CHAPTER IV

A REPRESENTATION OF A STIELTJES
TRANSFORMABLE GENERALIZED FUNCTIONS
4.1 Introduction:

The representation or a structure of a generalized function is the most essential problem in the theory of generalized functions. The classical Stieltjes transform defined by

$$F(s) = \int_0^\infty \frac{f(t)}{s+t} \, dt$$

(4.1.1)

has been extended to a certain class of generalized functions $S_\alpha(I)$ by Pandey [3]. The testing function space $S_\alpha(I)$ in [3] is defined over $R_+^*$. Pathak [4] has considered the $S_\alpha$ space defined over $R_+^n$ - the n-dimensional euclidian space of non-negative real numbers and $\alpha$ is a fixed arbitrary element of $R^n$. In his paper, Pathak [4] has proved a representation theorem for Stieltjes transformable generalized functions. The method of proof is analogous to the method employed in a structure theorem for Schwartz distributions [7 pp 272-274].

Somewhat different type of work is done by Kof [2]. He has solved the problem of finding those $f(s)$ for which there exists a $f(t)$ such that (4.1.1) holds. Necessary and sufficient conditions for exact representability by means of the transform
where \( A(t) \) is non-decreasing, have been formulated and proved, both for real functions and complex functions in [1]. In his paper Kof [2] has considered the responsibility of \( F(s) \) given by (4.1.1). The results obtained in [2] are used to prove theorems on interpolation for the solution of elliptic boundary-value functions in the refined scale of spaces by Stenzak [6].

Here first we have constructed a space \( \bar{J}_{c,d} \) which is obtained by making slight modification in the space \( J_{c,d} \) discussed fully in [8, 245-246]. The classical Stieltjes transform (4.1.1) has been extended to a certain class of generalized functions \( \bar{J}_{c,d} \) by Zemanian [8, p 245-246]. In this chapter we have obtained a representation formula for a class of Stieltjes transformable generalized functions defined on \( \bar{J}_{c,d} \). This representation formula shows that every element of the dualspace of \( \bar{J}_{c,d} \) is the linear combination of the finite order distributional derivative of continuous functions.

### 4.2 Construction of the Space \( \bar{J}_{c,d} \)

For the two fixed numbers \( c \) and \( d \), let \( \lambda_{c,d}(x) \) be a smooth function defined on \((0, \infty)\) such that

\[
\lambda_{c,d}(x) = \begin{cases} 
  x^c & \text{if } 2 \leq x < \infty \\
  x^d & \text{if } 0 \leq x \leq 1
\end{cases}
\]
Thus $\lambda_{c,d}(x)$ is so constructed such that it is infinitely differentiable on $(0, \infty)$. Thus $J_{c,d}$ is the space of all smooth functions $\varphi$ on $(0, \infty)$ with the semi-norm defined by

$$
\gamma_k(\varphi) = \sup_{0 < x < \infty} \left| \lambda_{c,d}(x) \left( x^k \right)^k \varphi(x) \right| < \infty
$$

with $k = 0, 1, 2, \ldots$

The kernels $\frac{1}{s+t}$ of (4.1.1) belongs to $J_{c,d}$ for $C < \frac{1}{2}$ and $d > -\frac{1}{2}$. As shown in [8, p.245] $J_{c,d}$ is a complete countably multinormed space. $D(I)$ the space of all smooth complex valued functions having compact support – is a subspace of $J_{c,d}$. Let $J_{c,d}'$ denote dual of $J_{c,d}$. The restriction of any member of $J_{c,d}'$ to $D(I)$ is in $D'(I)$. Thus $J_{c,d}'$ is a space of distributions.

