CHAPTER III

NONLINEAR FREE VIBRATIONS AND THERMAL BUCKLING OF ELASTIC PLATES AND SHELLS AT EVEVATED TEMPERATURE BY A SIMPLIFIED APPROACH.

I. NONLINEAR FREE VIBRATIONS AND THERMAL BUCKLING OF ELASTIC PLATES AT ELEVATED TEMPERATURE*

ABSTRACT

For heated thin elastic plates undergoing large amplitude vibrations, a modified energy expression has been considered and a pair of differential equations in the decoupled form has been derived. These equations are more advantageous than Berger's decoupled quasi-linear equations which fail to yield meaningful results for plates with movable edge conditions. With the help of the newly derived equations non-linear free vibrations of a circular plate at elevated temperature have been analysed in the present paper.

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3.1 DERIVATION OF FIELD EQUATIONS

For a circular plate of radius 'a' and thickness 'h', the sum of the membrane and bending energies undergoing large deflections in the absence of any external load may be expressed in the form

$$V = \frac{D}{2} \int_0^a \Big[\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + \frac{2\nu}{r} \frac{\partial w}{\partial r} \cdot \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \left(\frac{\partial w}{\partial r}\right)^2 +$$

$$+ \frac{12}{h^2} \left\{ e_1^2 + 2(\nu - 1)\, e_2 \right\} \Big] r dr \qquad \ldots (136)$$

Rearranging equation (136) as

$$V = \frac{D}{2} \int_0^a \Big[\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + \frac{2\nu}{r} \cdot \frac{\partial w}{\partial r} \cdot \frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r} \frac{\partial w}{\partial r}\right)^2 +$$

$$+ \frac{12}{h^2} \left\{ \bar{e}_1^2 + (1 - \nu^2) \frac{u^2}{r^2} \right\} \Big] r dr \qquad \ldots (137)$$

where

$$\bar{e}_1 = \frac{\partial u}{\partial r} + \frac{\nu u}{r} + \frac{1}{2} \left(\frac{\partial w}{\partial r}\right)^2 = \frac{(1 - \nu^2)\sigma_{rr}}{E} \qquad \ldots (138)$$

The kinetic energy KE of the plate is given by

$$KE = \frac{\rho h}{2} \iint_0^a \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right\} \, r dr dt \qquad \ldots (139)$$

Also the strain energy SE due to heating is given by [4]

$$SE = \int_0^a \int_{-h/2}^{h/2} \frac{E \alpha_t T}{1-\nu} \left\{ \bar{e}_1 - Z \nabla^2 W \right\} r dr dZ \qquad \ldots (140)$$

where S is the surface area of the plate and T (r,Z) the temperature distribution in the plate. If the term $(1-\nu^2)\frac{u^2}{r^2}$ in (137) is replaced by $\frac{\lambda}{4}\left(\frac{\partial w}{\partial r}\right)^4$, [Dutta and Banerjee (10)], λ being a factor depending on the Poisson's ratio of the plate material, decoupling will be possible. By doing so and applying Hamilton's principle to Lagrangian L = (KE - SE - V) and using Euler's variational equations, one gets the following pair of differential equations in the decoupled form

$$D \left[\nabla^4 w - \frac{6\lambda}{h^2} \left(\frac{\partial w}{\partial r}\right)^2 \left\{ \nabla^2 w + 2 \frac{\partial^2 w}{\partial r^2} \right\} \right] + \frac{\nabla^2 M_T}{1-\nu} +$$

$$+ \rho h \frac{d^2 w}{dt^2} - Cf(t) r^{\nu-1} \left\{ \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \frac{\partial w}{\partial r} \right\} = 0 \qquad \ldots (141)$$

$$\frac{12 D}{h^2} \bar{e}_1 = C f(t) r^{\nu-1} + \frac{N_T}{1-\nu} \qquad \ldots (142)$$

where C is a constant and f(t) is some function of time t and M_T and N_T are defined by equations (86) and (87).

3.2 METHOD OF SOLUTION

The membrane stress due to thermal loading is considered and so the temperature distribution is a function of the radial co-ordinate 'r' only [26]

$T(r,Z) = \tau_o(r)$ so that $M_T = 0$.

