CHAPTER - VI

ORTHONORMAL SERIES EXPANSION OF
GENERALIZED FUNCTIONS
6.1 **Introduction**

In this Chapter VI a test function space $A_{L_2, g}(I)$ of smooth functions is constructed and then the space of continuous linear functionals on $A_{L_2, g}(I)$ is investigated by using techniques given in Zemanian [142] and Pathak [99]. The elements of $A_{L_2, g}(I)$ possess orthonormal series expansion.

Finally an application to solve the differential equations of the form $p[L_x] u = h$, $a < x < b$ is considered.

6.2 **The Space $L_{2, g}(I)$.** We assume that function $g(x,y,t)$ (the source solution explained as 5.1 of Chapter V) as the smooth function of $x$ or $y$ with $\text{Re} t > 0$ and $g(x,y,t) \neq 0$ on any open interval. Let $\{\gamma_n(x)\}$ be a sequence over the interval $J$ relative to $g(x,y,t)$. Let $I$ denote the open interval $a < x < b < \infty$ and $a = -\infty$, $b = \infty$ may be allowed.

A function $\phi$ defined on $I$ is said to belong the space $L_{2, g}(I)$ if

$$\alpha_0(\phi) = \left( \int_0^\infty \left| \phi(x) \right|^2 g(x,y,t) \, dx \right)^{1/2} < \infty \quad \ldots \quad (6.2.1)$$

We assume that each $\gamma_n(x)$ (eignefunction) is a smooth function and belongs to $L_{2, g}(I)$.

If $\phi, \psi \in L_{2, g}(I)$, we define the bilinear form
\[(\phi, \psi) = \int_{a}^{b} \phi(x) \psi(x) g(x, y, t) \, dx \quad \ldots \quad (6.2.2)\]

It can be proved that the integral (6.2.2) exists. We shall prove that the space \(L_2, g(I)\) is countable multiflavored space.

6.2A Theorem The space \(L_2, g(I)\) is complete.

Proof - Let \(\{ \phi_n \}_{n=1}^{\infty}\) be a Cauchy sequence in \(L_2, g(I)\).

Let \(x_1\) be a fixed point of \(I\) and \(x\) a variable point of \(I\). Also let \(D^{-1}\) is the integral operator

\[D^{-1} = \int_{x_1}^{x_0} \ldots \, dz\]

Then for any smooth function \(\xi\) on \(I\)

\[D^{-1} D(\xi) = \xi(x) - \xi(x_1)\]

We know that \(g(x, y, t)\) is a smooth function with \(g(x, y, t) \neq 0\) on any open interval. Using Schwartz's inequality on the interval whose end points are \(x, x_1\), we may write

\[|D^{-1} g(x, z, t)(\phi_m - \phi_n)|^2 = \left| \int_{x_1}^{x} g(x, z, t) (\phi_m - \phi_n) \, dz \right|^2 \leq \int_{x_1}^{x} |g(x, z, t)|^2 \left| \phi_m - \phi_n \right|^2 \, dz \]

\[\ldots (6.2.3)\]
Now let denote an arbitrary open interval whose closure is compact in I. The first integral of (6.2.3) is bounded smooth function on every \( \land \) since \( g(x, z, t) \neq 0 \). The second integral converges to zero as \( m \to \infty, n \to \infty \). This shows that L.H.S. of (6.2.3) converges to zero. Hence every Cauchy sequence converges in \( L_2, g(I) \) and so the space \( L_2, g(I) \) is complete.

6.3 The Testing Function Space \( A_{L_2, g}(I) \)

Let \( L_x \) be the \( n^\text{th} \) order self adjoint linear differential operator such that

\[
L_x = \sum_{k=0}^{n} \theta_r(x) \frac{D^k_x}{D_x^k} \quad \text{for} \quad k = 0, 1, 2, \ldots \quad (6.3.1)
\]

defined on I. Here \( \theta_r(x) \) are smooth functions on I and \( \theta_r(x) \neq 0 \) for \( x \notin I \). Let \( \{ \psi_n(x) \} \) be a sequence of smooth functions satisfying

\[
L_x^k \psi_n = \lambda_n^k \psi_n \quad \text{for} \quad k = 0, 1, 2, \ldots \quad (6.3.2)
\]

where \( \lambda_n^k \) denote eigenvalues (as explained in 5.1). The testing function space depends upon choice of I, \( L_x, \{ \psi_n \} \) and the function \( g(x, y, t) \). \( A_{L_2, g}(I) \) consists of all functions \( \psi(x) \) that possess the following properties.

