CHAPTER V

ABELIAN THEOREMS
5.1 Introduction

If \( F(s) \) is the generalized Stieltjes transform of \( f(t) \), then in this chapter we will prove that the behaviour of generating function \( F(s) \) as \( s \to 0 \) \((s \to \infty)\) is related to the behaviour of determining function \( f(t) \) as \( t \to 0 \) \((t \to \infty)\). Such type of results are known as Abelian theorems. Sometimes these results are also called as initial value theorem and final value theorem respectively.

Widder [15] has proved the Abelian theorems when Stieltjes transform is defined as follows.

\[
F(s) = \int_0^\infty \frac{d\alpha(t)}{s+t} \quad ... (5.1.1)
\]

where \( \alpha(t) \) is a normalized function of bounded variation and \( s \) is a real variable. Following the Widder's method [15], Arya [1] gave the Abelian theorems for the generalized Stieltjes transform given by

\[
F(s) = \frac{\Gamma(2m+1)}{\Gamma(m+k+\frac{3}{2})} \frac{1}{s} \int_0^\infty 2F_1(2m+1,1,m-k+\frac{3}{2})d\alpha(t) \quad ... (5.1.2)
\]

Misra [12] has derived the initial value and final value Abelian theorems when the generalized Stieltjes transform is
defined by

\[ F(s) = \int_0^\infty \frac{f(t)}{(s+t)^{q+1}} \, dt \quad \text{Re} \, s > 0 \quad \ldots \quad (5.1.3) \]

Also in his paper Misra [12] has extended the above results to a certain class of generalized functions \( s' \). Carmichael [4] generalized these Abelian theorems to the complex plane. Thus showing that the Abelian theorems for the Stieltjes transform of the function hold in as general a setting as the Abelian theorems for the Laplace transform to which the Stieltjes transform is closely related.

Lavoine and Misra have extended (5.1.3) to a certain space of generalized functions and have obtained some Abelian theorems for distributional Stieltjes transformation for real \( s \) in their papers [9, 10] and Carmichael [5] has proved that the same Abelian theorems for the distributional Stieltjes transform hold good when the transform is a function of the complex variable \( s = \sigma + iw \).

Tiwari and Koranne [14] have extended Joshi's generalized Stieltjes transform given by (2.4.5) to generalized functions and proved some Abelian theorems. Joshi [7] gave Abelian theorems for (2.4.7). Some work on Abelian theorems for a distributional generalized Stieltjes transform is also done by Rao [13].
In this chapter we prove Abelian theorems for the generalized Stieltjes transform given by (5.1.4)

$$F(s) = \int_0^\infty s^{-1} H^{1,2}_{2,2} \left[ \frac{t}{s} \lambda \begin{vmatrix} (d_1,D_1), (1-b_1^2-B_1,\lambda B_1); \\ (e_1,E_1); (e_2,E_2) \end{vmatrix} \right] f(t) \, dt$$

... (5.1.4)

Of course, (5.1.4) is a particular case of (2.2.4) when we put $A = \mu = B = m = q = \alpha = \beta = \gamma = 1$, $\delta = 2$ and $n = p = o'$. It is also interesting to see how (5.1.4) is obtained after iterating Laplace transform (5.1.5) with the generalized Laplace transform (5.1.6). Let $F(s)$ be a Laplace transform of a function $\varnothing(u)$,

$$F(s) = \int_0^\infty H^{1,0}_{1,1} \left[ Su \begin{vmatrix} (b_1,B_1) \end{vmatrix} \varnothing(u) \right] \, du$$

... (5.1.5)

By (1.6.6) with $b_1 = 0$, $B_1 = 1$, the kernel of (5.1.5) reduces to $e^{-su}$. Again let $\varnothing(u)$ be the generalized Laplace transform of the function $f(t)$ given by

$$\varnothing(u) = \int_0^\infty H^{1,1}_{1,2} \left[ (tu)^\lambda \begin{vmatrix} (d_1,D_1) \\ (e_1,E_1)(e_2,E_2) \end{vmatrix} f(t) \right] \, dt$$

... (5.1.6)

For $\lambda = 1$, (5.1.6) is the generalization of the Laplace transform given by Joshi [8], which is discussed fully in section (3.1). Thus when (5.1.5) is iterated with (5.1.6), we get the generalized Stieltjes transform (5.1.4).

In Section 5.2, the classical initial value and final value Abelian theorems are established for $s \rightarrow 0$. In
Section 5.3, the said results are extended to a certain class of generalized functions. Lastly in Section 5.5, application of the Abelian Theorem is given.

