CHAPTER-I

INTRODUCTION

1.1 INTRODUCTION

This thesis broadly deals with (i) the probabilistic analysis of two-commodity continuous review inventory models and (ii) the determination of optimal continuous review inventory policies for two-commodity inventory systems.

Analysis of multi-commodity inventory systems in general is essential for any organisation because of the various types of item interdependencies, which may influence the over all cost of managing the inventories. Various factors responsible for item interdependencies, as identified by Silver (1981) include: (i)complementary demands: which means certain products tend to be demanded together, infact customer may not accept one without the other, (ii) substitutability of items: when an item of customer's choice is out of stock, the customer may opt for the available substitutes,(iii) co-ordinated control of total cost on inventory system, that is the simultaneous or joint control of inventories of various items taking into consideration the over all budgetory and space constraints which may save substantially the replenishment costs.

The various stochastic processes related to the above factors such as the arrival processes, demand processes, and/or inventory level processes will not, in general, be real-valued and hence the properties of vector-valued continuous time and discrete time stochastic processes are to be used for the study of transient and steady state distribution of the inventory level processes. For instance, in the case of complementary demands,the arrival processes themselves are dependent; in the case of substitutable items,even if the arrival processes are independent the demand processes may be dependent; and in the co-ordinated control situations the arrival processes may be independent, the demand processes may be independent, but the inventory level processes will be dependent.

Thus, in any one of the above cases the inventory level processes which are the central processes for the study of inventory models are to be treated as multi-variate continuous time stochastic processes. Because of the structural complexities involved in the inventory level processes, there is not enough work on multi-commodity continuous review stochastic inventory systems.

However, there are some important papers in this field, namely Balintfy(1964), Ignall(1966),Johnson(1967),Sivazlian(1967)

Glassey(1968), Ignall(1969), Sivazlian(1970), Schrady and Choe(1971), Goswick(1972), Naddor(1975), Sivazlian(1975), Deuermeyer(1979), Graves(1980), McKinney(1980), Deuermeyer (1980), Dogramali, et al (1981), Dixon and Silver (1982), Gascon (1984), Zipkin(1985), Roundy(1986), Eppen and Martin(1987), Mitchell(1988), Gascon and Leachman(1988), and Cohen et al(1992). Most of the above papers deal with periodic review models, since in this case one has to consider only bivariate or multivariate distributions instead of vector-valued processes.

Some work which is directly related to the topic of this thesis is by Balintfy (1964), Silver(1965), Sivazlian(1975) who studied multi-commodity continuous review models without substitutability and by Ignall and Veinott (1969) who considered periodic review models for substitutable items for the first time.

In the case of single commodity continuous review inventory models, an inventory policy is denoted by (s,S), where s is called the *re-order* point and S is called the *maximum inventory level*. Q = S-s being the ordering quantity. In m-commodity continuous review models, an inventory policy can be denoted by $\{(s_i, C_i, S_i),\ i=1,2,\dots,m\}$, where s_i, are called the *re-order points*, S_i, are called the *Maximum-inventory* levels, and C_i,

$s_\iota \le C_\iota \le S_\iota$ are called *Can-order points* for Commodity-ι, for $\iota=1,2,\ldots,m$. The operation of the policy $\left\{(s_\iota, C_\iota, S_\iota),\ \iota=1,2,\ldots,m\right\}$ is as follows:

As and when inventory level of any of the commodities reaches its re-order point,an order is placed for that commodity, and simultaneously order is placed for those other Commodities for which inventory level is less than or equal to the respective *Can-order points*; bringing the inventory levels of all the Commodities to their respective maximum inventory level , S_ι. Naturally, the quantity ordered may be a random variable for some or all the Commodities.

From the above description, it is clear that, even for two commodity models, that is , when m=2, the number of parameters of the policy is six; and in order to determine the optimal policies a non-linear objective function is to be minimized/maximized with respect to six integer-variables. This is another complexity involved in the analysis of multi-commodity systems, in addition to the dependence structure of the various stochastsic processes.

