Chapter 1

Preliminaries

1.1 Introduction

This chapter is devoted to recapitulating some of the basic definitions and results in categories which are required for the development of the thesis.

Definition 1.1.1: A category is a class $\mathcal{A}$, together with a class $\mathcal{M}$ which is a disjoint union of the form

$$\mathcal{M} = \bigcup \{[(X, A), (Y, B)] \mid [(X, A), (Y, B)] \in \mathcal{F} \times \mathcal{F} \}$$

where each $[A, B]_{\mathcal{A}}$ is a set. Furthermore, for each triple $(A, B, C)$ of members of $\mathcal{A}$, there is a function from $[B, C]_{\mathcal{A}} \times [A, B]_{\mathcal{A}}$ to $[A, C]_{\mathcal{A}}$ mapping $(\beta, \alpha)$ into $\beta \alpha$ (called the composition of $\beta$ by $\alpha$) subject to the following two axioms:

(i) **Associativity**: whenever the composition make sense we have

$$(\gamma \beta) \alpha = \gamma(\beta \alpha).$$
(ii) **Existence of identities:** For each \( A \in \mathcal{A} \), we have an element \( 1_A \in [A,A]_{\mathcal{A}} \) such that \( 1_A \alpha = \alpha \) and \( \beta 1_A = \beta \) whenever the composition make sense.

**Remark 1.1.2:** When there is no danger of confusion \([A,B]_{\mathcal{A}}\) is simply written as \([A,B]\). The members of \( \mathcal{A} \) are called objects and the members of \( \mathcal{M} \) are called morphisms or (arrows). If \( \alpha \in [A,B] \), we shall write \( \alpha : A \to B \) or sometimes \( A \xrightarrow{\alpha} B \). It is clear that the identity \( 1_A \) in \( A \) is unique.

**Example 1.1.3:**

1. **The category of sets:** The category \( \mathcal{F} \) whose class of objects is the class of all sets, where \([A,B]_{\mathcal{F}}\) is the class of all functions from \( A \) to \( B \).

2. **The category of topological spaces \( \mathcal{F} \):** Objects are topological spaces where \([A,B]_{\mathcal{F}}\) is the set of all continuous functions from \( A \) to \( B \).

3. **The category \( \mathcal{F}_0 \) of sets with base points:** It is the category whose objects are ordered pairs \((A,a)\) where \( A \) is a set and \( a \in A \). A morphism from \((A,a)\) to \((B,b)\) is a function \( f \) from \( A \) to \( B \) such that \( f(a) = b \).

**Definition 1.1.4:** A category \( \mathcal{A}' \) is called a **subcategory** of a category \( \mathcal{A} \) if

1. \( \mathcal{A}' \subset \mathcal{A} \).

2. \([A,B]_{\mathcal{A}'} \subset [A,B]_{\mathcal{A}}\) for all \( A,B \in \mathcal{A}' \).

3. The composition of any two morphisms in \( \mathcal{A}' \) is the same as their composition in \( \mathcal{A} \).
4. $1_A$ is the same in $\mathcal{A}'$ as in $\mathcal{A}$ for all $A \in \mathcal{A}'$.

If furthermore $[A, B]_{\mathcal{A}'} = [A, B]_{\mathcal{A}}$ for all $(A, B) \in \mathcal{A}' \times \mathcal{A}'$ then we say that $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$.

**Definition 1.1.5:** The *dual category* of a category $\mathcal{A}$ denoted by $\mathcal{A}^*$ has the same class of objects as $\mathcal{A}$, and is such that $[A, B]_{\mathcal{A}^*} = [B, A]_{\mathcal{A}}$ and the composition $\beta \alpha$ in $\mathcal{A}^*$ is defined as the compositions $\alpha \beta$ in $\mathcal{A}$.

**Definition 1.1.6:** A morphism $\theta : A \to B$ is called a *coretraction* if there is a morphism $\theta' : B \to A$ such that $\theta' \theta = 1_A$. In this case $A$ is called a retract of $B$. Dually, a morphism $\theta : A \to B$ is called a *retraction* if there is a morphism $\theta'' : B \to A$ such that $\theta \theta'' = 1_B$.

**Proposition 1.1.7.** If $\theta : A \to B$ and $\pi : B \to C$ are coretractions (retractions) then $\pi \theta$ is also a coretraction (retraction). On the other hand if $\pi \theta$ is a coretraction (retraction), then $\theta$ is a coretraction ($\pi$ is a retraction) but not necessarily $\pi$ (not necessarily $\theta$).

**Definition 1.1.8:** A morphism $\theta : A \to B$ is called an *isomorphism* if it is both a retraction and a coretraction. We say that “$A$ is isomorphic to $B$” if there is an isomorphism from $A$ to $B$. Then the relation “is isomorphic to” is an equivalence relation.