1. \( \psi(x) \) is defined complex valued and smooth function on I.
(ii) For each $k$, $\omega_k(\mathcal{E}) = \left( \int_a^b \left| \int_x^k \mathcal{E}(x) \right|^2 g(x, y, t) \, dx \right)^{1/2} \leq \infty$

$$\cdots \ (6.3.3)$$

(iii) For each $k$, $n = 0, 1, 2, \ldots$

$$\left( L^k \mathcal{E}, \psi_n \right) = \left( \mathcal{E}, L^k \psi_n \right) \quad \cdots \ (6.3.4)$$

It follows that $A_{L^2, g}(I)$ is a subspace of $L^2, g(I)$ and it is the linear space.

For fixed $y$ and $t$, if $g(x, y, t) = 1$, then we get the special case of $A_{L^2, g}(I)$ as the space $A_{L^2}(I)$ that has introduced by Zemanian [142].

Also,

$$\alpha_k(\mathcal{E}_1 + \mathcal{E}_2) = \left( \int_a^b \left| \int_x^{k} \mathcal{E}_1 + \mathcal{E}_2 \right|^2 g(x, y, t) \, dx \right)^{1/2}$$

$$= \left( \int_a^b \left| \int_x^{k} \mathcal{E}_1 + \int_x^{k} \mathcal{E}_2 \right|^2 g(x, y, t) \, dx \right)^{1/2}$$

$$= \left( \int_a^b \left| \left( \int_x^{k} \mathcal{E}_1 \right)^{1/2} + \left( \int_x^{k} \mathcal{E}_2 \right)^{1/2} \right|^2 \right)^{1/2}$$

$$\leq \left( \int_a^b \left| \int_x^{k} \mathcal{E}_1 \right|^2 g(x, y, t) \, dx \right)^{1/2} + \left( \int_a^b \left| \int_x^{k} \mathcal{E}_2 \right|^2 g(x, y, t) \, dx \right)^{1/2}$$

$$\leq \alpha_k(\mathcal{E}_1) + \alpha_k(\mathcal{E}_2).$$
Therefore each $\alpha_k$ is a seminorm with $\alpha_0$ is norm and the
topology is generated by $\{ \alpha_k \}_{k=0}^{\infty}$ and it follows that
$A_{L_2, g}(I)$ is a countable multinormed space as well as it
is a testing function space. Convergence in $A_{L_2, g}(I)$ implies
convergence in $L_{2, g}(I)$.

We can show that each $\gamma_n$ is a member of $A_{L_2, g}(I)$ for,

$$|\alpha_k(\gamma_n)|^2 = \left( \int_a^b |L^k_x \gamma_n|^2 g(x, y, t) \, dx \right)$$

$$= \left( \int_a^b \lambda_n^k |\gamma_n|^2 g(x, y, t) \, dx \right)$$

$$= |\lambda_n^k|^2 \left( \int_a^b |\gamma_n|^2 g(x, y, t) \, dx \right)$$

$$< |\lambda_n^k|^2 |\alpha_0(\gamma_n)|^2 < \infty$$

so the condition (ii) is satisfied.

We shall now prove that the space $A_{L_2, g}(I)$ is complete and
hence a Frechet space.

6.3A Theorem The space $A_{L_2, g}(I)$ is complete

Proof Let $\{ \theta_n \}_{n=1}^{\infty}$ be a Cauchy sequence in $A_{L_2, g}(I)$.

Then for each $k$, we have $\{ L^k_x \theta_n \}_{n=1}^{\infty}$ is a Cauchy sequence
in $L_{2, g}(I)$. By completeness of $L_{2, g}(I)$, there exists
$\xi_k \in L_{2, g}(I)$ which is the limit in $L_{2, g}(I)$ of $L^k_x \theta_n$. We
shall show that $\xi_k$ is everywhere (on $I$) equal to $L_x^k \xi_0$

where $\xi_0 \in L_2(I)$ is independent of $k$. Let $x_1$ be a fixed
point of $I$ and $x$ be a variable point of $I$. Also $D^{-1}$ is the
integral operator

$$D^{-1} = \int_{x_1}^{x_0} \ldots \, dz$$

Then for any smooth function $\xi$ on $I$

$$D^{-1} D(\xi) = \xi(x) - \xi(x_1)$$

Using Schwartz's inequality on the interval whose end points
are $x$ and $x_1$, we may write

$$\left| D^{-1} g(x, z, t) L_x^{k-1}(\phi_m - \phi_n) \right|^2 = \int_{x_1}^{x} g(x, z, t) L_x^{k}(\phi_m - \phi_n) \, dz \right|^2$$

$$\leq \int_{x_1}^{x} \left| g(x, z, t) \right|^2 \, dz$$

$$\times \int_{x_1}^{x} \left| L_x^{k-1}(\phi_m - \phi_n) \right|^2 \, dz$$

... (6.3.5)

