CHAPTER-IV

ANALYSIS OF TWO-COMMODITY INVENTORY MODEL FOR ONE-WAY SUBSTITUTABLE ITEMS

4.1 INTRODUCTION

In this chapter we develop a general Two-Commodity economic order quantity model under general assumptions that are mentioned in Chapter-I together with an assumption that an order is placed for both commodities simultaneously whenever inventory level of a specified commodity say Commodity 1 drops to its re-order point. Following is a brief summary of each section.

In Section 4.2, the description of the model and difference differential equations are given. The steady-state distribuion of the inventory levels, distribution of order levels and the objective functions are obtained in Section 4.3. In Section 4.4 an algorithm to obtain optimal $\langle Q_1, Q_2 \rangle$ is given. In Section 4.5, a solution interms of Laplace Transforms is obtained for the difference differential equations given in Section 4.2.

4.2 THE MODEL AND DIFFERENCE DIFFERENTIAL EQUATIONS

4.2.1 DESCRIPTION OF THE MODEL:

Consider a Two-Commodity inventory model in which depletions for Commodity-1 and Commodity-2 are due to demands only. The respective arrival processes $\{A_\iota(t) : t \geq 0\}$ are assumed to be independent Poisson processes with rate λ_ι, $\iota = 1,2$ as mentioned in the Chapter-I.

Suppose that the customers who have arrived for Commodity 2 will opt for Commodity 1, if the Commodity 2 is out of stock but not vice-versa. Therefore in this model shortages do not occur and one-way substitutability is enforced.

Here the replenishment of stock is accomplished whenever the inventory level of specified Commodity-1 reaches to its re-order point (zero), that is, an order is placed for both the commodities simultaneously so as to bring their respective levels to Q_1 and Q_2. Under this assumption, the quantity ordered $< Q_1, O_2 >$ is a random vector with possible values $< Q_1, n >$ for $n = 0,1,2\ldots,Q_2$, where O_2 is the quantity ordered for Commodity-2.

Suppose that the lead time is zero. As mentioned in previous chapters, here also, the processes $\{I_\iota(t) : t \geq 0\}$ corresponding to the inventory levels are governed by the arrival processes $\left\{A_\iota(t) : t \geq 0\right\}$ for $\iota = 1,2$. Further, the inventory level processes $\{I_\iota(t):t\geq 0\}$, $\iota=1,2$ are not independent. Hence, the bivariate process

$$\left\{I(t) = \langle I_1(t),\ I_2(t)\rangle : t \geq 0\right\}$$

with state space

$$\mathcal{X} = \left\{ \langle x,\ Y\rangle : x = 1,2,\ldots,Q_1;\ y = 0,1,2,\ldots,Q_2\right\}$$

and

$$S_k = \min\left\{t : t > S_{k-1},\ I_1(t) = Q_1 \text{ and } I_1(t-) \neq Q_1\right\} \text{ for } k=1,2,3,\ldots,$$

and $S_0 \equiv 0$ (i.e. S_k is the epoch at which the k-th order is placed) is to be analysed.

Under the above set up, the problem is to determine the optimum ordering quantities $\langle Q_1^*,\ Q_2^*\rangle$ so that the long run cost per unit time is minimum.

Also since, the depletions in inventory can occur due to demands only, the inventory level process can be described as follows:

The evolution of the inventory level process I is a single stream of arrivals which occurs according to the process $\left\{A_1(t) + A_2(t) : t \geq 0\right\}$ in which each demand is for Commodity-1 with probability p $[= \lambda_1/(\lambda_1+\lambda_2)]$ and for Commodity-2 with probability 1-p, if the commodity 2 is in stock; otherwise each customer will demand Commodity-1 with probability 1, the classification of demands being independent of each other. Clearly, $\{A_1(t)+A_2(t):\ t\geq 0\}$ is Poisson process with rate $(\lambda_1+\lambda_2)$.

From the above discussion it is obvious that the bivariate process I is a Markov-Renewal process, the transition diagram of the embedded Markov chain being:

$$\langle x, y\rangle \xrightarrow{p} \langle x-1, y\rangle, \quad \langle x, y\rangle \xrightarrow{q} \langle x, y-1\rangle, \quad \text{for } x = 2,3,\ldots,Q_1;\ y = 1,2,3,\ldots,Q_2$$

$$\langle 1, y\rangle \xrightarrow{p} \langle Q_1, Q_2\rangle, \quad \langle 1, y\rangle \xrightarrow{q} \langle x, y-1\rangle, \quad \text{for } y = 1,2,3,\ldots,Q_2$$

$$\langle x, 0\rangle \xrightarrow{1} \langle x-1, 0\rangle, \quad x = 2,3,\ldots,Q_1$$

$$\langle 1, 0\rangle \xrightarrow{1} \langle Q_1, Q_2\rangle$$

Since the sojourn times are exponential the Markov renewal Process reduces to a Markov process and hence I can be treated as a continuous time Markov process having transition function as given above. In the following sub-section, difference differential equations are given.

4.2.2 DIFFERENCE - DIFFERENTIAL EQUATIONS:

In this sub-section, the difference-differential equation for the probability of an inventory level is given. The derivation of these equations are similar to the earlier cases [cf. Chapter-II]. The set of differential equations are given by

$$P'_{x,y}(t) = -(\lambda_1+\lambda_2)\,P_{x,y}(t) + \lambda_1\,P_{x+1,y}(t) + \lambda_2\,P_{x,y+1}(t), \tag{4.2.1}$$

for $x = 1,2,\ldots,Q_1-1;\ y = 1,2,\ldots,Q_2-1,$

$$P'_{Q_1,y}(t) = -(\lambda_1+\lambda_2)\,P_{Q_1,y}(t) + \lambda_2\,P_{Q_1,y+1}(t), \tag{4.2.2}$$

for $y = 0,1,2,\ldots,Q_2-1,$

$$P'_{x,Q_2}(t) = -(\lambda_1+\lambda_2)\,P_{x,Q_2}(t) + \lambda_1\,P_{x+1,Q_2}(t), \tag{4.2.3}$$

for $x = 1,2,\ldots,Q_1-1,$

$$P'_{x,0}(t) = -(\lambda_1+\lambda_2)\,P_{x,0}(t) + (\lambda_1+\lambda_2)\,P_{x+1,0}(t) + \lambda_2\,P_{x,1}(t),$$

for $x = 1,2,\ldots,Q_1-1,$ (4.2.4)

and

$$P'_{Q_1,Q_2}(t) = -(\lambda_1+\lambda_2)P_{Q_1,Q_2}(t) + \lambda_1\sum_{y=1}^{Q_2}P_{1,y}(t) + (\lambda_1+\lambda_2)P_{1,0}(t). \tag{4.2.5}$$