5.2 Abelian Theorems

Theorem 5.2.1 (Initial Value Abelian Theorem)

If

(i) the generalized Stieltjes transform $F(s)$ of a complex valued function $f(t)$ is defined by (5.1.4)

(ii) $\lim_{t \to 0} \frac{f(t)}{t^\eta} = \alpha$ where $\eta$ is a real number and $\alpha$ is a complex number.

(iii) $\frac{f(t)}{t^\eta}$ is bounded on $y < t < \infty$ for all $y > 0$

(iv) $-1 - \lambda P_1 < \eta < -1 - \lambda Q_1$ where $P_1$ and $Q_1$ are real numbers as defined in (5.2.2) and (5.2.3)

Then

$$\lim_{s \to \infty} s^{-\eta} F(s) \lambda G(\lambda, \eta) = \alpha$$

where

$$G(\lambda, \eta) = \frac{\Gamma(1-e_2 - E_1 (\frac{\eta}{\lambda} + \frac{1}{\lambda}))}{\Gamma(e_1+E_1(\frac{\eta}{\lambda} + \frac{1}{\lambda})) \Gamma(1-d_1-D_1(\frac{\eta}{\lambda} + \frac{1}{\lambda})) \Gamma(b_1+B_1,-B_1(\eta+1))} \ldots (5.2.1)$$

For brevity, henceforth the function in (5.1.4) will be denoted by $H (\frac{t^\lambda}{s})$. 
Proof:

From the condition (ii)
\[ f(t) = O(t^n) \text{ as } t \to 0 \]

and from the asymptotic properties (1.5,1) and (1.5,3) of the H-function

\[ H \left( \frac{t}{s} \right)^{\lambda} = O \left( \left| \frac{t}{s} \right|^{\lambda P_1} \right) \text{ for small } x \]

where
\[ P_1 = \min \text{ Re} \left( \frac{e_1}{E_1}, \frac{e_2}{E_2} \right) \quad \ldots (5.2.2) \]

and

\[ H \left( \frac{t}{s} \right)^{\lambda} = O \left( \left| \frac{t}{s} \right|^{\lambda Q_1} \right) \text{ for large } x \]

where
\[ Q_1 = \max \text{ Re} \left( \frac{d_1-1}{B_1}, \frac{-b_1-B_1}{\lambda B_1} \right) \quad \ldots (5.2.3) \]

The condition (iv) follows and the Stieltjes transform \( F(s) \) of the function \( f(t) \) exists.

Consider,

\[ \int_0^\infty t^n s^{-1} H_2^{1,2} \left[ \left( \frac{t}{s} \right)^{\lambda} \right] \left( d_1, D_1, 1-b_1-B_1, \lambda B_1 \right) \left( e_1, E_1, e_2, E_2 \right) \right] dt \ldots (5.2.4) \]

\[ = \frac{1}{\lambda} \int_0^\infty s^n y^{\frac{n-1}{\lambda}} H_2^{1,2} \left[ y \left( d_1, D_1, 1-b_1-B_1, \lambda B_1 \right) \left( e_1, E_1, e_2, E_2 \right) \right] dy \]

(after putting \( \left( \frac{t}{s} \right)^{\lambda} = y \) in 5.2.4)

Now using the result (1.6.4), we get
\[
G(\lambda, \eta) \text{ is given in (5.2.1)}
\]

Now consider
\[
\left| s^{-\eta} F(s) - \frac{\alpha}{\lambda G(\lambda, \eta)} \right|
\]
(Using (5.2.5) we have,)

\[
= \left| s^{-\eta} \int_0^\infty s^{-1} H\left(\frac{t}{s}\right)^\lambda f(t) dt - \alpha s^{-\eta} \int_0^\infty t^\eta s^{-1} H\left(\frac{t}{s}\right)^\lambda dt \right|
\]
\[
= \left| s^{-\eta} \int_0^\infty s^{-1} H\left(\frac{t}{s}\right)^\lambda \left[ f(t) - \alpha t^\eta \right] dt \right|
\]
\[
= \int_0^\gamma s^{-\eta} \int_0^{s^{-1} H\left(\frac{t}{s}\right)^\lambda \left[ F(t) - \alpha t^\eta \right] dt \right|
\]
\[
+ \int_\gamma^\infty s^{-\eta} \int_0^{s^{-1} H\left(\frac{t}{s}\right)^\lambda \left[ f(t) - \alpha t^\eta \right] dt \right|
\]
\[
= I_1 + I_2
\]