Now let $J$ shows an arbitrary open interval whose closure
is compact in $I$. The first integral on R.H.S. of (6.3.5)
is bounded smooth function on every $J$ since $g(x, z, t) \neq 0$
on any open interval. The second integral converges to zero
as $m \to \infty, n \to \infty$. This shows that L.H.S. of (6.3.5)
converges to zero, uniformly on every $J$. ...
Now we have to obtain $L_x^{-1} L_x^{k-1} \phi_m$.

where

$$L_x^{-1} = \theta_x^{-1} D^{-n} \cdots \theta_0^{-1} D^{-n} \phi_0^{-1}$$

and

$$D^n = (D^{-1})^n$$

It follows that at each step the resulting quantity will converge uniformly on every $\sqrt{m}$ as $m \to \infty$.

Now $L_x^{-1} L_x^{k+1} \phi_m(x)$ vanishes at $x = x_1$ together with all derivatives of order less than

$$s = m_1 + m_2 + \ldots + m_j.$$ 

Thus,

$$L_x^k \phi_m(x) = L_x^{-1} L_x^{k+1} \phi_m(x) + \sum_{j \leq s} a_{m j} h_j(x) \ldots \ldots \ldots (6.3.6)$$

where the sum denotes the solution of the differential equation $L_x y = 0$ whose derivatives at $x_1$ of order less than $s$ are the same as those of $L_x^k \phi_m(x)$. Let us consider the equation

$$L_x y = 0$$

Let $h_j(x)$ denote the solution of this equation such that

$$h_j^i (x_1) = \delta_{ij}, \quad i = 0, 1, 2, \ldots, s-1$$

$$h_j^i = D^i (h_j)$$

$$\delta_{ij} = 0 \quad i \neq j$$

$$= 1 \quad i = j$$
Now by Hurewich [65, page 46] the $h_j(x)$ are linearly independent smooth functions on every $\sim$; also
\[ a_{mj} = D^j L^k_x \phi_m \text{ at } x_1. \]

Hence it follows that
\[
\sum_{j<k} a_{mj} h_j(x) \text{ converges in } L_2, g(\sim) \quad \ldots (6.3.7)
\]
for every $\sim$ for $L^k_x \phi_m$ converges in $L_2, g(I)$ and
\[
L^{-1}_x L^k_x \phi_m \text{ converges uniformly on every } \sim \text{ as } m \to \infty. \text{ As the } h_j \text{ are linearly independent, the coefficients } a_{mj}
\]
converges to a limit $a_j$ for every $j$. Consequently (6.3.7) and therefore $L^k_x \phi_m$ converge uniformly on every $\sim$.

Let $\varphi_k$ denote the uniform limit of $L^k_x \phi_m$. Then $\varphi_k$ is continuous function on $I$ for every $k$, and therefore we obtain from (6.3.6)
\[
\varphi_h(x) = L^{-1}_x \varphi_{h+1}(x) + \sum_{j<k} a_j h_j(x) \quad \ldots (6.3.8)
\]

We conclude that $\varphi_h$ is a smooth function and that
\[
\varphi_{k+1} = L_x \varphi_k; \text{ hence } \varphi_k = L^k_x \varphi_0. \text{ Also } \varphi'_h(x) = \varphi_h(x)
\]
almost everywhere on $I$ since $\varphi_k$ is the uniform limit on every $\sim$ of $\{L^k_x \phi_m\}_m$ and $\varphi'_k$ is the limit in $L_2, g(I)$ of $\{L^k_x \phi_m\}_m$. Thus both $L^k_x (\varphi_0)$ and $\varphi'_k$ are in the same equivalence class in $L_2, g(I)$. It follows that, for every $k$, $\alpha_k (\varphi_0) = \alpha_0 (L^k_x \varphi_0) < \infty$ and
\[ \omega_k (\varphi_0 - \varphi_m) = \alpha_0 (\int \varphi_0 - \int \varphi_m) \rightarrow 0, \text{ as } m \rightarrow \infty \]

In order to complete the proof, we have to show that

\[ (\int \varphi_0, \psi_n) = (\varphi_0, \int \psi_n) \text{ for every } n \text{ and } k. \]

For,

\[ (L^k \varphi_0, \psi_n) = \lim_{m \rightarrow \infty} (L^k \varphi_m, \psi_n) \]

\[ = \lim_{m \rightarrow \infty} (\varphi_m, L^k \psi_n) \]

\[ = (\varphi_0, L^k \psi_m) \]

It completes the proof.

