CHAPTER VI

ASYMPTOTIC EXPANSION OF GENERALIZED STIELTJES TRANSFORM
6.1 Introduction

Asymptotic analysis is that branch of Mathematics devoted to the study of the behaviour of functions at and near given points in their domains of definition.

To describe the behaviour of the function $f(z)$ as $|z| \rightarrow \infty$ within a sector $\alpha \leq \arg z \leq \beta$, it is in many cases sufficient to derive an expression of the form

$$f(z) = \phi(z) \left[ 1 + r(z) \right] \quad \text{(6.1.1)}$$

where $\phi(z)$ is a function of a simpler structure than $f(z)$ and $r(z)$ converges uniformly to zero as $|z| \rightarrow \infty$ within the given sector (6.1.1) is called the asymptotic representation of $f(z)$ for large $|z|$. Now let us see the definition of asymptotic series or asymptotic expression of $f(z)$ for large $|z|$. Let $f(z)$ be a function of the real or complex variable $z$, $\sum_{s=0}^{\infty} a_n z^{-s}$ - a formal power series, $R_n(z)$ - the difference between $f(z)$ and the $n^{th}$ partial sum of the series. Thus

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_{n-1}}{z^{n-1}} + R_n(z). \quad \text{(6.1.2)}$$
Suppose that for each fixed value of $n$

$$R_n(z) = O(z^{-n}) \text{ as } z \to \infty$$

Then we say that the series $\sum_{s=0}^{\infty} a_n z^{-s}$ is an asymptotic expansion of $f(z)$ and write

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots \quad z \to \infty.$$

Asymptotic series are very useful since by taking a finite number of terms, we can obtain an arbitrarily good approximation to the function $f(z)$ for large $|z|$.

It will not be out of place if we take a brief survey of the work done on the asymptotic expansion of Stieltjes transform.

Let $f$ be a locally integrable function on $[0, \infty)$ and let $S_f(z)$ denote the Stieltjes transform of $f$ given by

$$S_f(z) = \int_0^\infty \frac{f(t)}{t+z} \, dt \quad \ldots \quad (6.1.3)$$

where $z$ is a complex parameter in the cut plane $|\arg z| < \pi$.

If each of the moments of $f$,

$$\mu_s = \int_0^\infty t^s f(t) \, dt \quad s = 0, 1, 2 \ldots \quad \ldots \quad (6.1.4)$$

is finite, and if $x$ is positive then Olver [7,1.92] gave the following asymptotic expansion for $S_f(x)$
\[
S_f(x) = \sum_{s=0}^{n-1} (-1)^s \mu_s x^{-s-1} + \mathcal{E}_n(x) \quad \ldots (6.1.5)
\]
where
\[
|\mathcal{E}_n(x)| \leq x^{-n-1} \sup_{(0, \infty)} \left| \int_0^t \gamma^n f(\tau) d\tau \right| \quad \ldots (6.1.6)
\]

Woolcock [10,11] and Zimering [12] have considered the asymptotic behaviour of \( S_f(z) \) for large values of \( |z| \), when \( f(t) \) behaves like \( Ct^{-\alpha} \) as \( t \) tends to infinity, \( C \) being a constant and \( \alpha \geq 0 \). They obtain only the dominant terms of infinite asymptotic expansions. Recently Handelsman and Lew [3,4] developed a powerful technique which yields the asymptotic expansion, in the two limits \( \lambda \to 0^+ \) and \( \lambda \to +\infty \) for a class of functions defined by integrals
\[
I(\lambda) = \int_0^\infty h(\lambda t) f(t) dt \quad \ldots (6.1.7)
\]
where \( h(t) \) and \( f(t) \) are algebraically dominated near both \( 0^+ \) and \( +\infty \). The basic idea to get the asymptotic expansion of \( I(\lambda) \) in (6.1.7) is as follows:

Suppose that \( M[h,s], \ s = \sigma + i\tau \), is the Mellin transform of \( h(t) \) evaluated at \( s \) and \( M[f,1-s] \) is the Mellin transform of \( f(t) \) evaluated at \( 1-s \). Assume further that \( M[h,s] \) and \( M[f,1-s] \) are analytic in a strip containing the line \( \text{Re} s = c \), and the Parseval relation...
\[ \sum_{c \leq \text{Res} < d} \text{Res} \left[ x^{-s} M[h,s] M[f,1-s] + R \right] (6.1.9) \]

where

\[ R = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} x^{-s} M[h,s] M[f,1-s] ds (6.1.10) \]

If the integral (6.1.10) converges absolutely, then (6.1.9) provides a finite asymptotic expansion of \( I(\lambda) \) as \( \lambda \to \infty \) with exact remainder \( R \).