Here also one of our aims is to consider the objective function [which is a function of $\langle Q_1, Q_2\rangle$] as the steady-state expected cost per unit time, as has been done by Sivazlian and Stanfel (1975). The decision vector $\langle Q_1, Q_2\rangle$ is to be selected so as to minimize the objective function. In order to obtain the steady state expected cost per unit time, we need the steady state distribution and the distribution of the order level $\langle O_1, O_2\rangle$, which are derived in the next section.

4.3 STEADY STATE AND ORDER LEVEL DISTRIBUTIONS

The limiting distribution of the inventory level process $\{\underline{I}(t) = \langle I_1(t), I_2(t)\rangle : t \geq 0\}$ is obtained by using the Markov property of the process. The intensity rates for the inventory level process $\{\underline{I}(t) : t \geq 0\}$ on the state space $\mathcal{X}$ are

$$q_{\langle x,y\rangle,\langle i,j\rangle} = \begin{cases} \lambda_1 & , \text{ if } i = x - 1,\ j = y, \\ \lambda_2 & , \text{ if } i = x,\ j = y - 1, \\ 0 & , \text{ otherwise.} \end{cases} \tag{4.3.1}$$

Since the underlying process is finite and is clearly irreducible, recurrent, and aperiodic, it follows that the limiting distribution of the Markov process $\underline{I}$ exists and it is unique [cf. Adke and Manjunath (1984), pp-130)].

Let $\Pi(x, y)$ be the probability that exactly $\langle x, y\rangle$ units are in stock in the steady state for $\langle x, y\rangle \in \mathcal{X}$. Then it follows that

$$P_{x,y}(t) \longrightarrow \Pi(x,y), \text{ as } t \longrightarrow \infty$$

$$P'_{x,y}(t) \longrightarrow 0 \text{ , as } t \longrightarrow \infty$$

Now allowing $t \longrightarrow \infty$ in (4.2.1)-(4.2.5) we get,

$$\Pi(x, y) = p\,\Pi(x+1, y) + q\,\Pi(x, y+1), \tag{4.3.1}$$

for $x = 1,2,\ldots,Q_1 - 1$; $y = 1,2,\ldots,Q_2 - 1$,

$$\Pi(x, Q_2) = p\,\Pi(x+1, Q_2), \quad \text{for } x = 1,2,\ldots,Q_1-1, \tag{4.3.2}$$

$$\Pi(Q_1, y) = q\,\Pi(Q_1, y+1), \quad \text{for } y = 0,1,2,\ldots,Q_2-1, \tag{4.3.3}$$

$$\Pi(x, 0) = \Pi(x+1, 0) + q\Pi(x, 1), \quad \text{for } x=1,2,\ldots,Q_1-1, \tag{4.3.4}$$

and

$$\Pi(Q_1,Q_2) = p\sum_{y=1}^{Q_2} \Pi(1, y) + \Pi(1,0), \tag{4.3.5}$$

where

$p = \lambda_1/(\lambda_1 + \lambda_2)$, and $q = \lambda_2/(\lambda_1 + \lambda_2)$.

A solution to the above equations is obtained in the following theorem.

THEOREM 4.3.1: The steady state distribution Π of the inventory level I is given by

$$\Pi(x, y) = B[Q_1,Q_2;\ x,y;\ p]\Pi(Q_1, Q_2), \tag{4.3.6}$$

for $x = 1,2,\ldots,Q_1$; $y = 1,2,\ldots,Q_2$,

$$\Pi(x, 0) = \sum_{z=x}^{Q_1} B[Q_1,Q_2;\ z,1;\ p]\ q\ \Pi(Q_1, Q_2), \quad \text{for } x=1,2,\ldots,Q_1 \tag{4.3.7}$$

and

$$\Pi(Q_1, Q_2) = \left\{1+ \sum_{x=1}^{Q_1-1} p^{Q_1-x} + \sum_{y=1}^{Q_2-1} q^{Q_2-y} + \sum_{x=1}^{Q_1-1}\sum_{z=x}^{Q_1} B[Q_1,Q_2;\ z,1;\ p]\ q \right.$$

$$\left. + \sum_{x=1}^{Q_1-1}\sum_{y=1}^{Q_2-1} B[Q_1,Q_2;\ x,y;\ p]\right\}^{-1}. \tag{4.3.8}$$

The theorem is proved by establishing the following Lemmas.

LEMMA 4.3.1: The steady-state probability $\Pi(x, y)$ given in (4.3.6) satisfies (4.3.1) for $x = 1,2,\ldots,Q_1-1$; $y = 1,2,\ldots,Q_2-1$.

Proof: It follows from Lemma 2.3.1. □

LEMMA 4.3.2: The steady-state probability distribution $\Pi(x,y)$ given in (4.3.6) satisfies (4.3.2), for $x = 1,2,\ldots, Q_1-1$; $y = Q_2$, and (4.3.3) for $y = 1,2,\ldots,Q_2$; $x = Q_1$.

Proof: The proof of this Lemma follows from Lemma 2.3.2

LEMMA 4.3.3: The steady-state probability Π given in (4.3.6) satisfies (4.3.4).

Proof: The proof of this lemma follows from Lemma 3.3.3. □

LEMMA 4.3.4: The steady-state probability Π given in (4.3.6) and (4.3.7) satisfies (4.3.5).