For clamped circular plates the boundary conditions to be satisfied are $W = 0 = \frac{\partial w}{\partial r}$ at $r = a$, and for simply-supported edge conditions are

$$W = 0 = \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} \text{ at } r = a \qquad \ldots (143)$$

For movable edge condition we have $C = 0$ and for immovable edges the condition is $u = 0$ at $r = a$.

If now the problem is restricted to finding the fundamental mode of vibrations only, then the form of W satisfying the above conditions can be taken as

$$W = A\left[1 - 2P\frac{r^2}{a^2} + Q\frac{r^4}{a^4}\right]F(t) \qquad \ldots (144)$$

For clamped plates $P = Q = 1$ and for simply-supported edges $P = \frac{3+\nu}{5+\nu}$, $Q = \frac{1+\nu}{5+\nu}$.

Substituting the above expression for W in equation (141) one gets error function $\epsilon(r,t)$ which does not vanish in general, since the expression W is not the exact solution for equation (141).

Galerkin's procedure requires that the error function to be orthogonal over the domain of the plate i.e.,

$$\iint_S \epsilon(r,t)\, ds = 0 \qquad \ldots (145)$$

Equation (145) can be integrated after a lengthy but simple calculations.

The term $Cf(t)$ involving the constant C can be determined from equation (142) by integrating over the area of the plate surface after inserting the expression for W. Terms involving the in-plane displacement u can be eliminated by considering suitable expressions compatible with the boundary conditions. Equation (145) finally leads to the non-linear time equation

$$\ddot{F}(t) + \alpha F(t) + \beta F^3(t) = 0 \qquad \ldots (146)$$

which can be solved in terms of Jacobian elliptic functions [60].

3.3 CLAMPED CIRCULAR PLATE — IMMOVABLE EDGE

For a clamped circular plate with immovable edges equation (145) simplifies into

$$\frac{a^4 \rho h}{10} \ddot{F}(t) + D \left[\frac{32}{3} F(t) + 7.314 \lambda \left(\frac{A}{h}\right)^2 F^3(t) \right] +$$

$$+ 1.08\ Cf(t)\ a^{\nu + 1}\ F(t) = 0 \qquad \ldots (147)$$

Equation (142) can be integrated giving the term Cf(t) from the expression

$$4D \left(\frac{A}{h}\right)^2 F^2(t) = Cf(t) \frac{a^{\nu + 1}}{\nu + 1} + \frac{a^2 N_T}{2(1 - \nu)} \qquad \ldots (148)$$

Eliminating Cf(t) from (147) and (148) one gets the time-equation in the form

$$\ddot{F}(t) + \alpha F(t) + \beta F^3(t) = 0 \qquad \ldots (149)$$

where $$\alpha = \frac{10 D}{a^4 \rho h} \left[\frac{32}{3} - \frac{0.54(1 + \nu) a^2 N_T}{D(1 - \nu)} \right]$$

and

$$\beta = \left[4.32 (1 + \nu) + 7.314 \lambda \right] \left(\frac{A}{h} \right)^2 \qquad \ldots (150)$$

The solution of equation (149) has been given by Nash and Modeer in terms of Jacobian elliptic function of cosine type and non-dimensional time-period is given by

$$\frac{T^*}{T} = \frac{2\theta}{\pi} \left(1 + \frac{\beta}{\alpha} \right)^{-\frac{1}{2}} \qquad \ldots (151)$$

where

$$\frac{\beta}{\alpha} = 0.64 \left(\frac{A}{h} \right)^2 \Big/ \left(1 - N_T^* \right) \qquad \ldots (152)$$

where

$$N_T^* = 0.0658\, a^2 N_T \Big/ D(1-\nu) \qquad \ldots (153)$$

$$\lambda = (1-\nu^2)/5$$

for clamped plates [10].

3.4 BUCKLING CRITERION AND CRITICAL BUCKLING TEMPERATURE

For the pre-buckling state relative time-periods can be obtained from (154) and (155) by considering different values of N_T^* ($0 \leq N_T^* < 1$). Critical buckling temperature $(N_T^*)_{cr}$ is obtained when $N_T^* = 1$ and is given by

$$(N_T^*)_{cr} = 15.2 \qquad \ldots (154)$$

which is in agreement with the result obtained by Biswas [1976].

