CHAPTER II

NON-LINEAR FREE VIBRATION OF ELASTIC PLATES AT ELEVATED TEMPERATURE BY BERGER's METHOD

I. NON-LINEAR FREE VIBRATION OF TRIANGULAR PLATES AT ELEVATED TEMPERATURE*

2.1 GOVERNING EQUATIONS OF THE HEATED PLATES

Berger's approximate quasi-linear uncoupled differential equations governing the motion of the heated elastic plates are given by

$$D\,\nabla^4 W + K^2 \nabla^2 W + \rho h\, W_{,tt} + \frac{\nabla^2 M_T}{1-\nu} = 0 \qquad \ldots \quad (98)$$

$$\frac{N_T}{1-\nu} - 12\,\frac{D e_1}{h^2} = K^2 \qquad \ldots \quad (99)$$

where $e_1 = u_{,x} + v_{,y} + \frac{1}{2} W_{,x}{}^2 + \frac{1}{2} W_{,y}{}^2$ ... (100)

K^2 is independent of x and y but involves the time 't'.
In the present analysis for free flexural vibrations of heated plates equation (98) reduces to form

$$D\,\nabla^4 W + K^2 \nabla^2 W + \rho h\, W_{,tt} = 0 \qquad \ldots \quad (101)$$

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2.2 METHOD OF SOLUTION

(A) RIGHT-ANGLED ISOSCELES TRIANGLED PLATES

The origin of a simply-supported right-angled isosceles triangular plate is chosen at the vertex containing the right-angle with the equal sides of length 'a' along the co-ordinate axis [Fig. 4]

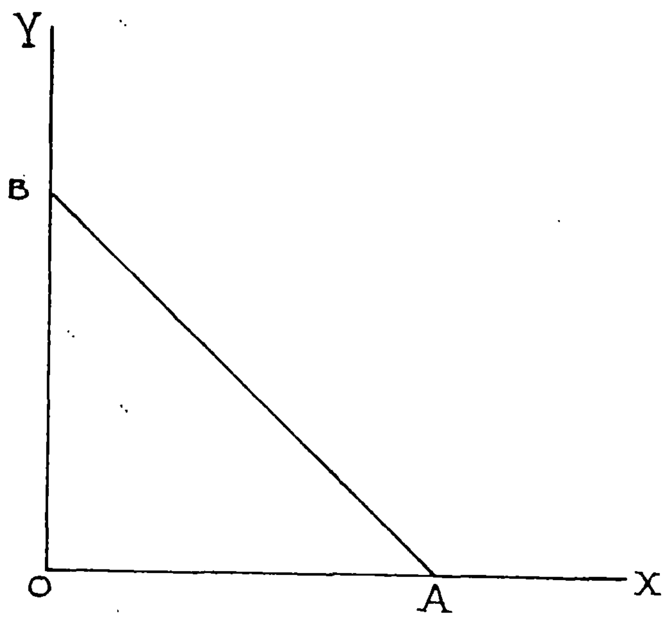


Figure 4 : Right-angled isosceles triangular plate of equal sides OA = OB = a .

For such a plate the in-plane and transverse boundary conditions are

$$u = W = W_{,xx} = 0 \quad \text{at} \quad x = 0$$

$$v = W = W_{,yy} = 0 \quad \text{at} \quad y = 0$$

$$W = W_{,\eta\eta} = 0 \qquad \text{at } x+y = a \qquad \ldots (102)$$

where

$$\frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}} \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right] \qquad \ldots (103)$$

Compatible with the above boundary conditions u, v and W are chosen in the form

$$u = \sum_{n=1,3\ldots}^{\infty} B_n \operatorname{Sin} \frac{n\pi x}{a} \left(\operatorname{Cos} \frac{n\pi y}{a} + \operatorname{Sin} \frac{n\pi x}{a} - \frac{n\pi}{4} \right) H(t) \qquad \ldots (104)$$

$$v = \sum_{n=1,3\ldots}^{\infty} B_n \operatorname{Sin} \frac{n\pi y}{a} \left(\operatorname{Cos} \frac{n\pi x}{a} - \operatorname{Sin} \frac{n\pi y}{a} + \frac{n\pi}{4} \right) G(t) \qquad \ldots (105)$$

$$W = \sum_{m=1,3\ldots}^{\infty} A_m \left(\operatorname{Sin} \frac{2\pi mx}{a} \cdot \operatorname{Sin} \frac{m\pi y}{a} + \operatorname{Sin} \frac{m\pi x}{a} \cdot \operatorname{Sin} \frac{2m\pi y}{a} \right) F(t) \qquad \ldots (106)$$