The space of continuous linear functionals on \( A_{L_2, g}(I) \) is denoted by \( (A_{L_2, g}(I))' \). It is also complete. The differential operator \( L_x \) on \( (A_{L_2, g}(I))' \) defined by

\[ \langle Lf, \varphi \rangle = \langle f, L\varphi \rangle, f \in \big( A_{L_2, g}(I) \big)', \varphi \in A_{L_2, g} \ldots (6.3.9) \]

Since \( L_x \) is self adjoint and continuous linear mapping of \( A_{L_2, g}(I) \) into itself. It is also continuous mapping of \( (A_{L_2, g}(I))' \) into \( (A_{L_2, g}(I))'' \).

6.3B. Some Properties of \( A \) and \( A' \)

(i) \( D(I) \) is a subspace of \( A_{L_2, g}(I) \) and convergence in \( D(I) \) implies its convergence in \( A_{L_2, g}(I) \) consequently,
the restriction of any $f \in A^*$ to $D(I)$ is in $D'(I)$. Moreover convergence in $A^*$ implies convergence in $D'(I)$.

(ii) Since $D(I) \subseteq A \subseteq L^2(I)$ and since $D(I)$ is dense in $L^2(I)$, $A_{L^2,g}(I)$ is also dense subspace of $L^2(I)$ consequently $\mathcal{E}(I)$ is a subspace of $(A_{L^2,g})'$. 

(iii) We imbed $L^2_g(I) \ (\text{and hence } A \subseteq L^2_g(I))$ into $(A_{L^2,g}(I))'$ by defining the member $f \in L^2_g(I)$ assigns to any $\varnothing \in A$ as

$$< f, \varnothing > = \int_a^b f(x) \varnothing(x)g(x,y,t)dx \ldots (6.3.10)$$

$f$ is clearly linear on $A$. Its continuity on $A_{L^2,g}(I)$ follows from the fact that if $\left\{ \varnothing_m \right\}_{m=1}^{\infty}$ converges in $A_{L^2,g}(I)$ to zero, then

$$\left| < f, \varnothing_m > \right| < \alpha_0(f) \alpha_0(\varnothing_m) \rightarrow 0 \text{ as } m \rightarrow \infty$$

Now, we prove the following lemmas that are related to series

6.3C **Lemma.** If $\varnothing \in (A_{L^2,g}(I))$ and $\left\{ \gamma_n \right\} \subseteq L^2_g(I)$ be orthonormal eigenfunctions of $L_x$ which form a basis for $L^2_g(I)$.

Then

$$\varnothing = \sum_{n=0}^{\infty} (\varnothing_n \gamma_n) \gamma_n \ldots (6.5.11)$$
where the series converges in $A$.

**Proof.** From the definitions of the space $A_{L^2, g}(I)$ we have $L_x^k \varphi \in L^2, g(I)$ for each $k = 0, 1, 2, \ldots$ and hence we may expand $L_x^k \varphi$ in a series of $\psi_n$. We have $L_x \psi_n = \lambda_n \psi_n$ where $\lambda_n$ are eigenvalues corresponding to eigfunctions $\psi_n$.

\[
L_x^k \varphi = \sum_{n=0}^{\infty} (L_x^k \varphi, \psi_n) \psi_n = \sum_{n=0}^{\infty} (\varphi, L_x^k \psi_n) \psi_n = \sum_{n=0}^{\infty} (\varphi, \lambda_n^k \psi_n) \psi_n = \sum_{n=0}^{\infty} (\varphi, \psi_n) L_x^k \psi_n
\]

The last series is convergent in $L^2, g(I)$

Therefore,

\[
\alpha_k \left[ \varphi - \sum_{n=0}^{N} (\varphi, \psi_n) \psi_n \right] = \alpha_0 \left[ L_x^k \varphi - \sum_{n=0}^{N} (\varphi, \psi_n) L_x^k \psi_n \right] \to 0 \text{ as } N \to \infty
\]

**Remark** The last result implies that $L_x \varphi$ satisfies the condition

\[(L \varphi, \psi) = (\varphi, L \psi)\]
where \( \emptyset, \psi \) are arbitrary members of \( \mathcal{A} \).