4.3 Theorem 4.3.1

Let $f$ be an arbitrary element of $J_{c,d}'$. Then there exists a finite number of bounded measurable functions, say $J_i'$ over $R_+$ such that for all $\varphi \in D(I)$

$$
\langle f, \varphi \rangle = \sum_{i=1}^{m+1} (-1)^{i+1} \langle J_i', \varphi \rangle,
$$

Proof: Since $f \in J_{c,d}'$ and $\varphi \in D$, by boundedness property of generalized functions, there exists a constant $C$ and a non-negative integer $m$ such that
\[
\langle f, \gamma \rangle \leq C \max_{0 \leq i \leq m} \gamma_i (\gamma)
\]
\[
= C \max_{0 \leq i \leq m} \sup_{0 < x < \infty} \left| \lambda(x)(xD_x)^i \sqrt{x} \gamma(x) \right|
\]
\[
\leq C_1 \max_{0 \leq i \leq m} \sup_{0 < x < \infty} \sup \left| \lambda(x) \right| \sup \left| (xD_x)^i \sqrt{x} \gamma(x) \right|
\]

for suitably chosen constant \( C_1 \). Here \( \lambda(x) \) stands for \( \lambda_{c,d}(x) \) defined in (4.2.1). Let us find the actual value of \((xD_x)^i \sqrt{x} \gamma(x)\)

\[
i = 1 \quad (xD_x) \sqrt{x} \gamma(x) = \frac{1}{2} x^{1/2} \gamma(x) + x^{3/2} \gamma'(x)
\]
\[
i = 2 \quad (xD_x)^2 \sqrt{x} \gamma(x)
= \frac{1}{4} \gamma(x)x^{1/2} + 2 x^{3/2} \gamma'(x) + x^{5/2} \gamma''(x).
\]

Hence by induction we can write for suitably chosen constants \( a_j, \ j = 0, \ldots, i \)
\[
(xD_x)^i \sqrt{x} \gamma(x) = \sum_{j=0}^{i} a_j \frac{j+1}{2} \gamma^j(x)
\]

Therefore we have,
\[
\left| \langle f, \gamma \rangle \right| \leq C_1 \max_{0 \leq i \leq m} \sup \left| \lambda(x) \right| \sum_{j=0}^{i} a_j \frac{j+1}{2} \gamma^j(x) \quad (4.3.1)
\]

for some constant \( C_2 \), we can write (4.3.1) as
\[
\left| \langle f, \gamma \rangle \right| \leq C_2 \max_{0 \leq i \leq m} \sup \left| \lambda(x) \right| \left| x^{i+\frac{1}{2}} \gamma^i(x) \right| \quad (4.3.2)
\]

Let \( \varphi_i(x) = \lambda(x) x^{i+\frac{1}{2}} \gamma(x) \quad 0 \leq i \leq m \)
(4.3.3) shows that $\varphi_i(x)$ is a member of $D$ for each $i$. Therefore,

$$\gamma(x) = [\lambda(x)]^{-1} x^{-i-\frac{1}{2}} \varphi_i(x) \quad (4.3.4)$$

Now consider,

$$\frac{d}{dx} \gamma(x) = \frac{d}{dx} [\lambda(x)]^{-1} x^{-i-\frac{1}{2}} \varphi_i(x)$$

$$= [\lambda(x)]^{-1} x^{-i-\frac{1}{2}} \varphi_i(x) + x^{-i-\frac{1}{2}} \varphi_i(-1)[\lambda(x)]^{-2} D_x \lambda(x) +$$

$$+ [\lambda(x)]^{-1} \varphi_i(-1-\frac{1}{2}) x^{-i-\frac{3}{2}}$$

$$= x^{-i-\frac{1}{2}} [\lambda(x)]^{-1} [D_x - \frac{D_x \lambda(x)}{\lambda(x)} - \frac{i+\frac{1}{2}}{x^2}] \varphi_i(x)$$

$$\leq x^{-i-\frac{1}{2}} [\lambda(x)]^{-1} \left[ |D_x \varphi_i(x)| + \left| \frac{D_x \lambda(x)}{\lambda(x)} \varphi_i(x) \right| + \left| \frac{i+\frac{1}{2}}{x} \varphi_i(x) \right| \right]$$