Combining equations (101) and (106) one gets

$$\frac{25\, D m^4 \pi^4}{a^4} F(t) - \frac{5 m^2 \pi^2 K^2}{a^2} F(t) = -\rho h F_{,tt} \qquad \ldots (107)$$

Integrating now equation (99) over the area of the plate and eliminating K^2 with the help of equation (107) one gets the non linear time differential equation as

$$F_{,tt} + \alpha F(t) + \beta F^3(t) = 0 \qquad \ldots (108)$$

where

$$\alpha = 25\, m^4 \pi^4 D \left[1 - \frac{a^2 N_T^*}{5(1-\nu) D m^2 \pi^2} \right] \Big/ a^4 \rho h \qquad \ldots (109)$$

$$\beta = \frac{75\, \pi^4 m^4 D}{a^4 \rho h} \sum_{m=1,3\ldots}^{\infty} m^2 \left(A_m / h \right)^2 \qquad \ldots (110)$$

and $\overline{N_T} = \frac{1}{A} \iint N_T \, dx dy$ is the mean value of N_T over the area A of the plate. The solution of the non linear time differential equation with initial conditions (7) can be expressed in terms of Jacobian elliptic function of dosine type (10) given by Nash and Modeer [60] and hence the ratio of the time-periods of non linear and linear vibration can be obtained as

$$T^*/T = \frac{2\theta}{\pi} \left[1 + \frac{3 \sum_{m=1,3,\cdots}^{\infty} m^2 (A_m/h)^2}{m^2 \left[1 - a^2 N_T^* / 5(1-\nu) D m^2 \pi^2 \right]} \right]^{-\frac{1}{2}} \qquad \ldots (111)$$

For free fundamental mode of vibrations without thermal loading the ratio of the time-periods reduces to

$$\frac{T^*}{T} = \frac{2\Theta}{\pi} \left[\frac{1}{\sqrt{1+3\left(\frac{A_1}{h}\right)^2}} \right] \qquad \ldots (112)$$

as obtained by Banerjee (107).

2.3 BUCKLING CRITERION

For the pre-buckling state non-dimensional time-periods T^*/T can be obtained from equation (111) by taking values of

$$\frac{a^2 N_T^*}{8\pi^2(1-\nu) D} = \lambda \text{ (say sufficiently near to unity.)}$$

Buckling occurs when $\lambda = 1$ and the critical buckling temperature $(N_T^*)_{cr}$ is obtained as

$$(N_T^*)_{cr} = 8\pi^2 D(1-\nu) / a^2 \qquad \ldots (113)$$

which is in agreement with the result obtained by Banerjee (107).

B. SIMPLY-SUPPORTED EQUILATERAL TRIANGULAR PLATE

Analysis of this section shall be carried out with the help of trilinear cordinates [Sen, B (1963)] let ABC be an equilateral triangle of sides '2a'. The centroid O on the undeflected middle surface is taken as the origin and the X- and Y-axes are taken perpendicular and parallel to the side BC.

If p_1, p_2, p_3 be the lengths on the sides CA, AB and BC respectively and 'r', the radius of the inscribed circle [Fig. 5] then

$$p_1 = r + \frac{x}{2} - \frac{\sqrt{3}}{2}\, y$$

$$p_2 = r + \frac{x}{2} + \frac{\sqrt{3}}{2}\, y$$

$$p_3 = r - x \qquad \ldots (114)$$

Hence $p_1 + p_2 + p_3 = 3r = \sqrt{3}a$

$$= K' \text{ (say)} \qquad \ldots (115)$$

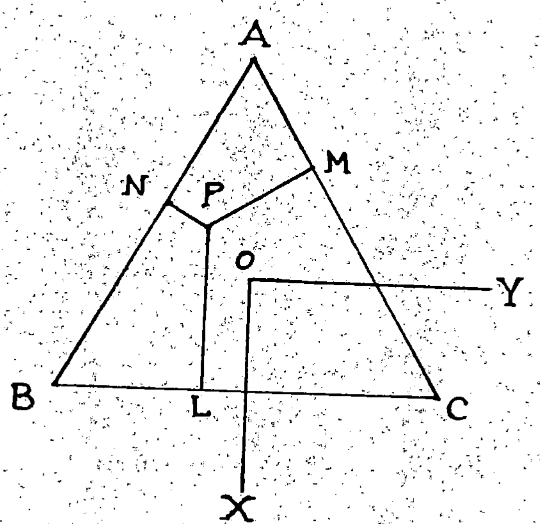


Figure 5 : Equilateral triangular plate of side 2a .