Once again, we consider the inventory policy $\left\{(s_\iota, C_\iota, S_\iota),\ \iota=1,2,\ldots,m\right\}$. If $C_\iota=s_\iota$, for $\iota = 1,2,\ldots,m$; the inventory policy is called the *individual ordering policy*. If $C_\iota = S_\iota$, for $\iota = 1,2,\ldots,m$; the inventory policy is called the *random joint ordering policy*. These are the only two policies which are considered in the literature so far for continuous review models.

The random joint ordering policy, in this thesis, is called as the Min-Policy in view of the fact that there can be a variety of joint ordering policies. For instance, even when m=2, the following alternate policies can be considered:

(a)Place the order for all the commodities when the inventory level of specified commodity say Commodity-1, reaches its re-order point.

(b)Max-Policy: Place the order only when state of the inventory levels of all the commodities reaches their respective re-order points.

When $m \geq 2$ there can clearly be a variety of other joint ordering policies, in addition to the policies mentioned above and earlier. The inventory model for two-commodity substitutable items can be described as follows:

Suppose that $\{A_\iota(t): t \geq 0\}$, $\iota = 1,2$ be the arrival processes of customers. Now, the i-th customer for Commodity-1, has probability p_{1i} of choosing Commodity-1 itself or can opt for

Commodity-2 with probability $1-p_{1i}$. Similarly, the i-th customer of Commodity-2 has probability p_{2i} of choosing the same commodity or can opt for other commodity with probability $1-p_{2i}$. The sequences $\{p_{1i}\}$ and $\{p_{2i}\}$ are called the substitutability factors. These probabilities, in addition to being dependent on the customer, can also depend explicitly on the individual choices of other earlier customers or implicitly through inventory levels of each item. Thus $\{p_{1i}\}$ and $\{p_{2i}\}$can themselves be random processes. Even in the simplest case, when p_{1i} and p_{2i} do not depend on i, that is, will be the same for all customers, and when they are not random, the demand Process $\{D_i(t):t \geq 0\}$ are dependent. Some of the joint inventory policies that are applicable for substitutable items are:

(i) placing the order for both the commodities whenever the inventory level of any commodity drops to its re-order point (Min-Policy);

(ii) Place the order whenever the inventory level of a specified commodity reaches its re-order point.

The description of the substitutability factors for m-commodity ($m \geq 2$) is more complicated. To the best of our knowledge there is no work on continuous review models for substitutable items. The substitutable inventory problem for single period inventory systems with stochastic demand was considered by Ignall and Veinott(1969), McGillivray and Silver(1978), Deuermeyer(1980), Parlar and Goyal(1984), Parlar(1985), Parlar(1988) and Pasternack and Drezner(1991).

Thus the analysis of multi-commodity inventory models with or without substitution differs from that of single commodity models. Essentially, the difference occurs because of the following reasons: one, the arrival process of customers $\{\underline{A}(t):t\geq 0\}$ may be independent, but the demand processes $\{\underline{D}(t): t\geq 0\}$ need not be, whereas the inventory level process $\{\underline{I}(t) : t\geq 0\}$ will be dependent; two, for Multi-Commodity inventory systems, there may be more than one replenishment policies; and third, the objective function (the long run cost per unit time) will, naturally, be a function of more than one variable.

In this thesis we have considered only two-commodity inventory models with and without substitutability. The following assumptions are common to all the models considered herein.

(i) arrival processes are independent Poisson processes,

(ii) depletion of inventory is due to demands only, that is, there is no deterioration,

(iii) instantaneous delivery of orders, that is, there is no lead time,

(iv) the re-order points of both the commodities, in general, taken to be zero, and hence the maximum inventory levels are denoted by Q_1 and Q_2 instead of S_1 and S_2.

Under the above assumptions, the bivariate process of inventory levels is studied for different inventory policies and for each policy, the long run cost per unit time is derived. Following is the brief summary of the thesis.

1.2 ORGANISATION OF THE THESIS

The thesis contains five chapters besides the present Chapter-I. This chapter provides the basic results needed in later chapters. These results may be found in Ross(1970) Sivazlian (1975), Adke and Manjunath (1984).