First consider \(I_1\)

\[
I_1 \leq s^{-\eta} \sup_{0 \leq t \leq y} \left| \frac{f(t)}{t^\eta} - \alpha \right| \left| \int_0^\infty s^{-1} t^\eta H\left(\frac{t}{s}\right)^\lambda dt \right|
\]
\[
= s^{-\eta} \sup_{0 \leq t \leq y} \left| \frac{f(t)}{t^\eta} - \alpha \right| \left| \frac{s^\eta}{\lambda G(\lambda, \eta)} \right| \quad (\text{by 5.2.5})
\]
Choos y so small that \( \left| \frac{f(t)}{t^n} - \alpha \right| < \frac{\epsilon}{2\lambda G(\lambda, \eta)} \)

in \( \alpha < t < y \), and for \( \epsilon > 0 \). Hence we have

\[ I_1 \rightarrow 0 \quad \ldots \quad (5.2.6) \]

Having fixed \( y \) in this way and using \( 5.2.3 \),

\[ I_2 \leq M \begin{array}{c} s^{-\eta-1-\lambda Q_1} \int_y^\infty t^{\lambda Q_1+\eta} \left[ \frac{f(t)}{t^n} - \alpha \right] dt \end{array} \]

for some constant \( M \). Since \( \frac{f(t)}{t^n} \) is a bounded quantity

in \( y < t < \infty \), for some constant \( M \),

\[ I_2 \leq M_1 s^{-\eta-1-\lambda Q_1} \frac{\lambda Q_1+\eta+1}{\lambda Q_1+\eta+1} \rightarrow 0 \quad \text{as} \quad s \rightarrow 0 \quad \ldots \quad (5.2.7) \]

Because \( \lambda Q_1 + \eta + 1 < 0 \) by \( (iv) \).

From \( (5.2.6) \) and \( (5.2.7) \) the result follows.

**Theorem 5.2.2 (Final Value Abelian Theorem)**

If

i) The condition \( (iv) \) of theorem 5.2.1 is satisfied.

ii) The complex valued function \( f(t) \) satisfies the following condition

\[ \lim_{t \rightarrow \infty} \frac{f(t)}{t^n} = \alpha \quad \text{for real} \ \eta \ \text{and complex} \ \alpha \]

iii) \( \frac{f(t)}{t^n} \) is bounded on \( \alpha < t < y \) for all \( y > 0 \).
then
\[ \lim_{s \to \infty} s^{-\eta} F(s) \lambda G(\lambda, \eta) = a. \]

\( G(\lambda, \eta) \) is as defined in (5.2.1).

**Proof:** Consider
\[
\left| s^{-\eta} F(s) - \frac{\alpha}{\lambda G(\lambda, \eta)} \right|
\]
( using (5.2.5) )
\[
= \left| \int_{0}^{\infty} s^{-1} H\left(\frac{t}{s}\right)^{\lambda} f(t) dt - \alpha \int_{0}^{\infty} t^{-\eta} s^{-1} H\left(\frac{t}{s}\right)^{\lambda} dt \right|
\]
\[
\leq \int_{0}^{\infty} s^{-\eta-1} H\left(\frac{t}{s}\right)^{\lambda} |f(t) - \alpha t^{\eta}| dt + \int_{0}^{\infty} s^{-\eta} \left| \int_{0}^{t} s^{-1} H\left(\frac{s}{s}\right)^{\lambda} f(t) - \alpha t^{\eta}\right| dt
\]
\[
J_1 + J_2
\]

First consider \( J_2 \). From the condition (ii) of the theorem, we can find a small positive number \( \varepsilon \) such that for \( y \) large enough,
\[
\sup_{y \leq t < \infty} \left| \frac{f(t)}{t^{\eta}} - \alpha \right| \leq \varepsilon \lambda G(\lambda, \eta)
\]
Hence we have,
\[
\left| J_2 \right| \leq \varepsilon \lambda G(\lambda, \eta) \left\| \int_{0}^{\infty} s^{-\eta-1} H\left(\frac{t}{s}\right)^{\lambda} t^{\eta} dt \right\|
\]
\[
= \varepsilon \lambda G(\lambda, \eta) \left\| \frac{s^{-\eta} s^{\eta}}{\lambda G(\lambda, \eta)} \right\| \text{ by (5.2.5)}
\]
Hence
\[
\left| J_2 \right| \leq \varepsilon
\]
Therefore $|J_2| \to 0$ as $G \to 0$ ... (5.2.8)