Two dimensional laplacian operator shall be obtained as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{\partial^2}{\partial p_3^2} - \frac{\partial^2}{\partial p_2 \partial p_3} - \frac{\partial^2}{\partial p_3 \partial p_1} - \frac{\partial^2}{\partial p_1 \partial p_2} \qquad \ldots (116)$$

The transverse displacement W satisfying the simply-supported boundary conditions

$$W = \nabla^2 W = o \quad \text{at} \quad p_1 = p_2 = p_3 = 0 \qquad \ldots (117)$$

is assumed in the form

$$W = \sum_{m=1,2,\ldots}^{\infty} A_m \left[\sin \frac{2\pi m p_1}{K'} + \sin \frac{2\pi m p_2}{K'} + \sin \frac{2\pi m p_3}{K'} \right] F(t) \qquad \ldots (118)$$

Also the following forms of u and v

$$u = \sum_{m=1,\ldots}^{\infty} \sqrt{3}\, B_m \left[\sin \frac{2m\pi(p_2+p_3)}{K'} + \sin \frac{2m\pi(p_1+p_3)}{K'} \right] H(t) \qquad \ldots (119)$$

$$v = \sum_{m=1,\ldots}^{\infty} B_m \left[\sin \frac{2m\pi(p_1+p_3)}{K'} - \sin \frac{2m\pi(p_2+p_3)}{K'} \right] G(t) \qquad \ldots (120)$$

can be chosen in conformity with the boundary conditions

$$\begin{aligned} u &= o && \text{at} \quad p_3 = o \\ \sqrt{3}\, v + u &= o && \text{at} \quad p_2 = o \\ \sqrt{3}\, v - u &= o && \text{at} \quad p_1 = o \end{aligned} \qquad \ldots (121)$$

Proceeding in the same way as laid down in the preceding section one arrives at the same type of differential equation (103) where

$$\alpha = \frac{D}{a^4 \rho h} \cdot \frac{16 m^4 \pi^4}{9a^4} \left[1 - \frac{3a^2 N_T^*}{4(1-\nu) D \pi^2 m^2} \right] \qquad \ldots (122)$$

$$\beta = \frac{D}{a^4 \rho h} \cdot \frac{16 m^2 \pi^4}{a^4} \sum_{m=1,\ldots}^{\infty} m^2 \left(\frac{A_m}{h}\right)^2 \qquad \ldots (123)$$

Non-dimensional time-periods T^*/T is given by equation (112) where α and β are to be replaced by equations (122) and (123).

