_c(z) \quad (3.5) \]

From incompressible continuity condition (3.2), we have

\[- \int r \frac{\partial}{\partial r} (r \nu_r) \, dr = \int \frac{\partial}{\partial z} (r \nu_z) \, dr \]

or

\[ \frac{\partial}{\partial z} \int_{R_c}^{R_0} r \, \nu_z \, dr = - \int_{R_c}^{R_0} r \frac{\partial}{\partial r} \nu_r \, dr \]

\[ = - \left. (r \nu_r) \right|_{R_c}^{R_0} \]

\[ = - \left| 0 - R_c \cdot 0 \right| \]

\[ = 0 \]

Further,

\[ \int_{R_c}^{R_0} r \, \nu_z \, dr = C = -R_n Q_0 \quad (3.6) \]

Where \( Q_0 \) is the leak-back, given by

\[ 2 \pi R_0 Q_0 = \pi R_0^2 U - \pi R_0^2 V_0 \quad (3.7) \]

\( V_0 \) is the average velocity of the fluid in lubricating zone.
From equation (3.7) \[ Q_0 = \frac{R_0}{2} (U_0 - V_0) \] (3.8)

Integrate equation (3.1) and get,

\[ v_z = \frac{1}{4\mu} \frac{dp}{dz} r^2 + A \log r + B \] (3.9)

Where \( A \) and \( B \) are constants to be determined with boundary conditions (3.3) (3.4), as

\[
A = \left[ -U_0 - \frac{1}{4\mu} \frac{dp}{dx} (R_0^2 - R_C^2) \right] \log \frac{R_0}{R_C}
\]

and

\[
B = \frac{1}{4\mu} \frac{dp}{dz} R_C^2 \left[ -u_0 - \frac{1}{4\mu} \frac{dp}{dx} (R_0^2 - R_C^2) \right] \log R_C \log \frac{R_0}{R_C}
\]

Put the value of \( A \) and \( B \) in equation (3.9) and get

\[
v_z = \frac{1}{4\mu} \frac{dp}{dz} \left[ r^2 - R_C^2 + \frac{(R_0^2 - R_C^2)}{\log \frac{R_C}{R_0}} \log \frac{r}{R_C} \right] + u_0 \frac{\log \frac{r}{R_C}}{\log \frac{R_C}{R_0}} \] (3.10)
If we put $R_0 = R_c + h$, then equation (3.10) becomes

$$v_z = \frac{1}{4\mu} \frac{dp}{dz} \left[ r^2 - R_c^2 + \frac{(2R_c h + h^2)}{\log \left( 1 + \frac{h}{R_c} \right)} \log \frac{r}{R_c} \right] + \frac{U_0 \log \frac{r}{R_c}}{\log \left( 1 + \frac{h}{R_c} \right)} \quad (3.11)$$

Integrating equation (3.11) and using equation (3.6), we obtain

$$\frac{dp}{dz} = \frac{16\mu R_0 Q_0}{R_0 - U_0} \left[ R_0^2 + \frac{R_0^2 - R_c^2(z)}{\log \frac{R_c(z)}{R_0}} \right] \quad (3.12)$$

Under zero-drag condition, we have,

Pressure force acting on the particle + viscous stresses experienced by the particle = 0

i.e. $\int_{-b}^{b} R_c^2(z) \frac{dp}{dz} dz - 2\pi \mu \int_{-b}^{b} R_c(z) \left( \frac{\partial v_z}{\partial r} \right)_{r=R_c(z)} dz = 0 \quad (3.13)$
If $\Omega$ is fluid volume, then from equation (3.1) we have

$$\int_{\Omega} \frac{dp}{dz} \ d\Omega = \mu \int_{\Omega} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \vec{v}_z}{\partial r} \right) \ d\Omega$$

$$\left( 3.14 \right)$$

$$\frac{1}{\mu} \left\{ \pi R_0^2 \left[ P(b) - P(-b) \right] - \pi \int_{-b}^{b} R_c^2(z) \frac{dp}{dz} \ dz \right\}$$