Although the above method is powerful, this does not give the explicit expression for the error terms associated with the expansion of \( I(\lambda) \). Therefore, R Wong [9] modified the above method slightly to obtain the explicit expressions for the remainder term. Using the theory of distribution McClure and Wong [6] gave the asymptotic expansion for (6.1.3) with the explicit expressions for the error terms.

Also Kusum Soni [8] has shown that Parseval relation for the Mellin transform can be used to obtain an explicit expression for the remainder in the asymptotic expansion of a class of integral transform given by (6.1.7)
In the next sections to follow, we give the asymptotic expansion of a generalized Stieltjes transform by using the method of Handelsman and Lew [3,4].

6.2 Asymptotic Expansion of a Generalized Stieltjes Transform

Here we will derive the asymptotic expansion of generalized Stieltjes transform given by (6.2.1).

\[ F(\lambda) = \int_0^\infty t^{-1} H_{2,2}^{2,1} \left[ \frac{\lambda}{t} \right] \begin{pmatrix} (-\beta, 1), (\alpha + \eta, 1) \\ (0, 1), (\eta, 1) \end{pmatrix} f(t) \, dt \quad (6.2.1) \]

(6.2.1) is a particular case of the generalized Stieltjes transform given by (2.2.3). Also (6.2.1) is obtained by iterating the following Laplace transforms. If \( F(\lambda) \) is the Laplace transform of the function \( \phi(x) \)

\[ F(\lambda) = \int_0^\infty e^{-\lambda x} \phi(x) \, dx \quad \ldots (6.2.2) \]

and \( \phi(x) \) is the generalized Laplace transform of the function \( f(t) \) given by Joshi [5]

\[ \phi(x) = \int_0^\infty H_{1,2}^{1,1} \left[ tx \right] \begin{pmatrix} (-\eta, 1) \\ (\beta, 1), (-\alpha - \eta, 1) \end{pmatrix} f(t) \, dt \quad \ldots (6.2.3) \]

then after iteration we get (6.2.1)

Let us now derive the asymptotic expansion of generalized Stieltjes transform given by

\[ F(\lambda) = \int_0^\infty t^{-1} H_{2,2}^{2,1} \left[ \frac{\lambda}{t} \right] \begin{pmatrix} (-\beta, 1)(\eta + \alpha, 1) \\ (0, 1), (\eta, 1) \end{pmatrix} g(t) \, dt \quad \ldots (6.2.4) \]
Assume that

\[ g(t) \sim \sum_{s=0}^{\infty} C_s t^{-a_s - \beta} \quad \text{as } t \to \infty \]

put \( t = \frac{1}{y} \)

Therefore,

\[ F(\lambda) = \sum_{s=0}^{\infty} C_s \int_0^\infty y^{2,1} H_{2,2} [ \lambda y \begin{pmatrix} (-\beta,1), (\eta+\alpha,1) \\ (0,1), (\eta,1) \end{pmatrix}] g(\frac{1}{y}) \frac{1}{y} dy \]

\[ = \sum_{s=0}^{\infty} C_s \int_0^\infty y^{2,1} H_{2,2} [ \lambda y \begin{pmatrix} (-\beta,1), (\eta+\alpha,1) \\ (0,1), (\eta,1) \end{pmatrix}] f(y) dy \quad \ldots (6.2.5) \]

where \( f(y) = g(\frac{1}{y}) y^{-1} \sum_{s=0}^{\infty} C_s y^{a_s + \beta + 1} \quad \text{as } y \to 0 \quad \ldots (6.2.6) \)

The integral (6.2.5) is of the form (6.1.7) and we can apply the result from [2,4,2.1] which is as follows:

If \( f(t) \sim \sum_{s=0}^{\infty} C_s t^{a_s} \quad \text{as } t \to 0^+ \) and

\[ h(t) \sim \sum_{s=0}^{\infty} b_s t^{-r_s} \quad \text{as } t \to \infty \quad \text{and} \quad r_s \neq a_s + 1 \]

for all \( s \), then

\[ F(\lambda) \sim \sum_{s=0}^{\infty} \frac{-r_s}{\lambda} b_s M[f : 1-r_s] + \sum_{s=0}^{\infty} C_s \lambda^{-a_s} M[h, 1+a_s] \]

as \( \lambda \to \infty \quad \ldots (6.2.7) \)

where \( M[f,z] \) denotes the Mellin transform of the function \( f \) with parameter \( z \).
Now from (6.2.5),
\[ h(t) = H_{2,2}^{2,1} \left[ t \mid (-\beta, l), (\eta + \alpha, l) \right] (0, l), (\eta, l) \]

Using (1.6.1)
\[ h(t) = H_{2,2}^{1,2} \left[ \frac{1}{t} \mid (l, l), (l - \eta, l) \right] (l + \beta, l), (l - \eta - \alpha, l) \]

But from (1.6.11)
\[ h(t) = \sum_{s=0}^{\infty} b_s t^{-(\beta + l + s)} \]

**Theorem 6.2.1:**

If \( f(t) \) and \( h(t) \) are given by (6.2.6) and (6.2.8) respectively and \( a_s \neq s+1 \) for any \( s \), then,
\[
F(\lambda) \sim \sum_{s=0}^{\infty} c_s \lambda^{-a_s - \beta} \frac{\Gamma(a_s + \beta) \Gamma(\eta + \alpha + \beta) \Gamma(l - a_s)}{\Gamma(\eta + \alpha + \beta + a_s)}
\]
\[
+ \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{\Gamma(1 + \beta + s) \Gamma(1 + \beta + \eta + s)}{\Gamma(\alpha + \beta + \eta + l + s)} \lambda^s \frac{M[f; -\beta - s]}{\Gamma(1 + \beta + \eta + s)} \ldots (6.2.9)
\]

**Proof:** First let us determine the value of \( M[h,a_s + \beta] \)
\[
M[h,a_s + \beta] = \int_0^\infty t^{a_s + \beta - 1} H_{2,2}^{2,1} \left[ t \mid (-\beta, l), (\eta + \alpha, l) \right] (0, l), (\eta, l) \ dt
\]

Using the result (1.6.4),
\[
M[h,a_s+\beta] = \frac{\Gamma(a_s+\beta) \Gamma(\eta+a_s+\beta) \Gamma(1-a_s)}{\Gamma(\eta+\alpha+a_s+\beta)}.
\]

From the result (6.2.7), the theorem follows.

### 6.3 Particular cases

1. If \( \alpha = \beta = 0 \) then (6.2.4) reduces to

\[
F(\lambda) = \int_0^\infty \frac{q(t)}{\lambda+t} dt
\]

Now if we put \( a_s = s+\delta' \) and use (6.2.9) then the asymptotic expansion of \( F(\lambda) \) for the case (6.3.1) is given by

\[
F(\lambda) \sim \sum_{s=0}^{\infty} c_s \lambda^{-s-\delta'} \Gamma(s+\delta') \Gamma(1-s-\delta')
+ \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \Gamma(1+s) \lambda^{-1-s} M[f,-s] \]

The result (6.3.2)(without error term) is obtained by Wong [9].

2. If we put \( \alpha=\eta=0 \) and \( \beta=\phi-1 \) in (6.2.1), we get

\[
F(\lambda) = \int_0^\infty t^{-1} H_{2,2}^{2,1} \left[ \frac{\lambda}{t} \right \ (1,0) (0,1) \ (0,1) \ (0,1) \right] f(t) dt
\]

Using (1.6.12)

\[
= \int_0^\infty t^{-1} H_{1,1}^{1,1} \left[ \frac{\lambda}{t} \right \ (1,0) (0,1) \right] f(t) dt.
\]

by using (1.6.7)

\[
F(\lambda) = \int_0^\infty t^{\alpha-1} \Gamma(\beta) \frac{f(t)}{(\lambda+t)^\beta} dt
\]
put \( t^{\phi-1} f(t) = g(t) \)

Therefore we get,

\[
F(\lambda) = \Gamma(\phi) \int_0^\infty g(t) \frac{dt}{(\lambda+t)^\phi} \quad \ldots \ (6.3.3)
\]

or

\[
F(\lambda) = \Gamma(\phi) S_g(\lambda) \quad \ldots \ (6.3.4)
\]

where \( S_g(\lambda) \) stands for Stieltjes transform of \( g(t) \).