Let $\varphi_i(x)$ is a function with compact support say $[A,B]$

Hence we get

$$\left| \frac{d}{dx} \gamma(x) \right| \leq C_3 \left| x^{-i-\frac{1}{2}} [\lambda(x)]^{-1} \left[ |D_x \varphi_i(x)| + \varphi_i(x) \right| \right|$$

where $C_3 = \max \left[ 1, \left| \frac{D_x \lambda(A)}{\lambda(A)} \right| + \left| \frac{i+\frac{1}{2}}{A} \right| \right]$

Hence by induction $j^{th}$ derivative of $\gamma(x)$ is given by

$$\left| \frac{d^j}{dx^j} \gamma(x) \right| \leq C_4 \left| x^{-i-\frac{1}{2}} [\lambda(x)]^{-1} \sum_{j=0}^{i} \frac{1}{j!} \varphi_i^j(x) \right|$$

for some suitably chosen constants $C_4$. 

Therefore, from (4.3.2), we have

\[ < f, \Psi > \leq C_5 \max_{0 \leq j \leq m} \sup_x |q_j^m(x)| \]  \hspace{1cm} (4.3.5)

We know that for any continuous function \( g \)

\[ \sup_x |g(x)| \leq \sup_x \left| \int_0^x g'(t) \, dt \right| \leq \| g' \|_{L^1} \]

where \( L^1 \) is essentially the space of all measurable functions \( f \) such that \( |f(x)| \) is integrable with

\[ \| f \|_{L^1} = \int |f(x)| \, dx \]

Therefore from (4.3.5) we get

\[ \left| < f, \Psi > \right| \leq C_5 \max_{1 \leq j \leq m+1} \| q_j^m \|_{L^1} \]  \hspace{1cm} (4.3.6)

Consider the product space \([L^1(0, \infty)]^{m+1}\) and the mapping \( J : D \rightarrow L^1 \) defined by

\[ J(\phi) = (\phi_j^m)_{1 \leq j \leq m+1} \]

clearly the mapping is into and \( J(D) \subseteq L^1 \).

Define a functional \( J' \) on \( D \) by

\[ J' : J(D) \rightarrow \mathbb{C} \quad \text{by} \quad J'(J(\phi)) = < f, \Psi > \]

By (4.3.5) it can be shown that \( J' \) is a continuous linear functional defined on \( D \). And by Hanh Banach theorem it can be extended to whole of \( L^1 \). Let the extension be also denoted by \( J' \).
\[ j' = \{ j_1', j_2', \ldots, j_{m+1}' \} \in [L_{\infty}^{m+1}]^* = L_{\infty}^{m+1} \]

\( L_{\infty}^{m+1} \) is isomorphic to the \((L_{1}^{m+1})^*\). Also we know that every linear functional defined in \( L_1 \) gives rise to a function in \( L_{\infty} \). Hence by using the Riesz Representation theorem [5,p.78] we get:

\[
\langle j', \varphi_j' \rangle = \langle f, \psi \rangle = \sum_{j=1}^{m+1} \langle j_1, \varphi_j \rangle
\]

Hence we obtain

\[
\langle f, \psi \rangle = \sum_{j=1}^{m+1} \langle j_1, \frac{d^j}{dx^j} \lambda(x) \times x^{j+\frac{1}{2}} \psi(x) \rangle \quad \text{from (4.3.3)}
\]

\[
= \sum_{j=1}^{m+1} \langle -1 \right)^j \lambda(x) \times x^{j+\frac{1}{2}} \frac{d^j}{dx^j} j_1', \psi(x) \rangle
\]

where \( j_1' \) are bounded measurable functions defined over \((0,\infty)\).
REFERENCES


\[ \text{A representation of Hankel transform of generalized function. } \text{J. Math. Analysis Appl. 34 (1971) pp 33-36} \]