Proof: Consider the right hand side of Equation (4.3.5),

$$p\sum_{y=1}^{Q_2}\Pi(1,y)+\Pi(1,0)=p\sum_{y=1}^{Q_2}B[Q_1,Q_2;\ 1,y;\ p]\Pi(Q_1,Q_2)$$
$$+\sum_{z=1}^{Q_1}B[Q_1,Q_2;\ z,1;\ p]\ q\ \Pi(Q_1,Q_2)$$

By replacing y by Q_2-y and z by Q_1-z in the right hand side of the above equation, we obtain,

$$\left\{\sum_{y=0}^{Q_2-1}[(Q_1+y-1))\ C\ (Q_1-1)]\ p^{Q_1}q^{y}\right.$$
$$\left.+\sum_{z=0}^{Q_1-1}[(Q_2+z-1))\ C\ (Q_2-1)]\ p^{Q_1-z}\ q^{Q_2}\right\}\Pi(Q_1,Q_2)$$

$$=\left\{\Pr[\,NB(Q_1,p)\le Q_1+Q_2-1]\ +\ \Pr[\,NB(Q_2,q)\le Q_1+Q_2-1]\right\}\Pi(Q_1,Q_2)$$

$$=\Pi(Q_1,Q_2).$$

Hence Π given in (4.3.6) and (4.3.7) satisfies (4.3.5).

LEMMA 4.3.5:

$$\Pi(Q_1,Q_2)=\left\{1+\sum_{x=1}^{Q_1-1}p^{Q_1-x}+\sum_{y=1}^{Q_2-1}q^{Q_2-y}+\sum_{x=1}^{Q_1-1}\sum_{z=x}^{Q_1}B[Q_1,Q_2;\ z,1;\ p]\ q\right.$$
$$\left.+\sum_{x=1}^{Q_1-1}\sum_{y=1}^{Q_2-1}B[Q_1,Q_2;\ x,y;\ p]\right\}^{-1}.$$

Proof: Since Π is a steady probability mass function,

$$\sum_{x=1}^{Q_1}\sum_{y=0}^{Q_2} \Pi(x,y) = 1.$$

That is,

$$\Pi(Q_1,Q_2) + \sum_{x=1}^{Q_1-1} \Pi(x,Q_2) + \sum_{y=1}^{Q_2-1} \Pi(Q_1,y) + \sum_{x=1}^{Q_1-1} \Pi(x,0) + \sum_{x=1}^{Q_1-1}\sum_{y=1}^{Q_2-1} \Pi(x,y)=1.$$

Substituting the quantities for Π and simplifying it reduces to, $\Pi(Q_1,Q_2)$ as given in (4.3.8). □

PROOF OF THEOREM 4.3.1:

The proof follows from Lemma 4.3.1-4.3.5.

LEMMA 4.3.6: The probability mass function of the order level $\langle Q_1,n\rangle$ is given by

$$\Pr[O_1=Q_1,O_2=n] = \begin{cases} [(Q_1+n-1)\ C\ (Q_1-1)]\ p^{Q_1}q^n/\Pr[NB(Q_1,p)\le Q_1+Q_2], & n=0,1,\ldots,Q_2, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to that of Lemma2.3.4.

In order to derive the expression for long run cost per unit time we need expected cycle time. Unlike other chapters, here the cycle time depends only on the evolution of the $I_1(t)$process. Therefore, in the following section we obtain the one dimensional marginal distribution of the demand process $D_1(t)$, $D_2(t)$, and the distribution of cycle time.

4.4 DISTRIBUTION OF CYCLE TIME AND DEMAND PROCESS:

4.4.1 DISTRIBUTIONS:

Consider an inventory model for substitutable items which can be described as follows:

Let $\{X_1(t): t\geq 0\}$ and $\{X_2(t): t\geq 0\}$ be any two independent Poisson processes with rates λ_1 and λ_2 respectively, which generates the time between demands for Commodity 1 and 2. Define another stochastic process $\{D_1(t): t\geq 0\}$ such that

$$D_1(t) = \begin{cases} X_1(t) & , \text{if } 0 \leq t < Y \\ X_1(t)+X_2(t)-X_2(Y), & \text{if } Y \leq t < \infty, \end{cases}$$

where Y is a random variable having a distribution function F(.). The process defined above is a Cox process under the assumption that $\{\lambda(t): t\geq 0\}$ is two state stochastic process. That is,

$$\lambda(t) = \begin{cases} \lambda_1 & ,\text{if } \leq t < Y \\ \lambda_1 + \lambda_2 & ,\text{if } Y \leq t < \infty . \end{cases} \quad (4.4.1)$$

Each time a demand occurs for Commodity-1, the decision is made whether or not to place an order. In view of this discussion the model in Section 4.2.2 can be described as follows:

If we have two types of commodities say Commodity 1 and 2, the Commodity-2 can be substituted by Commodity-1 whenever it is out of stock but not vice-versa, and Y is the epoch at which the Commodity 2 becomes out of stock. The operating doctorine considered is an (s,S) policy ($S > s \geq 0$), that is if the inventory level of Commodity-1 drops to its re-order point (zero) on some demand for Commodity 1, quantities Q_1 and Q_2 is ordered for Commodity 1 and 2 respectively, otherwise no order is placed.

Now the marginal probability distribution of inventory level for Commodity 1 and 2, and the expected value of the partial sums of an inter-demand times are obtained in the following theorems.

THEOREM 4.4.1: Let $\{D_1(t): t \geq 0\}$ be the Cox process with intensity process as defined in (4.4.1) and X_i be the time between (i-1)th and i-th demand for Commodity 1. Define $S_n^* = X_1 + X_2 + \ldots + X_n$, then the distribution functions of $D_1(t)$ and S_n^* are given by

$$\text{(i)} \quad \Pr[D_1(t)=j] = E_Y[\exp(-\Lambda(t))(\Lambda(t))^j/j!], \quad j=0,1,\ldots, \tag{4.4.2}$$

$$\text{(ii)} \quad \bar{F}_n^*(t) = \sum_{j=0}^{n-1} E_Y[\exp(-\Lambda(t))(\Lambda(t))^j/j!], \tag{4.4.3}$$

where Y is the epoch at which the demand rate changes.