3.5 PLATES WITH MOVABLE EDGES

For a circular plate with clamped movable edge $C = 0$ and T^*/T can be obtained in the form given by (154) where

$$\frac{\beta}{\alpha} = 0.1243 \left(\frac{A}{h}\right)^2 \qquad \ldots (155)$$

Critical buckling temperature for movable edge

With $C = 0$ in equation (148) one gets

$$F^2(t) = a^2 N_T \Big/ 8D(1-\nu)(A/h)^2 \qquad \ldots (156)$$

since N_T is time-independent so $F(t)$ must be a constant as seen in equation (157) and equation (147) leads to

$$(N_T^*)_{cr} = 64.105\, D\,(1-\nu)\big/a^2 \qquad \ldots (157)$$

3.6 NUMERICAL RESULTS

Table VI shows the variations of the ratio of time-periods for non-linear and linear vibrations for different values of non-dimensional amplitudes (A/h) and temperature parameter.

It is seen that the effect of temperature is to diminish the relative time-periods. The circular frequency $\Omega_o = \sqrt{\alpha}$ diminishes due to the presence of temperature. Moreover T^*/T

is seen to be independent of temperature parameter for plates with movable edge whereas critical buckling temperature for plates with movable edge exceeds four times than that for plates with immovable edge.

TABLE VI

Dependence of T^*/T on A/h and N_T^*

A/h	0	.2	.4	.6	.8	1
$\frac{T^*}{T}$ $(N_T^* = 0)$	1	.99	.97	.94	.90	.88 [Ref. 60]
	1	.99	.968	.937	.892	.876 (Present case)
$\frac{T^*}{T}$ $(N_T^* = .4)$	1	.98	.937	.904	.846	.808
$\frac{T^*}{T}$ $(N_T^* = .8)$	1	.952	.864	.764	.681	.562

II. NON-LINEAR FREE VIBRATIONS OF A SHALLOW SPHERICAL SHELL DUE TO THERMAL GRADIENT

Abstract

Using Berger's modified method non-linear free vibrations of an elastic shallow spherical shell with clamped immovable and movable edges when the shell is subjected to thermal gradient through the thickness have been analysed.

In this paper basic governing equations for the non-linear vibration analysis of a shallow spherical shell subjected to thermal gradient have been derived considering the modified energy expression and based on non-homogeneous theory, the flexural rigidity being the function of the radial distance r. Some numerical results showing the dependence of the ratios of non-linear and linear frequencies on the temperature co-efficients have been presented.

3.7 DERIVATION OF DYNAMIC FIELD EQUATIONS FOR A HEATED SPHERICAL SHELL

Considering a thin spherical shell of radius 'R' subjected to a steady thermal gradient. The temperature distribution 'T' is assumed to be a linear function of the radial distance 'r' and the modulus of elasticity 'E' is assumed to be a linear function of temperature 'T'.

As a result the stiffness parameter 'D' is a linear function of 'r' so that

$$T = T_o \left(1 - \frac{r}{R} \right) \qquad \ldots (158)$$

$$E = E_o(1 - mT) = E_o \left[1 - \mu \left(1 - \frac{r}{R} \right) \right] \qquad \ldots (159)$$

where $\mu = mT_o$; T_o the reference temperature, E_o, the reference modulus of elasticity and 'm' is the slope of variation of E with T.