**Definition 1.1.9:** A morphism $\alpha \in [A, B]$ is called a *monomorphism* if $\alpha f = \alpha g$ implies that $f = g$ for all $f, g : C \to A$. Similarly, if $f \alpha = g \alpha$ implies that $f = g$ for all $f, g : B \to C$ then $\alpha$ is called an *epimorphism*. 
Proposition 1.1.10. If $\alpha$ and $\beta$ are monomorphisms (respectively epimorphisms) and if $\beta \alpha$ is defined then $\beta \alpha$ is also a monomorphism (respectively epimorphism). On the other hand, if $\beta \alpha$ is a monomorphism (respectively epimorphism), then $\alpha$ is a monomorphism ($\beta$ is an epimorphism) but not necessarily $\beta$ (respectively $\alpha$).

Proposition 1.1.11. Every coretraction (respectively retraction) is a monomorphism (respectively epimorphism). However, the converse is not true.

Proposition 1.1.12. In the category of sets and sets with base point, a morphism is a monomorphism (respectively epimorphism) if and only if it is a one-one function (onto function).

Definition 1.1.13: A category is said to be balanced if every morphism which is both a monomorphism and an epimorphism is also an isomorphism.

Example 1.1.14: The category of sets (and sets with base point) is balanced.

Proposition 1.1.15. If $\alpha : A \to B$ is a coretraction (retraction) and is also an epimorphism (respectively monomorphism) then it is an isomorphism.

Definition 1.1.16: If $\alpha : A' \to A$ is a monomorphism, then $A'$ is called subobject of $A$. If $\alpha_1 : A_1 \to A$ and $\alpha_2 : A_2 \to A$ are monomorphisms, then we write $\alpha_1 \leq \alpha_2$ if there is a morphism $\gamma : A_1 \to A_2$ such that $\alpha_2 \gamma = \alpha_1$. If also $\alpha_2 \leq \alpha_1$, then we say that $A_1$ and $A_2$ are isomorphic subobjects of $A$. If $\alpha : A \to A'$ is an epimorphism, then $A'$ is said to be a quotient object of $A$. If $\alpha_1 : A \to A_1$ and $\alpha_2 : A \to A_2$ are epimorphisms, we write $\alpha_1 \leq \alpha_2$ if there is a morphism $\gamma : A_2 \to A_1$ such that $\gamma \alpha_2 = \alpha_1$. If also $\alpha_2 \leq \alpha_1$ then
we say that $A_1$ and $A_2$ are isomorphic quotient objects of $A$.

**Definition 1.1.17:** Given two morphisms $\alpha, \beta : A \to B$, we say that $u : K \to A$ is an equalizer for $\alpha$ and $\beta$ if

(i) $\alpha u = \beta u$ and

(ii) whenever $u' : K' \to A$ is such that $\alpha u' = \beta u'$, then there is a unique morphism $\gamma : K' \to K$ such that $u\gamma = u'$.

**Proposition 1.1.18.** If $u$ is an equalizer for $\alpha$ and $\beta$ then $u$ is a monomorphism. Any two equalizers for $\alpha$ and $\beta$ are isomorphic subobjects of $A$.

**Remark 1.1.19:** We shall denote the equalizer of $\alpha$ and $\beta$ as $\text{Equ}(\alpha, \beta)$. Then we say that $B \to \text{coequ}(\alpha, \beta)$ is the coequalizer of $\alpha$ and $\beta$ if it is the equalizer of these two morphisms in the dual category.

**Definition 1.1.20:** Let $\{u_i : A_i \to A\}_{i \in I}$ be a family of subobjects of $A$.

A morphism $u : A' \to A$ is called the intersection of the family if

(i) for each $i \in I$, we can write $u = u_i \gamma_i$ for some morphism $\gamma_i : A' \to A_i$.

(ii) if for every morphism $B \to A$ which factors through each $u_i$, factors uniquely through $u$.

If the intersection exists for every set of subobjects of every object in $\mathcal{A}$, then $\mathcal{A}$ is said to have intersections.

**Definition 1.1.21:** A morphism $u : I \to B$ is called the image of the morphism $f : A \to B$ if $f = uf'$ for some $f' : A \to I$, and if $u$ precedes any other
morphism into $B$ with the same property. If every morphism in a category $\mathcal{A}$ has an image, then $\mathcal{A}$ is said to have images.

**Definition 1.1.22:** Given two morphism $\alpha_1 : A_1 \to A$ and $\alpha_2 : A_2 \to A$ a commutative diagram is called a *pullback* for $\alpha_1$ and $\alpha_2$ if for every pair of morphism $\beta'_1 : P' \to A_1$ and $\beta'_2 : P' \to A_2$ such that $\alpha_1 \beta'_1 = \alpha_2 \beta'_2$, there exists a unique morphism $\gamma : P' \to P$ such that $\beta'_1 = \beta_1 \gamma$ and $\beta'_2 = \beta_2 \gamma$.