Proof: Conditioning on Y, $D_1(t)$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$, so that

$$\Pr[D_1(t)=j \mid Y] = \exp(-\Lambda(t))(\Lambda(t))^j/j!, \quad j=0,1,\ldots$$

where $\Lambda(t) = \int_0^t \lambda(u)\, du$,

$$= \lambda_1 t + \lambda_2(t-Y)\, I(Y\le t) \qquad (4.4.4)$$

Hence, the unconditional distribution of demands $D_1(t)$ during $(0,t]$ is given by,

$$\Pr[D_1(t)=j] = E_Y\left[\exp(-\Lambda(t))(\Lambda(t))^j/j!\right], \quad j=0,1,\ldots$$

We know that $S_n^* > t$ if and only if $D_1(t)<n$, therefore,

$$\bar{F}_n^*(t) = \Pr[S_n^* > t]$$

$$= \sum_{j=0}^{n-1} \Pr[D_1(t)=j]$$

$$= \sum_{j=0}^{n-1} E_Y\left[\exp(-\Lambda(t))(\Lambda(t))^j/j!\right], \quad n=1,2,\ldots$$

This concludes the proof.

THEOREM 4.4.2: The distribution of $D_2(t)$, the demand for Commodity-2 during $(0,t]$ is

$$\Pr[D_2(t)=j] = \exp(-\lambda_2 t)(\lambda_2 t)^j/j!, \qquad j=0,1,\ldots \qquad (4.4.5)$$

Proof: The proof of this theorem follows from the fact that $\{D_2(t): t\ge 0\}$ is a Poisson process.

THEOREM 4.4.3: If S_1 denotes the length of the first cycle, which is defined as $S_1 = \min\{t : t>0, I_1(t)=Q_1, \; I_1(t-)\neq Q_1\}$, then the expected length of the first cycle is given by

$$E(S_1) = \sum_{j=1}^{Q_1} a_{j+1}/\lambda_1 + \sum_{j=1}^{Q_1} (1-a_{j+1})/(\lambda_1+\lambda_2), \qquad (4.4.6)$$

where $a_j = {}_0\int^{\infty} \bar{F}(z/\lambda_1)\, dG_j(z)$.

Proof: From (4.3.2), the distribution of $S_1 \equiv S^*_{Q_1}$ is given by

$$\Pr[S^*_{Q_1} > t] = \Pr[D_1(t) < Q_1] .$$

That is,

$$\Pr[S^*_{Q_1} > t] = \sum_{j=0}^{Q_1-1} \Pr[D_1(t)=j] .$$

Hence, the expected length of the cycle S_1 is,

$$E(S_1) = \int_0^{\infty} \Pr[S_1>t]\; dt,$$

as S_1 is non-negative continuous random variable. Therefore,

$$E(S_1) = \sum_{j=0}^{Q_1-1} {}_0\int^{\infty} E_Y\left[\exp(-\Lambda(t))(\Lambda(t))^j/j!\right]dt.$$

Assuming that the integral can be taken under the expectation sign, we can write the above equation as,

$$E(S_1) = \sum_{j=0}^{Q_1-1} E_Y\left[{}_0\int^{\infty} \exp(-\Lambda(t))(\Lambda(t))^j/j!\right] dt.$$

That is from (4.3.3.),

$$E(S_1) = \sum_{j=0}^{Q_1-1} E_Y\Big[{}_0\int^{Y} \exp(-\lambda_1 t)(\lambda_1 t)^j/j!\ dt$$

$$+ {}_Y\int^{\infty} \exp[-\lambda_1 t - \lambda_2(t-Y)]\,[\lambda_1 t + \lambda_2(t-Y)]^j/j!\ dt\Big]$$

By substituting $\lambda_1 t = z$ and $\lambda_1 t + \lambda_2(t-Y) = u$, we get

$$E(S_1) = \sum_{j=0}^{Q_1-1} E_Y\Big[{}_0\int^{\lambda_1 Y} \exp(-z)(z)^j/j!\ dz/\lambda_1$$

$$+ {}_{\lambda_1 Y}\int^{\infty} \exp(-u)\ u^j\ du/j!(\lambda_1+\lambda_2)\Big]$$

$$E(S_1) = \sum_{j=0}^{Q_1-1} {}_0\int^{\infty}\left\{{}_0\int^{\lambda_1 Y} \exp(-z)\ z^j\ dz\ /\ j!\lambda_1\right\}\ f(y)dy$$

$$+ \sum_{j=0}^{Q_1-1} {}_0\int^{\infty}\left\{{}_{\lambda_1 Y}\int^{\infty} \exp(-u)\ u^j\ du\ /\ j!(\lambda_1+\lambda_2)\right\} f(y)dy$$

Interchanging the order of integration we get,

$$E(S_1) = \sum_{j=0}^{Q_1-1} {}_0\int^{\infty}\left\{ {}_{z/\lambda_1}\int^{\infty} f(y)\ dy \right\} \exp(-z)\ z^j\ dz\ /\ j!\lambda_1$$

$$+ \sum_{j=0}^{Q_1-1} {}_0\int^{\infty}\left\{ {}_0\int^{z/\lambda_1} f(y)\ dy\right\} \exp(-z)\ z^j\ dz\ /\ j!(\lambda_1+\lambda_2).$$

$$E(S_1)=\sum_{j=0}^{Q_1-1} {}_0\int^{\infty} \bar{F}(z/\lambda_1)\, dG_{j+1}(z)/\lambda_1+\sum_{j=0}^{Q_1-1} {}_0\int^{\infty} F(z/\lambda_1)dG_{j+1}(z)/(\lambda_1+\lambda_2),$$

where $G_j(.)$ is distribution function of a gamma random variable with shape parameter j. Or equivalently,

$$E(S_1) = \sum_{j=1}^{Q_1} a_j/\lambda_1 + \sum_{j=1}^{Q_1} (1-a_j)/(\lambda_1+\lambda_2),$$

where $a_j = {}_0\int^{\infty} \bar{F}(z/\lambda_1)\, dG_j(z)$

$$=E[\bar{F}(z/\lambda_1)] \tag{4.4.7}$$

Hence the proof is complete. □

4.4.2 EXPRESSION FOR THE OBJECTIVE FUNCTION:

Observe that the epochs of entrance of the inventory level process $\bar{I}$ into the state $<Q_1, Q_2>$ are regenerative epochs. Hence a sequence of inter-replenishment times $\{S_k - S_{k-1}: K \geq 1\}$ forms an ordinary renewal process. Therefore, $\{S_k - S_{k-1}: K \geq 1\}$ is a sequence of independent identically distributed random variables. Let $C(Q_1, Q_2, t)$ be the cost of maintaining the system for the duration $(0, t]$. Then $C(Q_1, Q_2, T_k)$ will be the cost of maintaining the inventory during the K-th cycle, where $T_k=S_k-S_{k-1}$. Once again, by using standard results in renewal theory [Cf. Ross 1970, pp 52], it can be shown that long run cost per unit time is given by

$$\lim_{t \to \infty} \frac{C(Q_1, Q_2, t)}{t} = \frac{E\,[C(Q_1, Q_2, S_1)]}{E\,(S_1)} \qquad (4.4.8)$$

Now it remains to evaluate the expression for the expected values on the right hand side of (2.3.14).