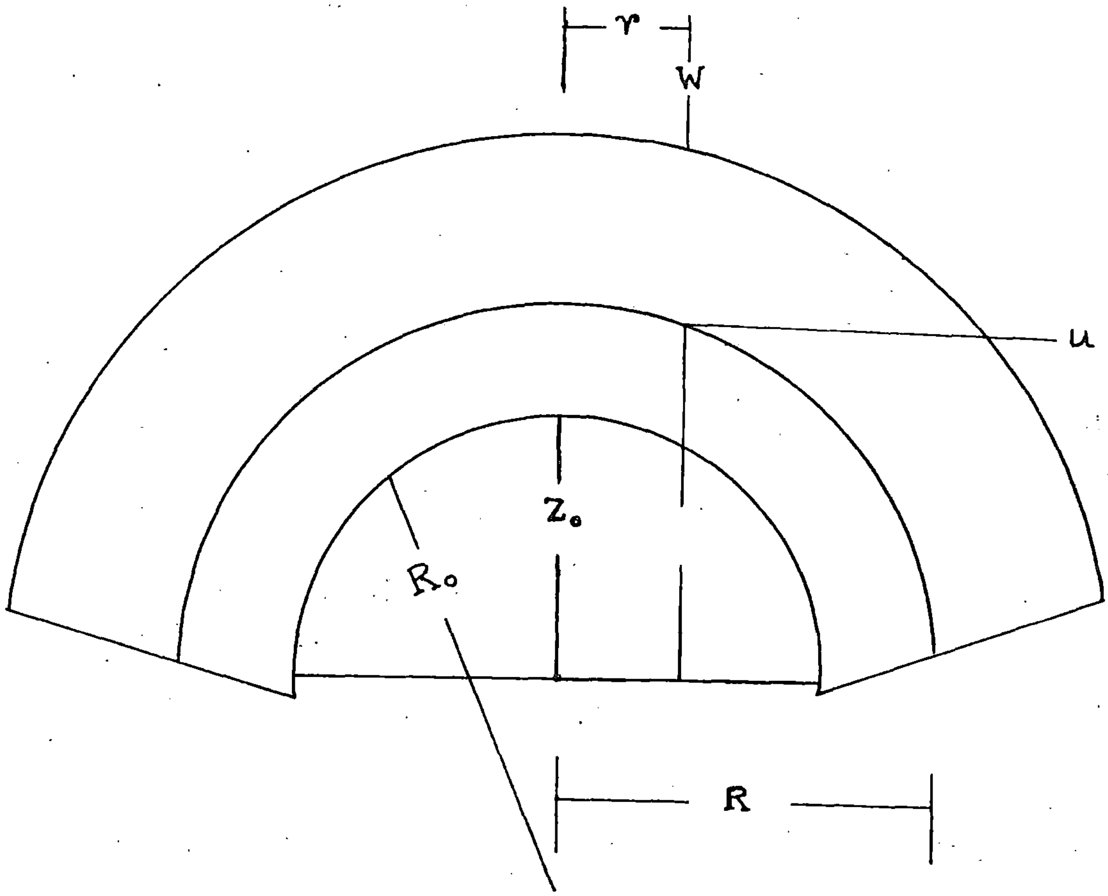

Figure 9 : The geometary of the shell .

The flexural rigidity D is given by the expression

$$D(r) = D_0 \left[1 - \mu \left(1 - \frac{r}{R} \right) \right] \qquad \ldots (160)$$

where D_0 is the reference flexural rigidity.

For the spherical shell with clamped edges, the co-ordinate system is employed as shown in Fig. 9 . The vertical component of the displacement of the middle surface is denoted by W, considered to be positive in the direction shown. The radial displacement of a point in the middle surface is denoted by u, measured horizontally as shown in the Fig. 9. The elevation of the middle surface of the shell above the base plane is denoted by $Z = Z(r)$ such that

$$Z = \frac{R^2}{2R_0} \left(1 - \frac{r^2}{R^2} \right)$$

The total potential energy of bending and stretching may be written as

$$V = \frac{1}{2} \iint_S D(r) \left[(\nabla^2 w)^2 - \frac{2(1-\nu)}{r} W_{,r} . W_{,rr} + \right.$$

$$\left. + \frac{12}{h^2} \left\{ e^2 + 2(\nu - 1)\, e_2 \right\} \right] r dr d\theta \qquad \ldots (161)$$

where

$$e = u_{,r} + \frac{u}{r} + \frac{1}{2} W_{,r}^{\;2} + W_{,r} \frac{dZ}{dr} \qquad \ldots (162)$$

$$e_2 = \frac{u}{r}\, u_{,r} + \frac{u}{2r} W_{,r}^{\;2} + \frac{u}{r} . W_{,r} \frac{dZ}{dr} \qquad \ldots (163)$$

where e and e_2 are the first and second strain invarient and h is the thickness of the shell.