In Chapter-II, we consider a Two-Commodity inventory model, in which the replenishment of stock is accomplished by using the Min-Policy, that is, an order is placed for both the commodities simultaneously whenever the inventory level of any one of the commodities drops to its re-order point bringing their respective levels to Q_1 and Q_2. Under this assumption, the quantity ordered $\langle O_1, O_2 \rangle$ is a random vector with possible values $\langle m, Q_2 \rangle$ or $\langle Q_1, n \rangle$ for $m = 0,1,2,\ldots,Q_1-1$ and $n = 0,1,2,\ldots,Q_2-1$; where O_ι is the quantity ordered for Commodity-ι, $\iota = 1,2$. We assume that there are no shortages of either commodity as well as

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that there is no lead time. For this model we have obtained the transient distribution of the inventory level in terms of Laplace Transforms, the steady state distribution of the inventory level, and the distribution of the order level. Also the objective function and an algorithm to find the optimum ordering quantities are given. At the end, an application to substitutable items is described. The same model was studied by Ignall (1969), but our results are different from Ignall's results. He has obtained the optimal ordering policy when the two products have the same maximum inventory levels and also that the holding costs are the same. He did not obtain the transient solution for the distribution of the inventory level.

Chapter-III deals with a model considered in Chapter-II, but the replenishment policy is different . Here in this chapter, order is placed if and only if the inventory level drops to re-order point for both the commodities. We call this as Max-Policy. In this model, the transient and the steady state distributions of the inventory levels are discussed. Objective functions have been given for the two cases namely, (i) when the sales are lost and (ii) when the unmet demands are backlogged. Also an algorithm is developed to compute the optimum ordering quantities.

In Chapter-IV, we have considered a two-commodity inventory system with one-way substitutability. In this model we assume that the customers who have arrived for Commodity-2 will opt for Commodity-1, if the Commodity-2 is out of stock, but not vice versa. Thus in this model, one-way substitutability is enforced. Hence the replenishment of stock is accompolished whenever the level of Commodity-1 drops to zero, to bring their respective levels to Q_1 and Q_2. Thus, the ordering quantity for Commodity-1 is fixed where as for Commodity-2 it is random. For this model, the transient distribution of the inventory level in terms of Laplace Transform, the steady state distribution of the inventory level, distribution of the order level, and an expression for the objective function are obtained.

Chapter-V deals with the problem of obtaining optimum ordering quantities for Commodity 1 and 2 with the assumption that an arrival will demand a Commodity-1 with probability p, which is $\lambda_1/(\lambda_1 + \lambda_2)$ and Commodity-2 with probability 1-p. An order is placed for both the commodities simultaneously whenever the inventory level of both the commodities drops to zero. In this model we have assumed that, if the customer demands a commodity which is out of stock, it will be substituted by another available commodity. This model is different from the

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model considered by Ignall and Veinott (1969), Deuermeyer (1980), McGillivray and Silver (1978), Goyal and Parlar (1984), Parlar (1988), and Pasternack and Drezner (1991) in the sense that all the above authors deal with single-period model with quantity demanded being random, where as our model deals with continuous review infinite horizon model with unit demand. Here also we have obtained the transient distribution of the inventory level, limiting distribution of the inventory level, and an expression for the long run cost per unit time.

In Chapter-VI, we have considered two-commodity inventory model where the depletions are due to demands only. The replenishment of stock is done by using the individual ordering policy, that is an order is placed for a commodity as and when the inventory level of that commodity drops to zero. Contrary to our expectation this model turned out be the most complicated of all the models. Hence only the steady-state distribution is obtained for the general case and transient distribution for some particular cases.

The table given on the next page summarises the results obtained in various chapters. Overall, this study deals with probabilistic analysis of inventory level process; the problem of determining optimal inventory policies is not considered in its full depth; as well as that of comparing the optimal policies of various types, such as min-policy, max-policy, etc.

It is hoped that many of the results contained in this thesis are first of its kind for two-commodity models with or without substitutability which of course needs to be extended in various directions.