For free fundamental mode of vibrations without thermal loading one gets,

$$T^*/T = \frac{2\theta}{\pi} \frac{1}{\sqrt{1+9(A/h)^2}} \qquad \ldots (124)$$

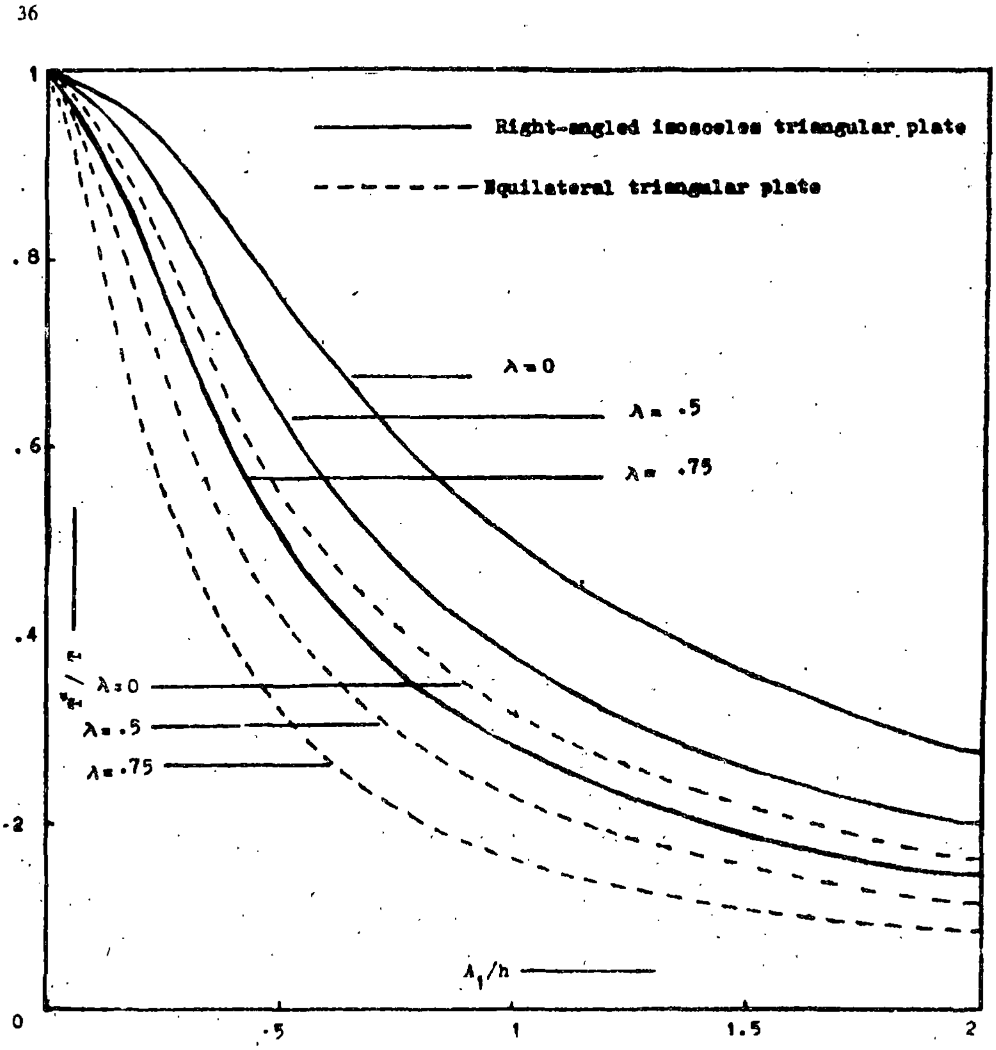
As in the previous case critical buckling temperature $(N_T^*)_{cr}$ is obtained in the form

$$(N_T^*)_{cr} = \frac{4(1-\nu) D \pi^2}{3a^2} \qquad \ldots (125)$$

as obtained by Datta [108].

2.4 NUMERICAL RESULTS AND DISCUSSION

Figure - 6 shows the variations of non-dimensional time-periods T^*/T for different values of non-dimensional amplitudes A/h and temperature parameter λ. It is seen that the effect of N_T^* is to diminish the non-dimensional time-periods. Also the circular frequency is given by the expression $W_0 = \sqrt{\alpha}$ and equations (109) and (122). Show that the circular frequency in each case diminishes due to the presence of N_T^*. It is seen from the figure that the non-dimensional time-periods are less for corresponding non-dimensional amplitudes in the cases of plates of more regular shapes. As it should be, the non-linear behaviour of the plates due to elevated temperature obtained here, is similar in nature as that of plates subjected to in-plane forces given by Biswas [25].



<u>Figure 6</u> : The variations of non-dimensional time-periods T^*/T for different values of non-dimensional amplitudes A_1/h and thermal loading parameter λ .

C. NON-LINEAR VIBRATIONS OF PARABOLIC PLATES AT ELEVATED TEMPERATURE*

Let us consider a parabolic plate with boundary given by

$$x^2 = \frac{a}{2}\,(2a - y) \;; \quad y = 0 \qquad \ldots (126)$$

For this plate shape clamped along the boundary the deflection W is expressed in the form

$$W = A\,\frac{y^2}{a^3}\left[\frac{a}{2}\,(2a - y) - x^2\right]^2 F(t) \qquad \ldots (127)$$