We now re-write the potential energy in the form

$$V = \frac{1}{2}\iint_S D(r)\left[(\nabla^2 w)^2 - \frac{2(1-\lambda)}{r} W_{,r}.W_{,rr}\right] r\,dr\,d\theta +$$

$$+ \frac{1}{2}\iint_S D(r)\left[\frac{12}{h^2}\bar{e}_1^{\,2} + \frac{12\lambda}{h^2}\left(\frac{1}{2}W_{,r}^{\,2} + W_{,r}\frac{dZ}{dr}\right)^2\right] r\,dr\,d\theta \quad \ldots (164)$$

where

$$\bar{e}_1 = u_{,r} + \frac{u}{r} + \frac{1}{2}W_{,r}^{\,2} + W_{,r}\frac{dZ}{dr}$$

and the term $\lambda\left\{\frac{1}{2}W_{,r}^{\,2} + W_{,r}\frac{dZ}{dr}\right\}^2$ has been replaced for the term $(1-r^2)\frac{u^2}{r^2}$ contained in the equation (161). Such replacement is physically possible as explained by Banerjee, B (1936).

The kinetic energy of the shell is given by

$$T' = \frac{\rho h}{2}\iint_0^R (u_{,t}^{\,2} + W_{,t}^{\,2})\, r\,dr\,dt$$

Forming the Lagrangian function $L = T' - V$; applying Hamilton's principle and using Euler's equations of the calculus variations, one gets the basic governing equations, decoupled in quasi-linear form, and expressed as

$$D(r)\Big[\nabla^4 w - \frac{12\lambda}{h^2}\Big(W_{,rr}\cdot W_{,r}^{\,2} - 3r\, W_{,rr}\cdot W_{,r}\cdot \frac{1}{R_o} - \frac{3W_{,r}^{\,2}}{R_o} +$$

$$+ r^2\, W_{,rr}/R_o^{\,2} + \frac{3r\, W_{,r}}{R_o^{\,2}} + \frac{W_{,r}^{\,3}}{2r} + \frac{1}{2}\, W_{,rr}\cdot W_{,r}^{\,2}\Big)\Big] +$$

$$+ \frac{dD(r)}{dr}\Big[2\Big(W_{,rrr} + \frac{W_{,rr}}{r}\Big) - \frac{W_{,r}}{2r^2} + \frac{\nu W_{,rr}}{r} -$$

$$- \frac{12\lambda}{h^2}\Big(\frac{W_{,r}^{\,3}}{2} - \frac{3}{2}\, r\cdot \frac{W_{,r}^{\,2}}{R_o} + r^2\frac{W_{,r}}{R_o^{\,2}}\Big)\Big] +$$

$$+ \frac{d^2D(r)}{dr^2}\Big[W_{,rr} + \frac{\nu W_{,r}}{r}\Big] + \rho h\, W_{,tt} -$$

$$- K^2 f(t)\Big[r^{\nu-1}\, W_{,rr} + \nu\, r^{\nu-2}\, W_{,r} - \frac{r^{\nu-1}(1+\nu)}{R_o}\Big] = 0 \qquad \ldots(165)$$

$$r^{1-\nu}\,\bar{e}_1 = \frac{K^2 h^2}{12}\, f(t) \qquad \ldots(166)$$

where K is a constant and $f(t)$ is a function of time.

3.8 METHOD OF SOLUTION

For a spherical shell with clamped immovable edges, one may assume

$$W = AF(t)\left(1 - \frac{r^2}{R^2}\right)^2 \qquad \ldots (166a)$$

where A stands for the maximum deflection in the positive direction.