SUMMARY TABLE

Model	State space of Inv. Level	Nature of Re-order epoch	Nature of Qty ord
Min-Plcy (Ch-II)	$\left\{\langle x,y\rangle : \begin{matrix} x=1,2,..,Q_1 \\ y=1,2,..,Q_2 \end{matrix}\right\}$	$S_{\iota 1}=\min\{t>0: A_\iota(t)=Q_\iota\}$ for $\iota=1,2$; $S_1=\min\{S_{11},S_{21}\}$	
		$S_k=\min\left\{t>S_{k-1}: I_\iota(t)=Q_\iota \text{ s.t } I_\iota(t-)\neq Q_\iota\right\}$ $\iota=1,2$ $k=2,3,.....$	Random
Max-Plcy (a) Backlog (Ch-III)	$\left\{\langle x,y\rangle : \begin{matrix} x\le Q_1 \\ y\le Q_2 \end{matrix} \;\&\langle x,y\rangle\neq\langle 0,0\rangle\right\}$	$S_{\iota 1}=\min\{t>0: A_\iota(t)=Q_\iota\}$ for $\iota=1,2$; $S_1=\max\{S_{11},S_{21}\}$	
		$S_k=\min\left\{t>S_{k-1}: I_\iota(t)=Q_\iota \text{ s.t } I_\iota(t-)\neq Q_\iota\right\}$ $\iota=1,2$ $k=2,3,.....$	Random

Model	State space of Inv. Level	Nature of Re-order epoch	Nature o Qty ord
(b)			
Lost-Sl (Ch-III)	$\left\{\langle x,y\rangle : \begin{array}{l} x=0,1,..Q_1 \\ \& \langle x,y\rangle \neq \langle 0,0\rangle \\ y=0,1,..,Q_2 \end{array}\right\}$	$S_{\iota 1}=\min\{t>0: A_\iota(t)=Q_\iota\}$ for $\iota=1,2$ $S_1=\max\{S_{11},S_{21}\}$ $S_k=\min\left\{t>S_{k-1}: I_\iota(t)=Q_\iota \text{ s.t } I_\iota(t-)\neq Q_\iota\right\}$ $\iota=1,2$	Fixed
One-Way** Substit. Item (Ch-IV)	$\left\{\langle x,y\rangle : \begin{array}{l} x=1,2,...,Q_1 \\ y=0,1,...,Q_2 \end{array}\right\}$	$S_k=\min\left\{t>S_{k-1}: I_1(t)=Q_1 \; I_1(t-)\neq Q_1\right\}$	Com-1 is fixed& Com-2 is random
Two-way Substit. Items. (Ch-V)	$\left\{\langle x,y\rangle : \begin{array}{l} x=0,1,..,Q_1 \\ y=0,1,..,Q_2 \\ \langle x,y\rangle \neq \langle 0,0\rangle \end{array}\right\}$	S_k is gamma r.v with parameter $(Q_1+Q_2, \lambda_1+\lambda_2)$.	Fixed
Indiv. Order. Policy (Ch-VI)	$\left\{\langle x,y\rangle : \begin{array}{l} x=1,2,...,Q_1 \\ y=1,2,..,Q_2 \end{array}\right\}$	$S_k=\min\{t>S_{k-1}: I(t)=\langle Q_1,Q_2\rangle\}$ $k=1,2,...\&\ S_0\equiv 0$	Fixed

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Remark: Analysis of the model for this policy without substitutability is not included in the thesis, because the analysis is similar to that done in Chapter-III.

1.3 NOTATION

$\{A_\iota(t):t\geq 0\}$	: Arrival process for Commodity-ι, ι=1,2.
$\{D_\iota(t):t\geq 0\}$	: Demand process for Commodity-ι, ι=1,2.
$\{I_\iota(t):t\geq 0\}$	: Inventory level process for Commodity-ι,ι=1,2
λ_ι	: Arrival rate of customers for Commodity-ι,
$\mathcal{X}$	: State space of inventory level process I(t).
S_k	: Epoch at which the k th order is placed.
$s_{\iota k}$	: Can order point of Commodity-ι, ι=1,2.
$P_{x,y}(t)$	: $Pr[I(t)=\langle x,y\rangle]$, the prob. that inventory level at time t is $\langle x,y\rangle$.
$P'_{x,y}(t)$	: $\frac{d}{dt}P_{x,y}(t)$
$\tilde{P}_{x,y}(s)$	: Laplace Transform of $P_{x,y}(t)$.
$[x\ C\ y]$	: $\binom{x}{y}$ or ${}^xC_y$
$B[Q_1,Q_2;\ x,y;\ p]$	: $\binom{Q_1+Q_2-x-y}{Q_1-x} p^{Q_1-x} q^{Q_2-y}$
p	: $\lambda_1/(\lambda_1+\lambda_2)$
q	: 1-p