Consider the inter-replenishment interval of length S_1. The cost associated with this interval is given by

$$C(Q_1, Q_2, S_1) = K + C_1Q_1 + C_2O_2 + H_1 \sum_{j=1}^{Q_1} (Q_1-j+1)\ (D_{1j} - D_{1j-1})$$

$$+ H_2 \sum_{j=1}^{O_2} (Q_2 - j + 1)\ (D_{2j} - D_{2j-1}) \qquad (4.4.9)$$

where $D_{ij} - D_{ij-i}$, $j=1,2,\ldots$ are the inter-demand times for Commodity-i, $i = 1,2$.

Now taking the expectation of (4.4.9),and substituting in (4.4.9) we get,

$$EC(Q_1, Q_2) = \Big\{ K + C_1Q_1 + C_2E(O_2) + \sum_{j=1}^{Q_1} (Q_1-j+1)\big[a_j/\lambda_1 + (1-a_j)/(\lambda_1+\lambda_2)\big]\Big\}\ [E(S_1)]^{-1}, \qquad (4.4.10)$$

where $E(S_1)$ is as defined in Equation (4.4.6).

Now, the problem is to minimize $EC(Q_1, Q_2)$ subject to the condition that Q_1 and Q_2 are non-negative integers. This is a non-linear integer programming problem. The objective function can be written as

$$\text{Min } EC(Q_1, Q_2) = \underset{Q_1}{\text{Min}} \left\{ \underset{Q_2}{\text{min}} \; EC(Q_1, Q_2) \right\} \qquad (4.4.11)$$

Given the parameters of the model one could compute the values for $EC(Q_1, Q_2)$ starting with $Q_1 = 1$, $Q_2 = 1$ and proceeding until Q_1 and Q_2 satisfy (2.3.17). The value of Q_1 and Q_2 for which the cost function changes its direction is naturally the optimum values of Q_1 and Q_2. In the following, we discuss some particular cases.

PARTICULAR CASES:

1. If the distribution of Y is degenerate at infinity, then there is no substitutability. Therefore, the optimum Q_1 is obtained as follows:

$$\bar{F}(u) = 1$$

hence, $E(S_1) = Q_1/\lambda_1$ and $a_j = 1$.

So, the cost function obtained in (4.4.10) reduces to,

$$EC(Q_1) = K\lambda_1/Q_1 + C_1\lambda_1 + H_1(Q_1+1)/2$$

Therefore, the optimum value of Q_1 say Q_1^* is such that

$$EC(Q_1^*) - EC(Q_1^*+1) \le 0$$

and $$EC(Q_1^*) - EC(Q^*_1-1) \le 0.$$

Thus,

$$Q_1^* = (2K\lambda_1/H_1)^{1/2},$$ which coincides with the basic EOQ.

2. If the distribution of Y is degenerated at zero, then there is substitutability right from the beginning. Therefore, the optimum value of Q_1 is

$$Q_1^* = (2K(\lambda_1+\lambda_2)/H_1)^{1/2},$$

which coincides with the basic EOQ for $\lambda = (\lambda_1+\lambda_2)$.

3. If the distribution of Y is exponential with mean λ_2^{-1}, then

$$F(u) = \begin{cases} 0 & \text{, if } u \le 0, \\ 1-\exp(-\lambda_2 u), & \text{if } u > 0. \end{cases}$$

Thus,

$$a_j = \int_0^\infty \bar{F}(z/\lambda_1) \exp(-z)\ z^{j-1} dz/(j-1)!$$

$$= \int_0^\infty \exp[(-\lambda_2/\lambda_1+1)z]\ z^{j-1} dz/(j-1)!$$

$$= (\lambda_1/(\lambda_1+\lambda_2))^j, \qquad (4.4.12)$$

$$\text{and } \sum_{j=0}^{Q_1-1} a_{j+1} = \sum_{j=1}^{Q_1} a_j$$

$$= (\lambda_1/\lambda_2)\left\{1 - (\lambda_1/(\lambda_1+\lambda_2))^{Q_1}\right\},$$

Therefore, the expected cycle time given in (4.4.6) reduces to,

$$E(S_1) = Q_1/(\lambda_1+\lambda_2) + \left\{1 - (\lambda_1/(\lambda_1+\lambda_2))^{Q_1}\right\}. \qquad (4.4.13)$$

Substituting these values in the cost expression given in (4.4.9), one can obtain optimum Q_1 and Q_2 by developing a computer program.

4.5 ALGORITHMS

In this section we give step by step procedures for simulating the inventory system and for determining the optimum re-order quantity $\langle Q_1, Q_2 \rangle$.

4.5.1: SIMULATION ALGORITHM:

The algorithm SIMULATION is basically to obtain the simulation of the inventory model. The successive steps are as follows:

STEP 1: Input the following parameters:

Q_ι :the maximum inventory level of Commodity-ι, $\iota=1,2$,

λ_ι :arrival rate of the customers for Commodity-ι, $\iota=1,2$,

H_ι :holding cost per item per unit time for Commodity-ι, $\iota=1,2$,

C_ι :unit cost of Commodity-ι, $\iota=1,2$,

K :fixed procurement cost,

T :the duration of simulation,

STEP 2: Set $P = \lambda_1/(\lambda_1+\lambda_2)$: $Q=1-P$: $I_1=Q_1$: $I_2=Q_2$ and TC=0.