Combining equations (101) and (127) and applying Galerkin's procedure one gets

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$$\int_{y=0}^{2a} \int_{x=-\sqrt{\frac{a}{2}(2a-y)}}^{\pm\sqrt{\frac{a}{2}(2a-y)}} \Big[D\Big\{24\frac{y^2}{a^6}AF(t)+\frac{2A}{a^6}(24x^2-5a^2+12ay)\ F(t)\Big\} +$$

$$+ K^2\Big\{2a^4-6a^3y-a^2y^2+2ay^3+12x^2y^2-4a^2x^2+6axy^2+$$

$$2x^4\Big\}\frac{AF(t)}{a^6}+\frac{A}{a^6}\rho h\ y^2\Big\{\frac{a}{2}(2a-y)-x^2\Big\}^2\ \ddot{F}(t)\Big]\ x$$

$$x\Big\{y^2\Big[\frac{a}{2}(2a-y)-x^2\Big]^2\Big\}dxdy = 0 \qquad \ldots (128)$$

Performing necessary integrations one arrives at the following equation

$$0.0233\ a^4\rho h\ \ddot{F}(t)+\Big[6.694\ D-2.0231\ a^2K^2\Big]F(t) = 0 \qquad \ldots (129)$$

in which K^2 is still unknown which is to be obtained by integrating equation (99) over the area of the plate leading to

$$\iint \frac{N_T}{1-\nu}dxdy - \frac{6D}{h^2}\iint \Big\{\Big(\frac{\partial w}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial y}\Big)^2\Big\}dxdy$$

$$= K^2\iint dx\,dy \qquad \ldots (130)$$

with limits of integration as in equation (128). Since u and v vanishes on the boundary of the plate for clamped immovable edges equation (130) ultimately leads to the result

$$\frac{1}{a^2}\iint\frac{N_T}{(1-\nu)D}dxdy - \frac{6}{h^2}(\ 1.18393)\ A^2\ F^2(t) = \frac{K^2}{D}(\ 2.66) \qquad \ldots (131)$$

Eliminating K^2 with help of equations (129) and (131) one gets the well known cubic equation of the form

$$\ddot{F}(t) + \alpha F(t) + \beta F^3(t) = 0 \qquad \ldots (132)$$

where

$$\alpha = \frac{D}{a^4 \rho h} \left[232.20 - 26.29 N_T^* \right] \qquad \ldots (133)$$

$$\beta = \frac{D}{a^4 \rho h} \left[137.57 \right] \left(\frac{A}{h} \right)^2 \qquad \ldots (134)$$

$$N_T^* = \frac{1}{a^2} \iint \frac{N_T}{D(1-\nu)} \, dx\,dy \qquad \ldots (135)$$

The solution of equation (132) with conditions (7) in terms of Jacobian elliptic function of cosine type are expressed in equation (10) and hence non-dimensional time-period is given by

$$\frac{T^*}{T} = \frac{2\theta}{\pi} \left[\frac{137.57 \, (A/h)^2}{232.20 - 26.29 N_T^*} \right]^{-\frac{1}{2}} \qquad \ldots (136)$$

2.4 NUMERICAL RESULTS

Variations of non-dimensional time-periods T^*/T for different values of non-dimensional amplitude A/h and temperature parameter N_T^* have been computed and presented graphically (Figure - 8).