Now
\[
(L\emptyset, \psi) = \int_a^b \psi \sum_{n=0}^{\infty} (\emptyset, \psi_n) (L_x \psi_n) g(x, y, t) \, dx
\]
\[
= \sum_{n=0}^{\infty} (\emptyset, \psi_n) \int_a^b \psi (L_x \psi_n) g(x, y, t) \, dx
\]
\[
= \sum_{n=0}^{\infty} (\emptyset, \psi_n) \int_a^b \psi_n (L_x \psi_n) g(x, y, t) \, dx
\]
\[
= (\emptyset, L_x \psi)
\]

Thus \( L_x \) is self-adjoint operator on \( \mathcal{A}_{L_2, g} (I) \).

6.3D Lemma Let \( \{a_n\} \) be a sequence of complex numbers.

Then \( \sum_{n=0}^{\infty} a_n \psi_n \) converges in \( \mathcal{A}_{L_2, g} \) if and only if

\[
\lim_{N \to \infty} \left\| \sum_{n=0}^{N} a_n \lambda_n \psi_n(x) (g(x, y, t))^{1/2} \right\|_2 < \infty
\]

... (6.3.12)

Proof We have
\[
\int_a^b \left| L_x^k \sum_{n=0}^{\infty} a_n \psi_n \right|^2 g(x, y, t) \, dx
\]
\[
= \int_a^b \left| \sum_{n=0}^{\infty} a_n \lambda_n \psi_n \right|^2 g(x, y, t) \, dx
\]
\[
= \int_a^b \left| \sum_{n=0}^{N} a_n \lambda_n \psi_n \right|^2 g(x, y, t) \, dx
\]
From which \((6.3.12)\) follows.

6.3E Lemma \(L_{2,g}^g(I)\) is a subspace of \(\mathcal{L}(I)\). If \(\{\phi_m\}\) converges in \(L_{2,g}^g(I)\) to the limit \(\emptyset\), then \(\phi_m\) also converges in \(\mathcal{L}(I)\) to the same limit \(\emptyset\). The proof is similar to that of [142, Lemma 9.3.4, p.263]. Conditions for mean convergence of eigen function expression of test functions are given below. In what we shall assume that

\[
\int_a^b |\psi_n(x)| \int_0^\infty g(x,y,t)dx = 0 \text{ (}m\text{)} \quad \text{as} \quad n \to \infty, \ m \in \mathbb{R}
\]

In particular

6.3F Lemma Let \(\{a_n\}\) be a sequence of complex numbers

Then \(\sum_{n=0}^{\infty} a_n \psi_n\) converges in \(L_{2,g}^g(I)\) if

\[
\sum_{n=1}^{\infty} (a_n^k \lambda_n^r)^p
\]

converges for each \(k = 0, 1, 2, \ldots\)

and \(r > \varphi + 1\)

Proof We have
207

\[ = \int_a^b \left( \sum_{n=1}^{N} a_n \lambda_n^{-1} \gamma_n \right) g(x, y, t) \, dx \]

\[ = \int_a^b \left( \sum_{n=1}^{N} \left( a_n \lambda_n \gamma_n \right) \right) g(x, y, t) \, dx \]

\[ \chi \left( \sum_{n=1}^{N} \left( a_n \lambda_n \gamma_n \right) \right) \]

In view of (6.3.13) the last series converges for \( r > p + 1 \).

Therefore

\[ \sum_{n=1}^{\infty} a_n \gamma_n \]

converges in \( A_{L_2} g \)

if \( \sum_{n=1}^{\infty} \left( a_n \lambda_n \right)^p \) is convergent for each \( k = 0, 1, 2, \ldots \)

6.4 Expansion of Generalized Functions

The following theorem provides orthonormal series expansion of generalized functions in \( A_{L_2} g \).