From (6.3.4), we get the asymptotic expansion for \( S_g(\lambda) \) after using the result (6.2.9) as

\[
S_g(\lambda) = \sum_{s=0}^{\infty} c_s \lambda^{-s-\phi} \frac{\Gamma(s+\phi+1) \Gamma(1-s-\phi)}{\Gamma(\phi)}
\]

\[
+ \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{\Gamma(q+s)}{\Gamma(\phi)} \lambda^{-(q+s)} M \left[ f, -q +1-s \right] \ldots (6.3.5)
\]

(6.3.5) (without error term) is exactly the same result obtained by R Wong [9, example 1].

(3) After putting \( \phi = 0, \eta = 2m, \alpha = \frac{1}{2} - m - k \) in (6.2.4) we get the generalized Stieltjes transform given by Arya[1]

\[
F(\lambda) = \sum_{s=0}^{\infty} t^{-1} \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})} 2F_1(2m+1, 1, m-k+\frac{3}{2}, \frac{t}{\lambda}) g(t) dt
\]

\ldots (6.3.6)

The theorem (6.2.1) yields the asymptotic expansion of \( F(\lambda) \) given by (6.3.6) as:
\( F(\lambda) \sim \sum_{s=0}^{\infty} c_s \lambda^{-s-\sigma} \frac{\Gamma(s+\sigma) \Gamma(2m+s+\sigma)}{\Gamma(m+\frac{1}{2} - k+s+\sigma)} \\
+ \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{\Gamma(1+s) \Gamma(1+2m+s)}{\Gamma\left( \frac{3}{2} + m - k + \sigma \right)} \lambda^{-1-s} M[f,-s] \quad (6.3.7) \)

Remark 1: As the result (6.2.1) does not give the error term, R Wong [9] has extended the above method slightly. Following the same method in [9], we get the following result for our generalized Stieltjes transform (6.2.4)

From (6.2.6) and (6.2.8), if

\[
\begin{align*}
\lim_{t \to 0} f(t) &= \sum_{s=0}^{n-1} c_s t^{\alpha+s-1} + f_n(t) \\
\lim_{t \to \infty} h(t) &= \sum_{s=0}^{n-1} b_s t^{-(\beta+s)} + h_n(t)
\end{align*}
\]

then as \( \lambda \to \infty \)

\[
F(\lambda) \sim \sum_{s=0}^{n-1} c_s \lambda^{-a-s-\beta} \frac{\Gamma(\alpha+s+\beta) \Gamma(\eta+\alpha+s+\beta)}{\Gamma(\eta+\alpha+\beta+\sigma)} \\
+ \sum_{s=0}^{n-1} \frac{(-1)^s}{s!} \frac{\Gamma(1+\beta+s) \Gamma(1+\beta+\eta+s)}{\Gamma(\alpha+\beta+\eta+1+s)} \lambda^{-1-\beta+s} M[f,-\beta-s] \\
+ \delta_n(\lambda)
\]

where \( \delta_n(\lambda) \) is the error term given by

\[
\delta_n(\lambda) = \int_0^\infty f_n(t) h_n(\lambda t) dt
\]

and \( \delta_n(\lambda) \to 0 \) as \( \lambda \to \infty \).
Remark 2: The method to derive an asymptotic expansion of $I(\lambda)$ for large $\lambda$ in [2; chapter 4] can be used to get an asymptotic expansion of $I(\lambda)$ for small value of $\lambda$ after interchanging the roles of $f$ and $h$ as in theorem 6.2.1. Proceeding on the same line [2, 4.6], we have the following result for (6.2.4).

If $f(t) = \sum_{s=0}^{\infty} b_s t^{-s-\beta}$ as $t \to \infty$

$$h(t) = \sum_{s=0}^{\infty} c_s t^{s+\beta-1}$$ as $t \to 0$

then,

$$F(\lambda) \sim \sum_{s=0}^{\infty} b_s \lambda^{\beta+s} \mathbb{M}[h, -\beta-s]
+ \sum_{s=0}^{\infty} c_s \lambda^{s+\beta-1} \mathbb{M}[f; a_s+\beta]$$ as $\lambda \to 0 +$
REFERENCES