$$= 2\pi R_n \int_{-b}^{b} \left( \frac{\partial \vec{v}_z}{\partial r} \right)_{r = R_0} \ dz - 2\pi \int_{-b}^{b} R_c(z) \left( \frac{\partial \vec{v}_z}{\partial r} \right)_{r = R_c(z)} \ dz$$

$$\left( 3.15 \right)$$

In view of equation (3.13), equation (3.15) can be written as

$$\pi R_n^2 \Delta P_0 = -2\pi \mu R_0 \int_{-b}^{b} \left( \frac{\partial \vec{v}_z}{\partial r} \right)_{r = R_0} \ dz$$

$$\left( 3.16 \right)$$

Using the quantity $\mu \overline{U}/a$ in the pressure and stress terms, the non-dimensional quantities becomes

$$\overline{b} = \frac{p a}{\mu \overline{U}}, \quad \overline{r} = \frac{r a}{\mu \overline{U}}, \quad \overline{U}_0 = \frac{U_0}{\overline{U}}, \quad \overline{v}_z = \frac{v_z}{\overline{U}}; \quad \overline{z} = \frac{z}{R_0}, \quad \overline{\varphi} = \frac{r}{a}$$

$$\overline{R}_c(z) = \frac{R_c(z)}{R_0}, \quad H = \frac{h_0}{R_0}, \quad \alpha = \frac{a}{b}, \quad \beta = \frac{a}{R_0}, \quad C_0 = \frac{2Q_0}{UR_0}$$

$$\left( 3.17 \right)$$
Velocity field $v_z$ and pressure gradient $dp/dz$ given by equations (3.11) and (3.12) take the forms,

$$
\bar{v}_z(\bar{r}, \bar{z}) = \frac{1}{4\beta} \frac{d\bar{p}}{d\bar{z}} \left[ \beta^2 \bar{r}^2 - \bar{R}_c^2(z) + \frac{1 - \bar{R}_c^2(z)}{\log R_c(z)} \log \frac{\beta \bar{r}}{R_c(z)} \right] + \bar{U}_0 \frac{\log \frac{\beta \bar{r}}{R_c(z)}}{\log \bar{R}_c(z)}
$$

(3.18)

$$
\frac{d\bar{p}}{d\bar{z}} = 8\beta \frac{C_0 - \bar{U}_0 \left(1 + \frac{1 - \bar{R}_c^2(z)}{2 \log \bar{R}_c(z)}\right)}{\left[1 - \bar{R}_c^2(z)\right] \left[1 + \bar{R}_c^2(z) + \frac{1 - \bar{R}_c^2(z)}{\log \bar{R}_c(z)}\right]} \tag{3.19}
$$

For the sake of convenience we omit bars in proceeding expressions.

Equation (3.18) gives

$$
\frac{\partial v_z}{\partial r} = \frac{1}{4\beta} \frac{dp}{dz} \left[ 2 \beta^2 r + \frac{1 - R_c^2(z)}{\log R_c(z)} \right] + \frac{U_0}{\log R_c(z)} \tag{3.20}
$$
with the help of equations (3.20) and (3.16) we obtains

\[
\Delta P_0 = 4\beta \int_{-\beta/a}^{\beta/a} \left[ \frac{2 + \frac{1 - R_c^2(z)}{\log R_c(z)}}{\log R_c(z)} \right] \left[ C_0 - U_0 \left( 1 + \frac{1 - R_c^2(z)}{2 \log R_c(z)} \right) \right] + \frac{U_0}{\log R_c(z)} \left( 1 + \frac{1 - R_c^2(z)}{\log R_c(z)} \right) \right] \, dz
\]