Now having fixed $y$ as above, consider $J_1$

$$|J_1| \leq \sup_{0 \leq t \leq y} \left| \frac{f(t)}{t^n} - \alpha \right| \int_0^y \left| s^{\eta - 1} H(s) \lambda t^n \right| \, dt$$

Due to condition (iii) and (5.2.2), for some constant $M$, we can write,

$$|J_1| \leq M \left| s^{\eta - 1 - \lambda \eta} \frac{\lambda \eta + 1}{\lambda \eta + 1} \right|$$

But from the condition (iv) of theorem 5.2.1, $-\eta - 1 - \lambda \eta < 0$. Hence,

$$|J_1| \to 0 \text{ as } s \to \infty \quad \ldots \quad (5.2.9)$$

From (5.2.8) and (5.2.9), the proof is complete.

5.3 Extension of Classical Abelian Theorem to the Space $J_{c,d}$

The space $J_{c,d}$ is fully discussed in section 3.2. However, let us recall the definition of the space $J_{c,d}$. $J_{c,d}$ is the space of all smooth functions $\varnothing(t)$ on $0 < t < \infty$ and the seminorm is defined as follows:

$$\delta_{c,d,k}(\varnothing) = \delta_k(\varnothing)$$

$$= \sup_{0 < t < \infty} \lambda_{c,d}(\log t)(tD_t)^k \sqrt{k} \varnothing(t) \quad < \infty$$

where

$$\lambda_{c,d}(\log t) = \begin{cases} t^c & 1 \leq t < \infty \\ t^d & 0 < t < 1 \end{cases}$$
J_c,d is a complete countably multinormed space. If $f \in J'_c,d$,
the dual space of $J_c,d$, then the Stieltjes transform $F$ of $f$
is defined by

$$F(s) = \langle f(t), s^{-1} H \left( \frac{t}{s} \right)^\lambda \rangle$$

For any complex $s$ not lying on the negative real axis.
And $s^{-1} H \left( \frac{t}{s} \right)^\lambda$ stands for the kernel function in (5.1.4)

**Result 5.3.1**: The kernel $K(s,t) = s^{-1} H \left( \frac{t}{s} \right)^\lambda$ is a member
of $J_c,d$ if $c + \frac{1}{2} + \lambda Q_1 < 0$ and $d + \frac{1}{2} + \lambda P_1 > 0$. $P_1$ and $Q_1$
are as defined in (5.2.2) and (5.2.3).

**Proof**: $K(s,t)$ is a smooth function and $K(s,t)$ is a member of
$J_c,d$ if and only if

$$\sup_{0 < t < \infty} \left| \lambda_{c,d} (\log t) (tD_t)^k \sqrt{t} K(s,t) \right| < \infty$$

But $(tD_t)^k \left[ \sqrt{t} K(s,t) \right] = \sum_{r=0}^{k} A_r t^{r+\frac{1}{2}} D_t^r K(s,t)$

$A_r$'s are some constants.

(Using 16.3)

$$D_t^r K(s,t) = \frac{s^{-1}}{t^r} H_{3,3}^1 \left[ (\frac{t}{s})^\lambda \right] \begin{pmatrix} \alpha, \lambda, (d_1 D_1), (1-b_1-B_1, \lambda B_1) \\ (e_1, E_1); (e_2, E_2), (r, \lambda) \end{pmatrix}$$

Therefore, we have to consider

$$\sup_{0 < t < \infty} \left| \lambda_{c,d} (\log t) \sum_{r=0}^{k} A_r t^{r+\frac{1}{2}} s^{-1} t^{1-r} H_{3,3}^1 \left( \frac{t}{s} \right)^\lambda \right|$$

The general term of the above is

$$\lambda_{c,d} (\log t) B_r t^{\frac{1}{2}} H \left( \frac{t}{s} \right)^\lambda$$

for some constants $B_r$. 
For large $t$,

\[
\sup_{1 \leq t < \infty} \left| t^c B_{tr} t^{1/2} H \left( \frac{t}{3} \right)^\lambda \right| < \infty
\]

and for small $t$,

\[
\sup_{0 < t < 1} \left| t^d B_{tr} t^{1/2} H \left( \frac{t}{s} \right)^\lambda \right| < \infty
\]

if

\[
c + \frac{1}{2} + \lambda \rho_1 < 0
\]

\[
d + \frac{1}{2} + \lambda \rho_1 > 0
\]

The above conditions are obtained by using (5.2.2) and (5.2.3).

**Remark:**

The above proof also follows from the result 8.2.1. Because the kernel function $s^{-1} H \left( \frac{t}{3} \right)^\lambda$ is a particular case of (3.2.3).