If $f : A \to B$ and $B'$ is a subobject of $B$, then the *inverse image* of $B'$ by $f$ is the pullback diagram given below:

![Pullback Diagram](image)

### 1.2 Functors

**Definition 1.2.1:** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *covariant functor* $T : \mathcal{A} \to \mathcal{B}$ is an assignment of an object $T(A) \in \mathcal{B}$ to each object $A \in \mathcal{A}$ and a morphism $T(\alpha) : T(A) \to T(B)$ to each morphism $\alpha : A \to B$ in $\mathcal{A}$, subject to the following conditions:

(i) **Preservation of Composition:** If $\alpha' \alpha$ is defined in $\mathcal{A}$, then
\[ T(\alpha' \alpha) = T(\alpha') T(\alpha). \]

(ii) **Preservation of Identities:** For each \( A \in \mathcal{A} \), we have \( T(1_A) = 1_{T(A)} \).

Replacing the condition \( \alpha : A \to B \) implies that \( T(\alpha) : T(A) \to T(B) \) and \( T(\alpha' \alpha) = T(\alpha') T(\alpha) \) by the condition \( \alpha : A \to B \) implies that \( T(\alpha) : T(B) \to T(A) \) and \( T(\alpha' \alpha) = T(\alpha) T(\alpha') \) in the above, we obtain the definition of a **contra-variant functor** from \( \mathcal{A} \) to \( \mathcal{B} \).

**Definition 1.2.2:** Let \( \mathcal{A} \) be a category and \( A \in \mathcal{A} \). Then we have a covariant morphism functor \( H^A : \mathcal{A} \to \mathcal{B} \) and a contra-variant morphism functor \( H_A : \mathcal{A} \to \mathcal{B} \) defined as follows:

If \( B \in \mathcal{B} \) and \( \alpha : B \to C \), then

(i) \( H^A(B) = [A, B] \) and \( H^A(\alpha) : [A, B] \to [A, C] \) is given by the rule \( (H^A(\alpha))(x) = ax \).

(ii) \( H_A(B) = [B, A] \) and \( H_A(\alpha) : [C, A] \to [B, A] \) is given by the rule \( (H_A(\alpha))(x) = x\alpha \).

**Definition 1.2.3:** Let \( T : \mathcal{A} \to \mathcal{B} \) be a covariant functor. Then \( T \) is called a **monofunctor (epifunctor)** if \( T(\alpha) \) is a monomorphism (epimorphism) in \( \mathcal{B} \) whenever \( \alpha \) is a monomorphism (epimorphism) in \( \mathcal{A} \).

If \( \mathcal{A} \) and \( \mathcal{B} \) are categories with zero objects, then \( T \) is said to be **zero preserving** if \( T(O) \) is a zero object in \( \mathcal{B} \), for \( O \) a zero object in \( \mathcal{A} \), \( T \) is said to be **kernel preserving** if \( T(u) \) is the kernel of \( T(\alpha) \) when \( u : k \to A \) is the kernel of \( \alpha : A \to B \).

The functor \( T \) is called a **faithful functor** if for every pair of objects \( A, B \in \mathcal{A} \),
the function \([A,B] \to [T(A), T(B)]\) induced by \(T\) is univalent.

A faithful functor which takes distinct objects into distinct objects is called an imbedding.

The functor \(T\) is said to be full if for every pair of objects \(A,B \in \mathcal{A}\), the function \([A,B] \to [T(A), T(B)]\) induced by \(T\) is onto and \(T\) is said to be representative if for every \(B \in \mathcal{B}\), there is an object \(A \in \mathcal{A}\) such that \(T(A)\) and \(B\) are isomorphic. A full, representative, faithful functor is called an equivalence.

**Definition 1.2.4:** An object \(P\) in a category \(\mathcal{A}\) is said to be Projective if for every diagram

\[
\begin{array}{ccc}
A & \rightarrow & A'' \\
\downarrow & & \downarrow \\
A' & \rightarrow & A''
\end{array}
\]

with \(A \rightarrow A''\) an epimorphism, there is a morphism \(P \rightarrow A\) making the diagram commutative. The category \(\mathcal{A}\) is said to have projectives if for each \(A \in \mathcal{A}\), there is an epimorphism \(P \rightarrow A\) with \(P\) projective.

**Proposition 1.2.5.** In the category of sets, every set is projective.

**Proposition 1.2.6.** In the category of sets, every nonempty set is injective.

**Definition 1.2.7:** An object \(U\) in a category \(\mathcal{A}\) is called a generator for \(A\), if for every pair of distinct morphisms \(\alpha, \beta : A \rightarrow B\), there is a morphism \(u : U \rightarrow A\) such that \(\alpha u \neq \beta u\). An object \(C\) is called a cogenerator for \(\mathcal{A}\) if it is a generator for the dual category.
Proposition 1.2.8. For the category of sets, any one element set is a generator and any two element set is a cogenerator.

Most of the definitions in general category are taken from [3]. They are defined in this thesis as applicable to the category of fuzzy groups.