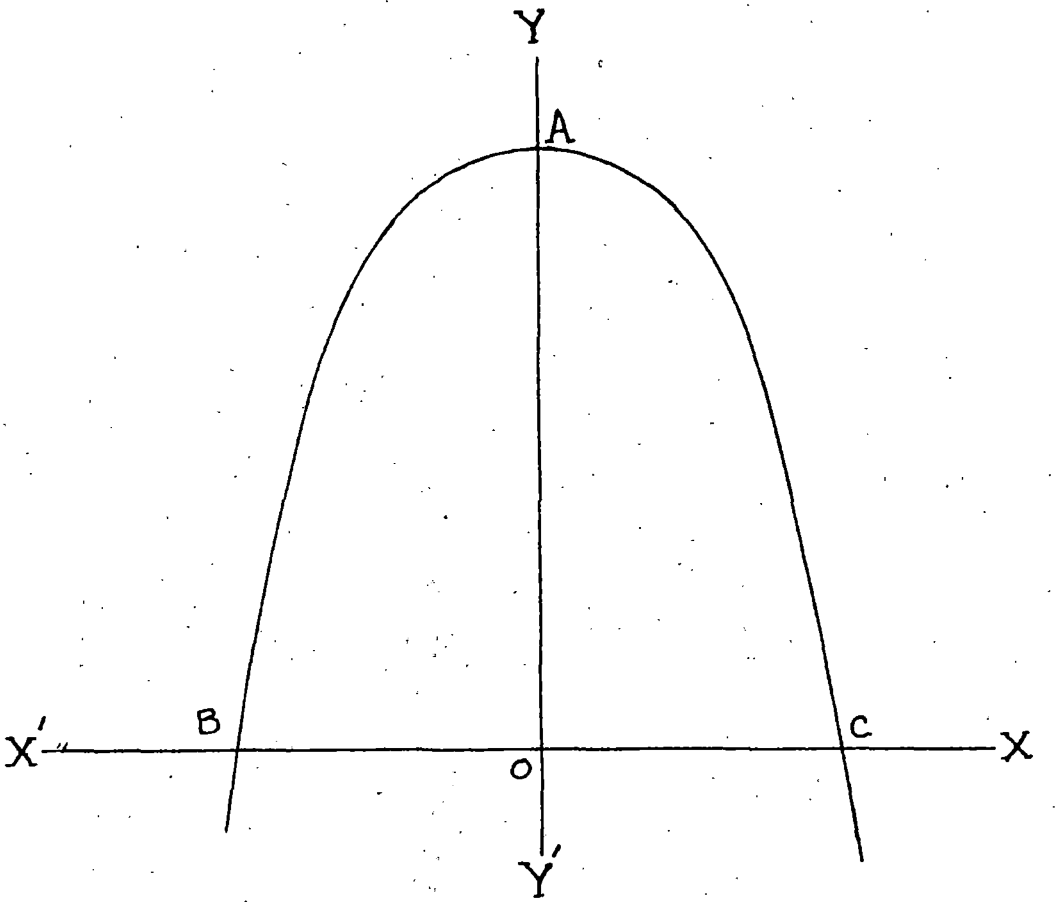

Figure 7: Geometary of the parabolic plate .

Figure 8 : The variations of relative time-periods T^*/T versus non-dimensional amplitudes A/h and thermal loading parameter N_T^* .

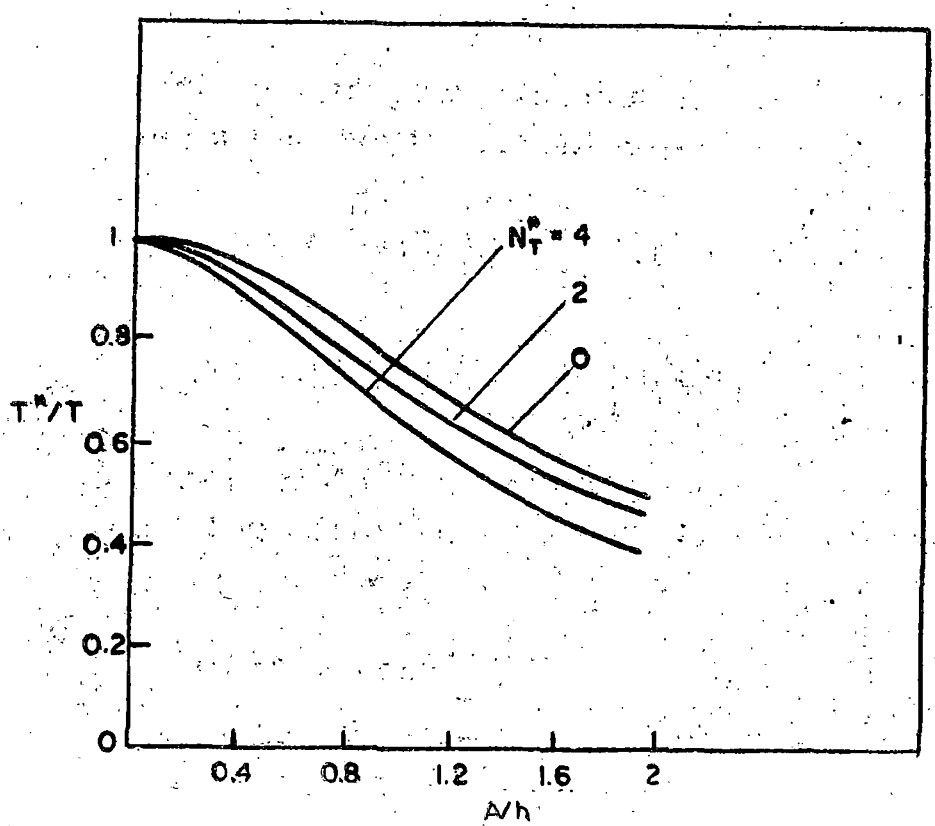

Figure 8 .