6.4 A Theorem If \( f \notin (A_{L_2}, g) \), then

\[ f = \sum_{n=0}^{\infty} \langle f, \gamma_n \rangle \gamma_n \] \( \cdots \) (6.4.1)

where the series converges in \( \lambda \).
Proof For any $\emptyset \in A$ and using Lemma 6.3

$$< f, \emptyset > = < f, \sum_{n=0}^{\infty} (g_n^*, \psi_n) \psi_n > = \sum_{n=0}^{\infty} < f, \psi_n > < \emptyset, \psi_n >$$

$$= < \sum_{n=0}^{\infty} (f_n^*, \psi_n) \psi_n, \emptyset > \quad \ldots (6.4.2)$$

Thus right hand side of (6.4.2) converges for every $\emptyset \in A$ which means series in (6.4.1) converges in $A'$. The series (6.4.1) may be considered as an inversion formula for a certain generalized transformation of $f \leftrightarrow (A_{L^2, g})'$ defined by

$$F(n) = T_f = < f, \psi_n >, \quad n = 0, 1, 2, \ldots \quad \ldots (6.4.3)$$

The inverse transformation given by (6.4.1) can be expressed as

$$f = T^{-1} F(n) = \sum_{n=0}^{\infty} F(n) \psi_n \quad \ldots (6.4.4)$$

where the series converges in $(A_{L^2, g})'$.

6.4. B Theorem (The uniqueness Theorem)

If $f, h \in A'$ and if their transforms satisfy $F(n) = H(n)$ for every $n$, then $f = h$ in the sense of equality in $A'$.

The proof follows on using (6.4.4)

The next theorem characterizes a sequence $\{ b_n \}$ so that

$$b_n = < f, \psi_n >, \quad f \leftrightarrow (A_{L^2, g})'$$
6.4 C Theorem

Let \( \{ b_n \} \) be a sequence of complex numbers. Assume that
\[
\left( \int_a^b \left( \gamma_n(x) \right)^p g(x,y,t) \, dx \right)^{1/p'} = o(n^p), \quad n \to \infty
\] ... (6.4.5)

Then for the convergence of
\[
\sum_{n=0}^{\infty} b_n \gamma_n
\] ... (6.4.6)

the condition for some non-negative integer
\[
\sum_{\lambda_n \neq 0} \lambda_n^{-q} r^{-n} p_n < \infty, \quad n > 1
\] ... (6.4.7)

is sufficient if \( r = p, \quad l = p = \frac{1}{2} \) and necessary if \( r > p + 1, \quad l = p' = \frac{1}{2} \).

Furthermore if \( f \) denotes the sum in \( (\mathbb{L}_2, g) \)' of (6.4.6)

then \( b_n = \langle f, \gamma_n \rangle \).

Proof Let \( \mathcal{G} \in \mathbb{L}_2, g \) then by Lemma 6.3C,
\[
\sum_{n=0}^{\infty} a_n \gamma_n
\]
where \( a_n = \langle \mathcal{G}, \gamma_n \rangle \) converges in \( \mathbb{L}_2, g \). Therefore,
\[
|a_n| = \left| \int_a^b \mathcal{G}(x) \gamma_n(x) g(x,y,t) \, dx \right| \leq c_0(\mathcal{G}) x
\]
\[
x \left( \int_a^b \left| \gamma_n(x) \right|^p g(x,y,t) \, dx \right)^{1/p'}
\]
Here we can choose $p' = 2$ that is, $\frac{1}{p'} = \frac{1}{2}$ consequently

$$a_n = o(n^p), \ n \to \infty \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (6.4.8)$$

Therefore by Lemma 6.3D

$$\lim_{N \to \infty} \left\| \sum_{n=0}^{N} n^p \lambda_n^k \gamma_n(x) (g(x, y, z))^{1/p'} \right\|_{p'} < \infty$$

... (6.4.9)

Here we choose $p' = 2$

Now for $n > 1$

$$\sum_{n=0}^{\infty} \left| b_n \gamma_n \right| = \sum_{n=0}^{\infty} \left| \int_a^b b_n \gamma_n(x) \phi(x) g(x, y, t) \, dx \right|$$

$$\leq \int_a^b \sum_{n=0}^{\infty} \left| b_n \lambda_n^{-k} n^p \phi \right| \left| \lambda_n n^p \gamma_n(x) \right|^p \, x \ g(x, y, t) \, dx$$

$$\leq \phi(x) \left[ \sum_{n=0}^{\infty} \left| b_n \lambda_n^{-k} \right|^p \right]^{1/p}$$

$$\leq C \phi(x) \left[ \sum_{n=0}^{\infty} \left| b_n \lambda_n^{-k} \right|^p \right]^{1/p}$$

$$\leq C \phi(x) \left[ \sum_{n=0}^{\infty} \left| \lambda_n n^p \gamma_n(x) \phi(x) g(x, y, t) \right|^{p'} \right]^{1/p'}$$