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U_s	:	$\lambda_1/(\lambda_1+\lambda_2+s)$
V_s	:	$\lambda_2/(\lambda_1+\lambda_2+s)$
$\Pi(x,y)$	:	$Pr[I = \langle x,y \rangle]$, under steady state
$E(I_\iota)$	:	Expected inventory level under steady state.
$G(u,Q_\iota)$	:	Distribution function of gamma random variable with parameter (λ_ι, Q_ι).
$\langle O_1, O_2 \rangle$	:	Ordering quantity for Commodity 1 and 2.
T_k	:	Length of the k-th cycle
H_ι	:	Holding cost per unit per unit time for Commodity-ι, $\iota=1,2$.
C_ι	:	Unit cost of Commodity-ι, $\iota=1,2$.
K	:	Fixed ordering cost.
P_ι	:	Shortage cost per unit per unit time for Commodity-ι.
f_{S_1}	:	Density function of the first cycle time.
$NB(k,\theta)$	:	Negative binomial random variable with parameter (k,θ).
$P[NB(k,\theta)=x]$	:	$\binom{x-1}{k-1} \theta^k (1-\theta)^x, \quad x=k,k+1,\ldots$

1.4 SOME PRELIMINARY RESULTS

Following are the some of the results which are often used in this thesis.

LEMMA 1.4.1: $\int_0^\infty \exp(-st)\, P'_{x,y}(t) = s\,\tilde{P}_{x,y}(s) - P_{x,y}(0)$,

where $P_{x,y}(0) = \begin{cases} 1 & \text{, if } \langle x,y\rangle = \langle Q_1, Q_2\rangle \\ 0 & \text{, otherwise.} \end{cases}$

Proof: Proof of the result can be found in Sivazlian and Stanfel(1975).

LEMMA 1.4.2: $\int_m^\infty \exp(-u)\, u^{k-1}/(k-1)!\,du = \sum_{x=0}^{k-1} \exp(-m)\, m^x/x!$

Proof: Proof of the lemma follows by integration by parts.

LEMMA 1.4.3: $\Pr[NB(k,\theta) \le x-1] = \Pr[NB(x-k,1-\theta) \ge x]$.

Proof: Proof is straight forward.

LEMMA 1.4.4: $\sum_{v=0}^{u} [(x+v-1)\, C\, v][(u-v+y)\, C\, (u-v)] = [(x+y+u)\, C\, u]$.

Proof: Proof can be found in Johnson and Kotz(1969).

LEMMA 1.4.5: $\Pr[NB(k,\theta) \le n] = \Pr[B(n,\theta) \ge k]$, where $B(n,\theta)$ is binomial random variable with parameter (n,θ).

This is a standard result.

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LEMMA 1.4.6: Let $\{X(t):t\geq 0\}$ be a finite Markov process with state space $\mathcal{X} = \{1,2,...,m,m+1,...M\}$ such that $C = \{1,2,...,m\}$ is an ergodic class and $T = \{m+1,m+2,...,M\}$ is a transient class, every state of which leads to all states in C. Then $\alpha_{jk} = \lim_{t \to \infty} p_{jk}(t) = \alpha_k$ exists and is independent of j for all $k \in \mathcal{X}$.

Proof: Proof of this can be found in Adke and Manjunath(1984).

1.5 ASSUMPTIONS

Following are the some set of assumptions made in this thesis.

Assumption 1: The arrival process $\{A_\iota(t):t \geq 0\}$ is assumed to be independent Poisson process with intensity $\lambda_\iota, \iota=1,2$

Assumption 2: Instantaneous delivery of orders. That is lead time is negligible.

Assumption 3: There is no deterioration of items. That is depletion of inventory takes place due to demand only.

Assumption 4: Replenishment of stock is accomplished using (s,S) policy.

Assumption 5: At time t=0,inventory level of Commodity 1 and 2 is Q_1 and Q_2 respectively.