STEP 3: Generate an arrival epoch of a customer during (0,T]using the exponential distribution with mean $(\lambda_1+\lambda_2)^{-1}$.

STEP 4: Suppose that A(J), J=1,2,... denotes the time instant of the arrivals of J-th customer. For each J, generate the type of commodity chosen by the customer. Accordingly reset $I_1= I_1-1$, or $I_2= I_2-1$ if $I_2>0$, otherwise $I_1= I_1-1$.

STEP 5: At every depletion epoch verify whether on hand inventory of Commodity-1 is zero (reached re-order point). If so an order is to be placed for $\langle Q_1,O_2\rangle$ units immediately so as to bring their level to $\langle Q_1,Q_2\rangle$.

STEP 6: After deciding the nature of customer at depletion epoch, the total cummulative cost must be computed. The total cost will be equal to initial cost plus the cost we get multiplying by number of items in stock by holding cost for both the commodities. On the other hand whenever the order is placed for $<Q_1, O_2>$ units then procurement cost incurred for placing an order must be added to the initial cost and inventory holding cost of both the commodities.

STEP 7: The procedure (execution) is stopped as and when the depletion epoch is T or larger. The average cost per unit time for this particular combination is computed.

STEP 8: The above steps from (2)-(7) are repeated to get required number of realizations up to T and for each realization average cost per unit time is computed. An average of these values over the different realizations is taken as an estimate of the long run cost per unit time.

STEP 9: The difference between this estimate and the value of the objective function at Q_1 and Q_2 corresponds to the accuracy of an approximation.

4.5.2 OPTIMIZATION ALGORITHM:

The algorithm OPTIMIZATION gives the optimum ordering quantities Q_1 and Q_2 so that the objective function given in (4.4.10) is minimum. Following are the main steps of the algorithm:

STEP 1: Input the following parameters of the objective function:

Q_ι :the maximum inventory level of Commodity-ι, $\iota=1,2$,

λ_ι :arrival rate of the customers for Commodity-ι, $\iota=1,2$,

H_ι :holding cost per item per unit time for Commodity-ι, $\iota=1,2$,

C_ι :unit cost of Commodity-ι, $\iota=1,2$,

K :fixed procurement cost,

STEP 2: Compute $A_J = \int_0^\infty \bar{F}(z/\lambda_1)\, dG_J(z)$ for given F(.) analytically if possible, otherwise use any standard numerical integration procedure to obtain A_J.

STEP 3: Compute $E(S_1)$ given in (4.4.6).

STEP 4: Initialise $Q_1=0$: OPTCQ1Q2=99999# : TEMPCQ1Q2=999#

STEP 5: Do While (OPTCQ1Q2≥TEMPCQ1Q2)

5.1 SET Q1=Q1+1 :OPTCQ1Q2=TEMPCQ1Q2

5.2 MINIMIZE EC(Q1,Q2) with respect to Q2.

5.3 Recalculate

$$\text{TEMPCQ1Q2} = \left\{K + C1*Q1 + C2*EO2 + \sum_{J=1}^{Q1} (Q1-J+1)\left[A_J/\lambda_1 + (1-A_J)/(\lambda_1+\lambda_2)\right]\right\} \times E(S_1)^{-1}$$

End Do

STEP 6: Output

OPTIMUMQ1 = Q1-1

OPTIMUMQ2 = Q2-1

OPTIMUMCOST=OPTCQ1Q2

STEP 7: End of the algorithm.

Sub steps of the algorithm can be deveoped on the lines of Chapter-II. In what follows, the transient distribution in terms of Laplace Transforms for the inventory level is derived.

4.6 TRANSIENT SOLUTION TO THE DIFFERENTIAL EQUATIONS

For this model, we obtain expression for $P_{x,y}(t)$ in terms of its Laplace Transforms. In this section we derive the same and verify that this infact satisfies the Laplace Transform of the equations.

The Laplace Transform of Equations (4.2.1) - (4.2.5) are given by

$$s\tilde{P}_{x,y}(s) = -(\lambda_1 + \lambda_2)\tilde{P}_{x,y}(s) + \lambda_1\tilde{P}_{x+1,y}(s) + \lambda_2\tilde{P}_{x,y+1}(s),$$

for $x = 1,2,\ldots,Q_1 - 1;\ y = 1,2,\ldots,Q_2-1,$

$$s\tilde{P}_{Q_1,y}(s) = -(\lambda_1 + \lambda_2)\tilde{P}_{Q_1,y}(s) + \lambda_2\tilde{P}_{Q_1,y+1}(s),$$

for $y = 0,1,2,\ldots,Q_2 - 1,$

$$s\tilde{P}_{x,Q_2}(s) = -(\lambda_1 + \lambda_2)\tilde{P}_{x,Q_2}(s) + \lambda_1\tilde{P}_{x+1,Q_2}(s),$$

for $x = 1,2,\ldots,Q_1 - 1,$

$$s\tilde{P}_{x,o}(s) = -(\lambda_1+\lambda_2)\tilde{P}_{x,o}(s) + (\lambda_1+\lambda_2)\tilde{P}_{x+1,o}(s) + \lambda_2\tilde{P}_{x,1}(s),$$

for $x=1,2,\ldots,Q_1-1$,

and

$$s\tilde{P}_{Q_1,Q_2}(s) - 1 = -(\lambda_1+\lambda_2)\tilde{P}_{Q_1,Q_2}(s) + \lambda_1\sum_{y=1}^{Q_2}\tilde{P}_{1,y}(s) + (\lambda_1+\lambda_2)\tilde{P}_{1,o}(s).$$

The above set of equations can be written as,

$$\tilde{P}_{x,y}(s) = U_S\tilde{P}_{x+1,y}(s) + V_S\tilde{P}_{x,y+1}(s), \qquad (4.6.1)$$

for $x = 1,2,\ldots,Q_1-1$; $y = 1,2,\ldots,Q_2-1$,

$$\tilde{P}_{Q_1,y}(s) = V_S\,\tilde{P}_{Q_1,y+1}(s), \qquad (4.6.2)$$

for $y = 0,1,2,\ldots,Q_2-1$,

$$\tilde{P}_{x,Q_2}(s) = U_S\,\tilde{P}_{x+1,Q_2}(s), \qquad (4.6.3)$$

for $x = 1,2,\ldots,Q_1-1$,

$$\tilde{P}_{x,o}(s) = [U_S+V_S]\tilde{P}_{x+1,o}(s) + V_S\tilde{P}_{x,1}(s), \qquad (4.6.4)$$

for $x=1,2,\ldots,Q_1-1$,

and

$$\tilde{P}_{Q_1,Q_2}(s) = U_S\sum_{y=1}^{Q_2}\tilde{P}_{1,y}(s) + [U_S+V_S]\tilde{P}_{1,o}(s) + 1/(\lambda_1+\lambda_2+s),$$