$$\leq C \phi(x) \left[ \sum_{n=0}^{\infty} \left| \lambda_n n^p \gamma_n(x) \phi(x) g(x, y, t) \right|^{p'} \right]^{1/p'}$$

$$\leq C \phi(x) \left[ \sum_{n=0}^{\infty} \left| \lambda_n n^p \gamma_n(x) \phi(x) g(x, y, t) \right|^{p'} \right]^{1/p'}$$
The first series on the right converges in view of (6.4.7) and the second converges by (6.4.9). This shows that (6.4.6) converges in \( \mathbb{A}_{L_2, g} (I)' \). Now assume that (6.4.6) converges in \( \mathbb{A}_{L_2, g} (I)' \). Then we show by contradiction that (6.4.7) converges for some \( q \). The rest part of the proof is similar to that of Theorem 9.6.1 [142].

Finally assume that (6.4.6) converges in \( \mathbb{A}_{L_2, g} (I)' \) to its sum \( f \), say. Then
\[
< f, \gamma_m > = \sum_{n=0}^{\infty} b_n \gamma_n, \gamma_m = b_m
\]
by orthogonality of \( \gamma_n \). This verifies the last statement of the Theorem.

The following theorem provides another characterization of elements of \( \mathbb{A}_{L_2, g} (I)' \).

6.4 D Theorem A necessary and sufficient condition \( f \) to be a member of \( \mathbb{A} ' \) is that there be some non-negative number \( q \) and \( h \in L_{2, g} (I) \) such that
\[
f = L^q h + \sum_{n=0}^{\infty} c_n \gamma_n
\]
where \( c_n \) denotes complex constants.

Sufficient Part. Since \( L_x \) is a continuous linear mapping \( \mathbb{A}_{L_2, g} (I)' \) into \( \mathbb{A}_{L_2, g} (I)' \) and \( L_{2, g} (I) \subseteq (\mathbb{A}_{L_2, g} (I)') \).

it follows that \( L_x^k (h) \in (\mathbb{A}_{L_2, g} (I))' \). Also since
Necessary Part

Let \( f \notin \sum_{n=0}^{\infty} F(n) \varphi_n \). Set \( H(n) = \sum_{\lambda_n \neq 0}^{\infty} \lambda_n^{-q} F(n) \) where \( \lambda_n \neq 0 \) and \( q > 0 \) such that

\[
\sum_{\lambda_n = 0}^{\infty} \lambda_n^{-q} F(n)^2 \text{ converges}
\]

Also set \( H(n) = 0 \) when \( \lambda_n = 0 \). Hence \( \sum_{n=0}^{\infty} |H(n)|^2 \) converges and by the Riesz-Fischer Theorem [121], there exists a \( h \notin L_2, g(I) \) such that \( H(n) \to (h, \varphi_n) \).

Moreover, since \( \varphi_n \in A \) the definition of \( L_x \) on \( A' \) yields

\[
(h, \lambda_n \varphi_n) = (h, L_x \varphi_n) = (L_x h, \varphi_n)
\]

Altogether then, we may write

\[
f = \sum_{n=0}^{\infty} F(n) \varphi_n = \sum_{\lambda_n \neq 0}^{\infty} \lambda_n^{-q} H(n) \varphi_n + \sum_{\lambda_n = 0}^{\infty} F(n) \varphi(n)
\]

\[
= \sum_{n=0}^{\infty} (L_x^q h, \varphi_n) \varphi_n + \sum_{\lambda_n = 0}^{\infty} F(n) \varphi(n)
\]

\[
= L_x^q h + \sum_{\lambda_n = 0}^{\infty} F(n) \varphi(n)
\]
6.5 Application

We know that $L_x$ is a continuous linear mapping of $(A_{L_2, g}(I))'$ into itself. Therefore for every $f \in A'$

$$L_x^k = \sum_{n=0}^{\infty} \langle f, \gamma_n \rangle L_x \psi_n = \sum_{n=0}^{\infty} \langle f, \psi_n \rangle \lambda_n^k \psi_n \quad \ldots \quad (6.5.1)$$