Insurting equation (90) into equation (166) integrating over the surface area of the shell one gets the constant K in the form

$$\frac{K^2 h^2}{12} f(t) = \frac{16\, A F(t)\, R^{1-\nu}}{R_o(5-\nu)(7-\nu)} + \frac{128\, A^2 F^2(t)\, R^{-1-\nu}}{(5-\nu)(7-\nu)(9-\nu)} \qquad \ldots (167)$$

To solve equation (165), one may use Galerkin's error minimising technique. Substituting equation (166a) into (165) and using equation (167) one gets, the following time differential equation of the form,

$$\frac{d^2}{dt^2}\left\{F(t)\right\} + \alpha F(t) + \beta F^2(t) + \gamma F^3(t) = 0 \qquad \ldots (168)$$

where

$$\alpha = \frac{10\,D_o}{\rho h\,R^4}\left[10.666 + \frac{3.2\lambda R^4}{R_o^2 h^2} + \frac{1536\,R^4}{h^2 R_o^2 (3+\nu)(5+\nu)(5-\nu)(7-\nu)}\right.$$

$$\left. -\mu\left\{3.657+1.22\nu + \frac{0.9335\;R^4}{R_o^2 h^2} - \frac{1536\,(2\nu-13)\,R^4}{R_o^2 h^2 (3+\nu)(5+\nu)(5-\nu)(7-\nu)(8-\nu)}\right\}\right]$$

... (169)

$$\beta = \frac{10\,D_o}{\rho h\,R^4}\left(\frac{A}{h}\right)\left[\frac{9.6\lambda R^2}{R_o h} + \frac{12288\,R^2}{R_o h\,(3-\nu)(25-\nu^2)(7-\nu)(9-\nu)}\right. +$$

$$+ \frac{124576\,R^2}{R_o h\,(3+\nu)(5+\nu)(7+\nu)(5+\nu)(7-\nu)} - \mu\left\{\frac{3.46\,\lambda\,R^2}{R_o\,h}\right. +$$

$$+ \frac{36854\,(\nu^2 - 15\nu - 55)\,R^2}{R_o h\,(3+\nu)(5+\nu)(5-\nu)(6-\nu)(7-\nu)(8-\nu)(9-\nu)(10-\nu)} -$$

$$\left.\left. - \frac{24576\,(2\nu - 13)\,R^2}{R_o h\,(3+\nu)(5+\nu)(7+\nu)(5-\nu)(6-\nu)(7-\nu)(8-\nu)}\right\}\right]$$

$$\gamma = \frac{10D_0}{\rho h R^4}\left(\frac{A}{h}\right)^2 \left[7.3145\,\lambda + \frac{196608}{(3+\vartheta)(25-\vartheta^2)(49-\vartheta^2)(9-\vartheta)} \right.$$

$$\left. -\mu\left\{2.95\lambda - \frac{589824(\vartheta^2-15\vartheta+55)}{(3+\vartheta)(25-\vartheta^2)(49-\vartheta^2)(6-\vartheta)(8-\vartheta)(9-\vartheta)(10-\vartheta)}\right\}\right] \quad \ldots(170)$$

3.9 EVALUATION OF λ

The factor λ which occurs in the foregoing equations can be determined from minimum potential energy and given by [Banerjee, B (1986)]
& Datta, S (ref. 10)

$$\lambda = 2\vartheta^2 \quad \text{(for clamped edge)}$$

$$\lambda = \vartheta^2 \quad \text{(for simply-supported edge)} \quad \ldots(171)$$

3.10 SOLUTION OF TIME-DIFFERENTIAL EQUATION

The solution of the time-differential equation (163) with initial conditions (7) has been given by Bhattacharjee [16] and the ratio of non-linear and linear vibrational frequencies is given by

$$\frac{\omega^*}{\omega} = \left[1 + \left(\frac{A}{h}\right)^2 \left\{ \frac{3}{4} \frac{\gamma}{\alpha} - \frac{5}{6} \left(\frac{\beta}{\alpha}\right)^2 \right\} \right]^{\frac{1}{2}} \quad \ldots(172)$$

where ω^* and ω are the non-linear and linear frequencies.

3.11 NUMERICAL RESULTS

Variations of the non-dimensional frequency ratios for the non-linear and linear vibrations have been presented for different variations of non-dimensional amplitudes and temperature co-efficient μ. Results for both geometries $\left[\xi = R^2/2 R_o h = 1 \text{ and } \xi = 1/2 \right]$ have been presented graphically. Immovable edge conditions have been considered in the analysis and results for $\xi = 1$ have been compared with those of Banerjee and Sinharay (1985) when $\mu = 0$. Deviations in the results can be observed because equation (7) of the above reference appears to the fallacious. Correct form should be the equation (167) of the present thesis.