To extend the preceding results to the space $J_{c,d}$, we require the notion of the value of the distribution at a point. This concept is introduced by Lojasiewicz [11].

**Definition:** Let $T$ be a distribution defined in a neighbourhood of a point $x_0$. We say that $T$ has a value $c$ at $x_0$ i.e., $T(x_0) = c$ if the distributional limit $T(x_0 + \lambda x)$ exists in a neighbourhood of $x_0$ and if it is a constant function $c$. 
Theorem 5.3.1 (Distributional Initial Value Abelian Theorem)

If

(i) \( f \in J_{c,d} \)

(ii) \( \frac{f(t)}{t^n} \to \alpha \) as \( t \to 0^+ \) in the sense of Lojasiewicz

(iii) \( -1 - \lambda P_1 < \eta < -1 - \lambda Q_1 \) where \( P_1 \) and \( Q_1 \) are as defined in (5.2.2) and (5.2.3)

then

\[
\lim_{s \to 0} s^{-\eta} F(s) \lambda = \alpha
\]

\( G(\lambda, \eta) \) is as defined in (5.2.1).

Proof: Consider

\[
s^{-\eta} F(s) = \frac{\alpha}{\lambda G(\lambda, \eta)}
\]

using (5.2.5), we have

\[
= s^{-\eta} \langle f(t), s^{-1} H(\frac{t}{s})^\lambda \rangle - \alpha s^{-\eta} < t^n, s^{-1} H(\frac{t}{s})^\lambda >
\]

\[
= s^{-\eta} < f(t) - \alpha t^n, s^{-1} H(\frac{t}{s})^\lambda >
\]

\[
= s^{-\eta} G(s)
\]

where \( G(s) = < f(t) - \alpha t^n, s^{-1} H(\frac{t}{s})^\lambda > \)

Now from the boundedness property of generalized functions, there exists a positive constant \( M \) and a non-negative integer \( r \) such that
\[ |G(s)| \leq M \max_{0 \leq k \leq r} \sup_{0 < t < \infty} \left| \delta_k [K(s,t)] \right| \]

\[ \leq M_1 \sup_{0 < t < \infty} \lambda c_{d} (\log t) t^{\frac{1}{2}} s^{-1/2} H \left( \frac{s}{t} \right) \]

for suitably chosen constant \( M_1 \). Now from asymptotic properties of \( H \)-function (5.2.2) and (5.2.3), we have,

\[
\left| s^{-\eta} G(s) \right| \leq \sup_{0 < t < 1} \frac{\lambda P_1 + \frac{1}{2} + d}{s^{\eta+1+\lambda P_1}} + \sup_{1 \leq t < \infty} \frac{\lambda Q_1 + \frac{1}{2} + c}{s^{\eta+1+\lambda Q_1}}
\]

since \( s > 0, t > 0 \),

\[
\left| s^{-\eta} G(s) \right| \leq M_2 + M_3 s^{-1-\eta-\lambda Q_1}
\]

\[
\leq M_4 s^{-1-\eta-\lambda Q_1}
\]

for some suitably chosen constants \( M_2, M_3 \) and \( M_4 \). But from the condition (iii) of the theorem \( 1 - \eta - \lambda Q_1 > 0 \).

Therefore \( \left| s^{-\eta} G(s) \right| \xrightarrow{s \to 0} 0 \) as \( s \to 0 \).

This establishes the theorem.

Before extending final value Abelian theorem to the space \( J_{c,d} \), let us have some notations.

**D** - The space of smooth functions having compact support

**I** - The interval \((0, \infty)\)

**E(I)** - The space of smooth functions on \( I \).

**E'(I)** - The space of distributions having compact support with respect to \( I \).
Since $D \subset J_{c,d} \subset E$. It follows that $J_{c,d}$ is dense in $E$.

**Theorem 5.3.2 (Distributional Final Value Abelian Theorem)**

If

(i) $f \in J_{c,d}^{'}$ and $f$ can be decomposed into $f = f_1 + f_2$ where $f_1$ is an ordinary function and $f_2 \in E'(I)$. $f_1$ satisfies the hypothesis of theorem 5.2.2.

(ii) $\frac{f(t)}{t^\eta} \to \alpha$ as $t \to \infty$ in the sense of Lojasiewicz.

(iii) $-1-\lambda P_1 < \eta < -1-\lambda Q_1$ where $P_1$ and $Q_1$ are as defined in (5.2.2) and (5.2.3).