Now consider the differential equation

$$P(L_x)u = h, \quad a < x < b \quad \ldots \quad (6.5.2)$$

where $P$ is a polynomial, $h \in A'$ is given and $u \in A'$ is unknown to be determined. Using (6.5.2) and applying transformation (6.4.3) we have

$$P(\lambda_n) U(n) = H(n) \quad \ldots \quad (6.5.3)$$

where $T_u = U$, $Th = H$

If $P(\lambda_n) \neq 0$ for any $n$, we can divide by $P(\lambda_n)$ and then apply the inverse transformation (6.4.4) to get the equation

$$u = \sum_{n=0}^{\infty} \frac{H(n)}{P(\lambda_n)} \psi_n \quad \ldots \quad (6.5.4)$$

By Theorem 6.4A and applying transformation (6.4.3) the solution exists in $(A_{L_2, g}(I))'$. If $P(\lambda_n) = 0$ for some $\lambda_n$ say for $\lambda_{n_k}$ for $k = 0, 1, 2 \ldots m$ then solution exists in

$(A_{L_2, g}(I))'$ if and only if $H(\lambda_n) = 0$ for $k = 1, 2 \ldots m$
In this case a solution to (6.5.2) is
\[ u = \sum_{n=0}^{\infty} \frac{H(n)}{p(\lambda_n)} \psi_n \] ... (6.5.5)
which is no longer unique in \((A_{L_2, g(I)})^*\) and we can add to it any complementary solution
\[ u = \sum_{k=1}^{\infty} a_k \psi_{nk} \]
where \(a_k\) are arbitrary constants.

Using 6.5 we can solve the following differential equations

**Example 1** Let
\[ L_x = -\frac{d^2}{dx^2} + x^2 \] ... (i) ... (i)
Then the eigenvalues of the operator are
\[ \lambda_n = 2n + 1 \] ... (ii)
and the corresponding normalized eigenfunctions
\[ \psi_n(x) = (\frac{1}{2^n n! \pi^{1/2}})^{1/2} e^{-x^2/2} H_n(x) \] ... (iii)
where \(H_n(x)\) is the Hermite polynomial of degree \(n\).

Consider the equation
\[ \frac{\partial^2}{\partial x^2} u(x,t) - x^2 u(x,t) = \psi_n(x) \lambda_n e^{\lambda_n t} \]
\[ (-L_x)u(x,t) = \psi_n(x) \lambda_n e^{\lambda_n t} \] ... (iv)
Applying (6.5.3), we have,

\[ (-\lambda_n)Tu = \lambda_n e^{\lambda_n t} T \]

\[ Tu = - e^{\lambda_n t} \langle \psi_n', \psi_n \rangle \]

\[ Tu = - e^{\lambda_n t} = H(n), \text{ say} \]

By (6.5.4)

\[ u = \sum_{n=0}^{\infty} \frac{-\psi_n(x) H(n)}{P(\lambda_n)} \]

\[ u = \sum_{n=0}^{\infty} \frac{1}{2^{n/2} (n!)^{1/2} \pi^{1/4}} e^{-x^2/2} H_n(x) e^{(2n+1)t} \]

Example 2

\[ L_x = - \frac{xd^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} + \frac{\alpha^2}{4x} \]

The eigenvalues of \( L_x \) are

\[ \lambda_n = n + \frac{\alpha + 1}{2} \]

and corresponding normalized eigenfunctions

\[ \psi_n(x) = \left[ \frac{n!}{\varphi(n)} \right]^{1/2} (-)^{\pi/2} x^{\alpha/2} L_n^\alpha(x) \]

where \( L_n^\alpha(x) \) is the Laguerre polynomial of degree \( n \) and

\[ \varphi(n) = \frac{n!}{\sqrt{(n+\alpha+1)}} \]
The heat equation is

\[ x \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} = \left( \frac{x}{4} + \frac{\alpha^2}{4x} \right) u(x,t) \]

\[ = \lambda_n e^{\lambda_n t} \Psi_n(x), \quad \lambda_n \in \Lambda \]

\[ (-L_x) u(x,t) = \lambda_n e^{\lambda_n t} \psi_n(x) \quad (v) \]

By applying (6.5.3) we have

\[ Tu = -e^{\lambda_n t} < \psi_n, \psi_n > \]

\[ Tu = -e^{\lambda_n t} = H(n) \]

By (6.5.4)

\[ u = \sum_{n=0}^{\infty} \frac{\psi_n(x) H(n)}{P(\lambda_n)} \]

\[ u = -\sum [ Q(n) ]^{1/2} e^{-\pi/2} \frac{x}{\alpha/2} L_n^\alpha(x) e^{\lambda_n t} \quad \ldots (vi) \]

\[ \text{ooo} \]