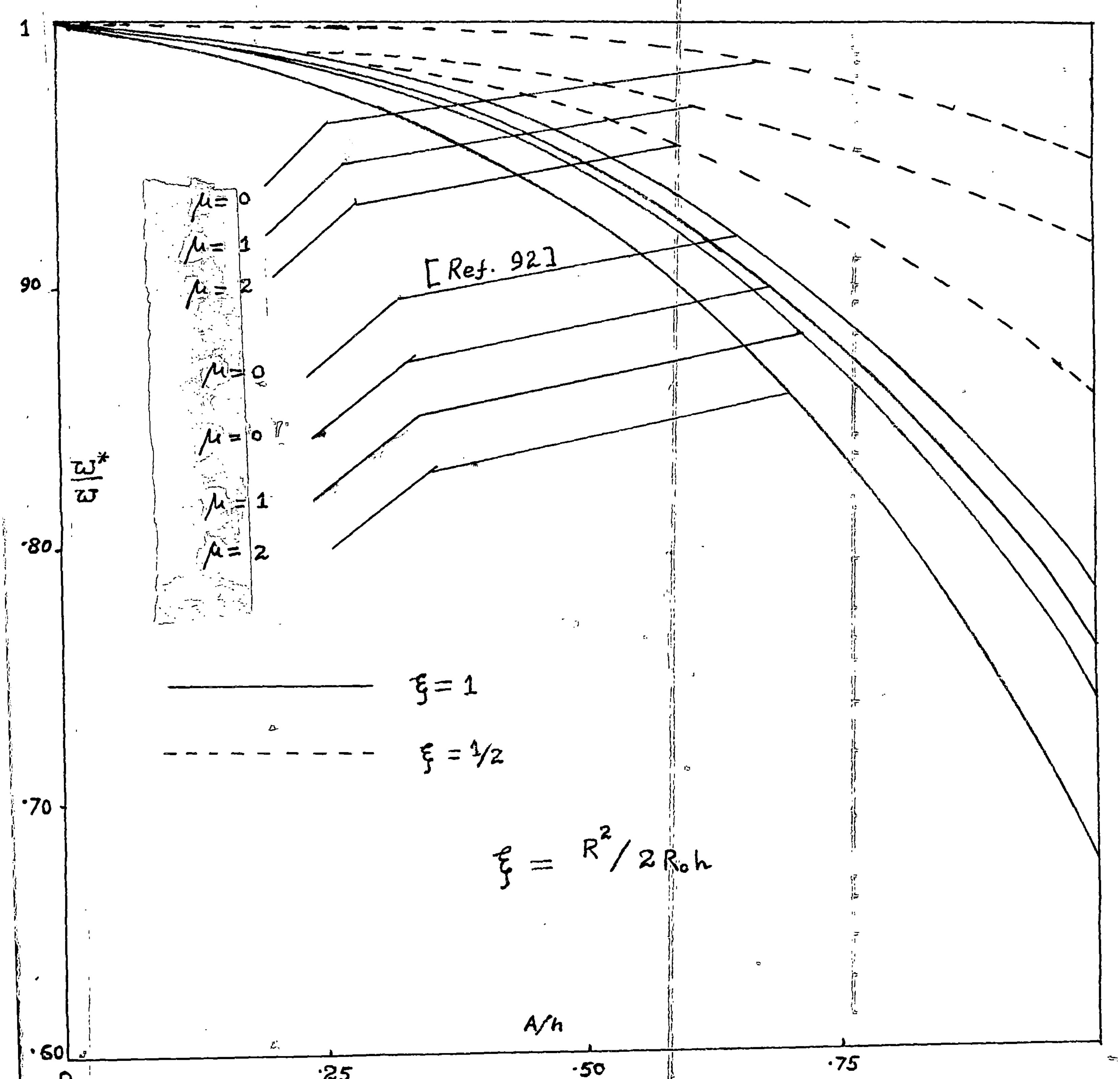

FIGURE - 10 : The Variations of the non-dimensional frequency ratios for Variations of non-dimensional amplitudes and temperature co-efficient μ.