(iv) $\lambda Q_1 < q < \lambda P_1$ and $-\eta - 1 - q < 0$ where $q$ is as defined in (5.3.2).

then

\[
\lim_{s \to \infty} s^{-\eta} F(s) = \lim_{t \to \infty} \frac{f(t)}{t^\eta \lambda G(\lambda, \eta)}.
\]

**Proof:** Since $f = f_1 + f_2$, we have

\[ F(s) = F_1(s) + F_2(s). \]

But \[ F_2(s) = \left< f_2(t), s^{-1} H \left( \frac{t}{s} \right) \right> \]

We have the result [16 sec.3.3] that if $f \in E'(I)$ then there exists a constant $C$ and a non-negative integer $q$ such that for all $\phi \in D(I)$

\[ |\left< f, \phi \right>| \leq C \sup_{0 < t < \infty} \left| D_t^q \phi(t) \right| \quad \ldots (5.3.2) \]
The \( q \) is called the order of \( f \).

Support of \( f_2 \) is a compact subset of \( I \). Let \( \lambda(t) \in D(I) \) be identically equal to one on a neighbourhood of support of \( f_2 \).

\[
\left| F_2(s) \right| = \left| \langle f_2(t), \lambda(t) s^{-1} H(t/s) \rangle \right| \\
\leq C \sup_{0 < t < \infty} \left| D_t^q [\lambda(t) s^{-1} H(t/s)] \right|
\]

where \( q \) is the order of \( f_2 \).

\[
\left| F_2(s) \right| \leq C \sup_{0 < t < \infty} \sum_{\nu=0}^{q} \left( \begin{array}{c} q \\ \nu \end{array} \right) \left| D_t^{q-\nu} \lambda(t) D_t^\nu s^{-1} H(t/s) \right|
\]

Using the result (1.6.3)

\[
D_t^\nu H_2,2^1 (t/s) = t^{-\nu} H_3,3^1 (t/s)
\]

For suitably chosen constant \( C_1 \), we have

\[
\left| F_2(s) \right| \leq C_1 \sup_{0 < t < \infty} s^{-1} t^{-q} H_3,3^1 (t/3)^\lambda
\]

put \( t/s = z \)

\[
\left| F_2(s) \right| \leq C_1 \sup_{0 < z < \infty} s^{-1} (sz)^{-q} H_3,3^1 (z)^\lambda
\]

Now under the condition (iv) of theorem,

\[
\sup_{0 < z < \infty} z^{-q} H_3,3^1 (z)^\lambda
\]

is bounded. Hence for some constant \( C_2 \)
\[ \left| F_2(s) \right| \leq c_2 s^{1-q} \]
\[ \left| s^{-\eta} F_2(s) \right| \leq C_2 s^{1-q-\eta} \rightarrow 0 \text{ as } s \rightarrow \infty \quad \ldots \quad (5.3.3) \]

\[
\lim_{s \to \infty} s^{-\eta} F(s) = \lim_{s \to \infty} s^{-\eta} F_1(s) + \lim_{s \to \infty} s^{-\eta} F_2(s)
\]
\[= \lim_{s \to \infty} s^{-\eta} F_1(s) \text{ from (5.3.3)}. \]

since \( f_1 \) is an ordinary function which satisfies the hypothesis of theorem 5.2.2, the required result follows.

5.4 Particular Cases of Abelian Theorem

The generalized Stieltjes transform given by (5.1.4) is quite a good generalization covering almost all existing generalizations. Hence this result of Abelian theorem unifies all the result of Abelian theorem given so far. We discuss below only a few cases.

(1) If in (5.1.4)
\[ D_1 = B_1 = E_1 = E_2 = 1 \]
\[-b_1 = e_2, \quad d_1 = e_1 = 0, \quad \lambda = 1 \]

\[ F(s) = \int_0^\infty s^{-1} H_{1,1}^{1,1} \left[ \frac{t}{s} \right] (0,1) f(t) \, dt \]

Using (1.6.6)
\[ F(s) = \int_0^\infty \frac{f(t)}{s+t} \, dt \quad \ldots \quad (5.4.1) \]

then initial value Abelian theorem 5.2.1 takes the following form
Theorem 5.4.1: If

(i) The Stieltjes transform $F(s)$ of a complex valued function $f(t)$ is defined by (5.4.1).

(ii) $\lim_{t \to 0} \frac{f(t)}{t^n} = \alpha$ where $\eta$ is a real number and $\alpha$ is a complex number.

(iii) $\frac{f(t)}{t^n}$ is bounded on $y \leq t < \infty$ for all $y > 0$.