(3.20)

with the help of equations (3.20) and (3.16) we obtains

\[
\Delta P_0 = 4\beta \int_{-\beta/a}^{\beta/a} \left[ \frac{2 + \frac{1 - R_c^2(z)}{\log R_c(z)}}{\log R_c(z)} \right] \left[ C_0 - U_0 \left( 1 + \frac{1 - R_c^2(z)}{2 \log R_c(z)} \right) \right] + \frac{U_0}{\log R_c(z)} \left( 1 + \frac{1 - R_c^2(z)}{\log R_c(z)} \right) \right] \, dz
\]

(3.21)

As, \( \Delta P_0 = P\left( \frac{-\beta}{\alpha} \right) - P\left( \frac{\beta}{\alpha} \right) \)  \hspace{1cm} (3.22)

Then,

\[
\Delta P_0 = 8\beta \int_{-\beta/a}^{\beta/a} \left[ \frac{C_0 - U_0 \left( 1 + \frac{1 - R_c^2(z)}{2 \log R_c(z)} \right)}{\log R_c(z)} \right] \left( 1 + \frac{1 - R_c^2(z)}{\log R_c(z)} \right) \, dz
\]

(3.23)
If we put,

\[ D_{11} = 4 \beta \int_{-\beta / \alpha}^{\beta / \alpha} \frac{2 + \frac{1 - R_c^2(z)}{\log R_c(z)}}{1 - R_c^2(z)} \frac{1}{1 + R_c^2(z) + \frac{1 - R_c^2(z)}{\log R_c(z)}} \, dz \]  

\[ (3.24) \]

\[ D_{12} = 4 \beta \int_{-\beta / \alpha}^{\beta / \alpha} \frac{\left(2 + \frac{1 - R_c^2(z)}{\log R_c(z)}\right) \left(1 + \frac{1 - R_c^2(z)}{2 \log R_c(z)}\right)}{1 - R_c^2(z)} \frac{1}{1 + R_c^2(z) + \frac{1 - R_c^2(z)}{\log R_c(z)}} - \frac{1}{2 \log R_c(z)} \, dz \]  

\[ (3.25) \]

\[ D_{21} = 8 \beta \int_{-\beta / \alpha}^{\beta / \alpha} \frac{dz}{(1 - R_c^2(z)) \left(1 + R_c^2(z) + \frac{1 - R_c^2(z)}{\log R_c(z)}\right)} \]  

\[ (3.26) \]

\[ D_{22} = 8 \beta \int_{-\beta / \alpha}^{\beta / \alpha} \frac{\left(1 + \frac{1 - R_c^2(z)}{2 \log R_c(z)}\right)}{(1 - R_c^2(z)) \left(1 + R_c^2(z) + \frac{1 - R_c^2(z)}{\log R_c(z)}\right)} \, dz \]  

\[ (3.27) \]

then equation (3.21) takes the form

\[ D_{11} C_0 + \Delta P_c = D_{12} U_0 ; \]  

\[ (3.28) \]
\[ D_{21}C_0 + \Delta P_0 = D_{22}U_0; \]  

(3.29)

For \( U_0 = 1 \), above equations give

\[ C_0 = \frac{D_{12} - D_{22}}{D_{11} - D_{21}}; \quad \Delta P_0 = D_{12} - D_{11}C_0 \]  

(3.30)

\[ \frac{U_0}{V_0} = (1 - C_0)^{-1}; \quad \text{effective viscosity} \quad \eta = \frac{\alpha U_0}{16 V_0} \Delta P_0 \]  

(3.31)

We have calculated the value of \( U_0/V_0 \) and \( \eta \) and compared the calculated value from the results obtained by other authors. The results are given in form of Tables.

Variation of velocity field \( v_z \) is obtained from equation (3.18). For different values of \( \alpha \) (= 1.5, 1.0, 0.5) and \( \beta \) (= 0.90, 0.95, 0.99), the variation of \( v_z \) with respect to gap thickness \( H \) have been shown in the Tables – 3.1 and 3.2.
<table>
<thead>
<tr>
<th>S.N.</th>
<th>Shape of Particle</th>
<th>$\beta$</th>
<th>$\alpha$</th>
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<td>(14)</td>
<td>(15)</td>
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# Table-3.2

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REFERENCES