(4.6.5)

where

$$U_S = \lambda_1/(\lambda_1+\lambda_2+s) \quad \text{and} \quad V_S = \lambda_2/(\lambda_1+\lambda_2+s).$$

In the following theorem an expression for $\tilde{P}_{x,y}(s)$ is given and is verified that it satisfies the system of equations given above

THEOREM 4.6.1: A solution to the above equations is given by

$$\tilde{P}_{x,y}(s) = B[Q_1,Q_2;\ x,y;\ p](U_S+V_S)^{Q_1+Q_2-x-y}(\lambda_1+\lambda_2+s)^{-1}/(1-\tilde{f}_{S_1}(s)) \tag{4.6.6}$$

for $x=1,2,\ldots,Q_1-1;\ y=1,2,\ldots,Q_2-1,$

$$\tilde{P}_{x,o}(s) = \sum_{z=x}^{Q_1}[Q_1+Q_2-z-1\ C\ (Q_2-1)]\,q^{Q_2}\,p^{Q_1-z}(U_S+V_S)^{Q_1+Q_2-x}$$

$$\times\ (\lambda_1+\lambda_2+s)^{-1}(1-\tilde{f}_{S_1}(s))^{-1}, \tag{4.6.7}$$

and

$$\tilde{f}_{S_1}(s) = \lambda_1\sum_{y=1}^{Q_2}B[Q_1,Q_2;\ 1,y;\ p](U_S+V_S)^{Q_1+Q_2-1-y}(\lambda_1+\lambda_2+s)^{-1}$$

$$+\sum_{z=1}^{Q_1}B[Q_1,Q_2;\ z,1;\ p]\ q\ (U_S+V_S)^{Q_1+Q_2-z}(\lambda_1+\lambda_2+s)^{-1}. \tag{4.6.8}$$

In order to prove the theorem , we shall establish the following Lemmas.

LEMMA 4.6.1: The expression for $\tilde{P}_{x,y}(s)$ given in (4.6.6)satisfies Equation (4.6.1), for $x = 1,2,\ldots,Q_1-1$; $y = 1,2,\ldots,Q_2-1$.

Proof: The proof of the lemma follows from Lemma 2.5.1.

LEMMA 4.6.2: The expression for $\tilde{P}_{x,y}(s)$ given in (4.6.6) satisfies Equations (4.5.2) and (4.5.3), for $x=Q_1$, $y=1,2,\ldots,Q_2-1$ and $y=Q_2$, $x=1,2,\ldots,Q_1-1$ respectively.

Proof: It follows from Lemma 2.5.2.

LEMMA 4.5.3: The expressions for $\tilde{P}_{x,y}(s)$ and $\tilde{P}_{x,o}(s)$ given in (4.6.6) and (4.6.7) satisfy Equations (4.6.4), for $x=1,2,\ldots,Q_1-1$.

Proof: Consider the right hand side of (4.5.4),

$$[U_S+V_S]\tilde{P}_{x+1,0}(s) + V_S\tilde{P}_{x,1}(s)$$

$$= \Big\{ [U_S+V_S] \sum_{z=x+1}^{Q_1} [Q_1+Q_2-z-1 \; C \; (Q_2-1)]\, q^{Q_2} p^{Q_1-z} (U_S+V_S)^{Q_1+Q_2-x-1}$$

$$+ V_S[Q_1+Q_2-x-1 \; C \; (Q_1-x)]\, q^{Q_2-1} p^{Q_1-x} (U_S+V_S)^{Q_1+Q_2-x-1}\Big\}$$

$$\times \quad (\lambda_1+\lambda_2+s)^{-1}(1-\tilde{f}_{S_1}(s))^{-1}$$

$$= \left\{ \sum_{z=x+1}^{Q_1} [Q_1+Q_2-z-1 \; C \; (Q_2-1)] \, q^{Q_2} \; p^{Q_1-z} (U_S + \; V_S)^{Q_1+Q_2-x} \right.$$

$$\left. + [Q_1+Q_2-x-1 \; C \; (Q_2-1)] \, q^{Q_2} \; p^{Q_1-x} (U_S + \; V_S)^{Q_1+Q_2-x} \right\}$$

$$x \quad (\lambda_1+\lambda_2+s)^{-1} (1-\tilde{f}_{S_1}(s))^{-1}$$

$$= \tilde{P}_{x,o}(s), \qquad \text{for } x=1,2,\ldots,Q_1-1 \qquad \square$$

LEMMA 4.6.4: The expressions for $\tilde{P}_{x,y}(s)$ in (4.6.6) and (4.6.7) satisfy Equation (4.6.5).

Proof: Consider Equation (4.6.5),

$$\tilde{P}_{Q_1,Q_2}(s) = U_S \sum_{y=1}^{Q_2} \tilde{P}_{1,y}(s) + (U_S+V_S) \; \tilde{P}_{1,o}(s) + (\lambda_1+\lambda_2+s)^{-1}$$

Substituting for the respective quantities from (4.6.6) and (4.5.7), the above equation reduces to

$$(\lambda_1+\lambda_2+s)^{-1}(1-\tilde{f}_{S_1}(s))^{-1}$$

$$= \left\{ U_S \sum_{y=1}^{Q_2} B[Q_1,Q_2; \; 1,y; \; p] \, (U_S + \; V_S)^{Q_1+Q_2-y-1} \right.$$

$$\left. +(U_S + \; V_S) \sum_{z=1}^{Q_1} B[Q_1,Q_2; \; z,1; \; p] \, q(U_S + \; V_S)^{Q_1+Q_2-z-1} \right\}$$

$$x \; (\lambda_1+\lambda_2+s)^{-1}(1-\tilde{f}_{S_1}(s))^{-1} + (\lambda_1+\lambda_2+s)^{-1}$$

This reduces to (4.6.8). $\square$

PROOF OF THEOREM 4.6.1: Proof of Theorem 4.6.1 follows from Lemma 4.6.1 - 4.6.4.