(iv) $-1 - \lambda P_1 < \eta < -1 - \lambda Q_1$ where

$$P_1 = \min \text{ Re} \left(0, e_2\right)$$

$$Q_1 = \max \text{ Re} \left(-1, -e_2 - 1\right)$$

then

$$\lim_{s \to 0} s^{-\eta} F(s) G(1, \eta) = \alpha.$$  

where

$$G(1, \eta) = \frac{1}{F'(-\eta) F'(-\eta)}$$

(2) If we choose the parameters as,

$$\lambda = 1, \quad d_1 = e_2, \quad D_1 = E_2, \quad b_1 = \Phi, \quad B_1 = 1, \quad e_1 = 0$$

and $E_1 = 1$ we get,

$$H(s) = \int_0^\infty s^{-1} H_{1,1}^{1,1} \left[ \begin{array}{c} \Phi \\ \Phi \end{array} \right] (-\Phi, 1) \ f(t) \ dt$$

Using (1.6.7)

$$H(s) = \int_0^\infty s^{-\Phi} \frac{f(t)}{(s+t)^{\Phi+1}} \ dt$$
\[
\frac{H(s)s^{-\eta}}{\Gamma(\eta+1)} = F(s) = \int_0^{\infty} \frac{f(t)}{(s+t)^{\eta+1}} \, dt 
\]

(5.4.2)

The generalization (5.4.2) is considered by Misra [12]. For this, Final Value Abelian Theorem 5.2.2 reduces to:

**Theorem 5.4.2**: If

(i) The generalized Stieltjes transform \( F(s) \) of a complex valued function \( f(t) \) is defined by (5.4.2)

(ii) The complex valued function \( f(t) \) satisfies the following condition

\[
\lim_{t \to \infty} \frac{f(t)}{t^\eta} = \alpha \text{ for real } \eta \text{ and complex } \alpha.
\]

(iii) \( \frac{f(t)}{t^\eta} \) is bounded on \( 0 < t < \gamma \) for all \( \gamma > 0 \).

(iv) \(-1-\lambda P_1 < \eta < -1-\lambda Q_1\) where

\[
P_1 = \min \Re \left( 0, \frac{e_2}{E_2} \right) \quad Q_1 = \max \Re \left( \frac{e_2-1}{E_2}, -\frac{\eta+1}{E_2} \right)
\]

then

\[
\lim_{s \to \infty} s^{-\eta} H(s) G(1,\eta) = \alpha
\]

where

\[
G(1,\eta) = \frac{1}{\Gamma(\eta+1)\Gamma(\eta-\eta)}
\]

But \( \lim_{s \to \infty} s^{-\eta} H(s) G(1,\eta) \)
This is precisely the result proved by O.P. Misra [12 Thm. 4.1.1].

5.5 Application of Abelian Theorem

L.R. Bragg and J.W. Dettman have given following three sets of initial boundary value problems for partial differential equations.

\[ P_1 \quad \frac{\partial}{\partial t} u(x,t) = P(x,D) u(x,t) \quad t > 0 \]
\[ u(x,0) = \phi(x) \]

\[ P_2 \quad \frac{\partial^2}{\partial t^2} v(x,t) = P(x,D) v(x,t) \quad t > 0 \]
\[ v(x,0) = \phi(x) \]
\[ v_x(x,0) = 0 \]

\[ P_3 \quad \frac{\partial^2}{\partial y^2} w(x,y) + P(x,D) w(x,y) = 0 \quad y > 0 \]
\[ w(x,0) = \phi(x) \]

\[ P(x,D) \text{ is the linear partial differential operator.} \]

In their paper [2], the authors have established the relation between the solutions of \( P_1 \) and \( P_2 \). The relation between the
solutions of $P_1$ and $P_2$. The relation between the solutions of $P_1$ and $P_3$ is established in [3]. John W. Dettman in his paper [6] found the following relationship between $P_2$ and $P_3$.

$$W(x,y) = \frac{2y}{\pi} \int_0^\infty \frac{v(x,z)}{y^2+z^2} \, dz$$

$$V(x,t) = \lim_{\epsilon \to 0} \frac{w(x, \sqrt{-t^2-i\epsilon}) - w(x, \sqrt{-t^2+i\epsilon})}{\sqrt{-t^2-i\epsilon}}$$

Here we show that as a function of $z$, $v(x,z) \in J^1_c,d$.

$w(x,y)$ is the distributional generalized Stieltjes transform of $v(x,z)$

$$i.e. \quad w(x,y) = \langle v(x,z), \frac{2y}{\pi} \frac{4}{(y^2+z^2)} \rangle$$

or

$$\frac{\pi}{2} w(x,y) = \langle v(x,z) \frac{y}{(y^2+z^2)} \rangle$$

The object of this article is to relate the distributional solution $v(x,t)$ of $P_2$ with the soln. $w(x,y)$ of $P_3$.

**Lemma 5.5.1** - The kernel function $\frac{y}{y^2+z^2}$ as a function of $z$

is a member of $J^1_c,d$.

**Proof**: The kernel of (5.1.4) $K(y,z)$ is

$$K(y,z) = y^{-1} H_{1,2}^{1,2} \left[ \frac{z}{y} \right] (d_1,D_1), (1-b_1-B_1; \lambda B_1); (e_1,E_1); (e_2,E_2)$$

put $d_1 = e_2, D_1 = E_2, \lambda = 2, b_1 = 1 - \frac{1}{\lambda}, B_1 = \frac{1}{\lambda}, e_1 = 0, E_1 = 1$
\[ K(y, z) = y^{-1} H_{1,1}^{1,1} \left[ \frac{(z^2)}{(y^2)} \right] (0,1) \]

By (1.6.7)
\[ K(y, z) = \frac{y}{y^2 + z^2} \]

Under the conditions of the result (5.3.1) the proof follows.

**Lemma 5.5.2.** \( W(x, y) \) is an analytic function of \( y \) for all \( y \) not lying on the -ve real axis.

**Proof :** It can be proved in a similar way proved by A.H. Zemanian [16 thm. 3.3.1].

**Theorem 5.5.1 :** If \( V(x, z) \) is the distributional solution of \( P_2 \) then \( w(x, y) \) is the solution of \( P_3 \).

**Proof:** First we will show that
\[ \frac{\partial^2}{\partial y^2} w(x, y) + p(x, D)w(x, y) = 0 \quad y > 0 \quad \ldots (5.5.1) \]
Consider
\[ \frac{\partial^2}{\partial y^2} w(x, y) = \frac{\partial^2}{\partial y^2} < v(x, z), \frac{2y}{\pi(y^2 + z^2)} > \]
\[ = < v(x, z), \frac{\partial^2}{\partial y^2} \frac{2y}{\pi(y^2 + z^2)} > , \text{ by lemma (5.5.2)}, \]
\[ = < v(x, z), \frac{2}{\pi} \left[ -\frac{-2y^5 - 2y^2z^4 - 4y^2z - 4yz^4 + 4y^5}{(y^2 + z^2)^4} \right] > \quad \ldots (5.5.2) \]
Also,

\[ P(x,D)w(x,y) = P(x,D) < v(x,z), \frac{2y}{\pi(y^2+z^2)} > \]

\[ = < P(x,D) v(x,z), \frac{2y}{\pi(y^2+z^2)} > \]

the above step is justified due to [16, sec.2.6(2)]

But from P2, \( P(x,D) v(x,z) = \frac{\partial^2}{\partial z^2} v(x,z) \).

Hence,

\[ P(x,D) w(x,y) = < \frac{\partial^2}{\partial z^2} v(x,z), \frac{2y}{\pi(y^2+z^2)} > \]

\[ = < v(x,z), \frac{\partial^2}{\partial z^2} \frac{2y}{\pi(y^2+z^2)} > \]

\[ = < v(x,z), \frac{2y}{\pi} \left[ \frac{-2y^4 - 2z^4 - 4y^2z^2 + 8z^2y^2 + 8z^4}{(y^2+z^2)^4} \right] > \quad \text{...(5.5.3)} \]

From (5.5.2) and (5.5.3), (5.5.1) is established.

To prove the boundary condition \( w(x,0) = \emptyset(x) \) in \( J_c \),
we will use the Abelian theorem 5.3.1 (Initial Value).

Take \( \eta = 0 \), then (5.3.1) reduces to

\[ \lim_{z \to 0} v(x,z) = \lim_{y \to 0} v(x,y) = \frac{\eta}{2} \quad \text{and} \quad \frac{\eta}{2} \quad \text{...(5.3.1)} \]

Because \( G(2,\eta) = G(2,0) = \frac{1}{\pi} \frac{1}{\Gamma(1/2)\Gamma(1/2)} = \frac{1}{\pi} \)

Therefore

\[ \lim_{y \to 0} w(x,y) = w(x,0) = \lim_{z \to 0} v(x,z) = v(x,0) = \emptyset(x) \quad \text{(from P2)} \]

This completes the proof.
REFERENCES


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