In the above theorem, we have given an expression for $\tilde{P}_{x,y}(s)$, $\langle x,y\rangle \in \mathcal{X}$ and shown that it is solution to the system of equations (4.6.1)-(4.6.5). However, it is infact possibel to derive the expression for $\tilde{P}_{x,y}(s)$, $\langle x,y\rangle \in \mathcal{X}$ by using renewal theory argument. Hence, in the following sub-section we have derived the same.

DERIVATION OF THE EXPRESSION FOR $\tilde{P}_{x,y}(s)$:

In this sub-section we have derived the expression for $P_{x,y}(s)$, for $\langle x,y\rangle \in \mathcal{X}$.

THEOREM 4.6.2: For $\langle x,y\rangle \in \mathcal{X}$, the Laplace Transform of $P_{x,y}(t)$ is given by

$$\tilde{P}_{x,y}(s) = \tilde{g}_{x,y}(s)/(1 - \tilde{f}_{S_1}(s)) \qquad \forall \langle x,y\rangle \in \mathcal{X}$$

where,

$$\tilde{g}_{x,y}(s) = B\,[Q_1,Q_2;\ x,y;\ p]\quad (U_S+V_S)^{Q_1+Q_2-x-y}(\lambda_1+\lambda_2+s)^{-1},$$

for $x=1,2,\ldots,Q_1$ and $y=1,2,\ldots,Q_2$, (4.6.9)

$$\tilde{g}_{x,0}(s) = \sum_{z=x}^{Q_1} B\,[Q_1,Q_2;\ z,1;\ p]\ q\ (U_S+V_S)^{Q_1+Q_2-x}\ (\lambda_1+\lambda_2+s)^{-1},$$

for $x=1,2,\ldots,Q_1$, and (4.6.10)

$$\tilde{f}_{S_1}(s) = \lambda_1 \sum_{y=1}^{Q_2} \tilde{g}_{1,y}(s) + \tilde{g}_{1,o}(s). \qquad (4.6.11)$$

Proof: From (2.5.7), $P_{x,y}(s)$ is,

$$\tilde{P}_{x,y}(s) = \tilde{g}_{x,y}(s)/(1 - \tilde{f}_{S_1}(s)) \qquad \forall \langle x,y\rangle \in \mathcal{X}$$

First we shall obtain the numerator, that is

$$\tilde{g}_{x,y}(s) = {}_0\int^{\infty} \exp(-st)\; g_{x,y}(t)\; dt.$$

From Theorem 2.5.3, it follows that

$$\tilde{g}_{x,y}(s) = B\,[Q_1,Q_2;\; x,y;\; p]\;\; (U_S + V_S)^{Q_1+Q_2-x-y}\; (\lambda_1+\lambda_2+s)^{-1}, \qquad (4.6.12)$$

for $x=1,2,\ldots,Q_1$, and $y=1,2,\ldots,Q_2$.

Now we shall obtain the Laplace Transform of $g_{x,o}(t)$, from (2.5.9) $g_{x,o}(t)$ is given by

$$\begin{aligned} g_{x,o}(t) &= \Pr[\, I(t) = \langle x,\ 0\rangle,\ S_1>t] \\ &= \Pr[\, A(t) = \langle Q_1-x,\ Q_2\rangle,\ S_1>t] \\ &= \sum_{r=0}^{Q_1-x} {}_0\int^{t} \Pr[\, A(u) = \langle r,\ Q_2-1\rangle \mid A_1(u)+A_2(u)=r+Q_2-1,\ S_{21}=u\,] \\ &\quad \times \Pr[\,(r+Q_2)\text{th arrival is for Commodity-2}]\ f_{S_{21}}(u)] \\ &\quad \times \Pr[\, A_1(t-u)+A_2(t-u)=Q_1-x-r]\ du \end{aligned}$$

Using the fact $A_1(t)$ and $A_2(t)$ are two independent Poisson arrival processes and after algebraic simplifications, the above expression for $g_{x,o}(t)$ reduces to

$$g_{x,o}(t)=\sum_{r=o}^{Q_1-x} [r+Q_2-1 \, C \, Q_2-1] q^{Q_2} p^r \exp[-(\lambda_1+\lambda_2)t] [(\lambda_1+\lambda_2)t]^{Q_1+Q_2-x} \times [(Q_1+Q_2-x)!]^{-1} .$$

Therefore the Laplace Transform of $g_{x,o}(t)$ is

$$\tilde{g}_{x,o}(s) = \sum_{r=o}^{Q_1-x} [r+Q_2-1 \; C \; Q_2-1] \; q^{Q_2} p^r \; (U_S+ V_S)^{Q_1+Q_2-x} \; (\lambda_1+\lambda_2+s)^{-1}.$$

Substituting $z=Q_1-r$ on the right hand side of the above equation, it leads to

$$\tilde{g}_{x,o}(s) = \sum_{z=x}^{Q_1} B \; [Q_1,Q_2; \; z,1; \; p] \; q \; (U_S+ V_S)^{Q_1+Q_2-x} \; (\lambda_1+\lambda_2+s)^{-1}. \tag{4.6.13}$$

Therefore (4.6.6) follows.

We now proceed to find $\tilde{f}_{S_1}(s)$. We have from (4.3.10),

$$\tilde{P}_{Q_1,Q_2}(s) = U_S \sum_{y=1}^{Q_2} \tilde{P}_{1,y}(s) + (\lambda_1+\lambda_2+s)^{-1} \; \tilde{P}_{1,o}(s) + (\lambda_1+\lambda_2+s)^{-1},$$

by virtue of (2.5.7) and (4.6.12), it is equivalent to

$$\tilde{f}_{S_1}(s) = \lambda_1 \sum_{y=1}^{Q_2} \tilde{g}_{1,y}(s) + \tilde{g}_{1,o}(s).$$

This concludes the proof.