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Short Communication

Non-linear free vibration of orthotropic circular plates at elevated temperature

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Abstract

Non-linear behaviour of an axisymmetric orthotropic circular plate at elevated temperature for both clamped movable and immovable edges has been studied, using generalised dynamical field equations (in the von Karman sense) derived in terms of displacement components. Relative time-periods for linear and non-linear vibrations are seen to depend on relative amplitudes and thermal loading parameter. Critical buckling temperatures for both the boundary conditions have been obtained in the limiting case and corresponding results for isotropic plates have been compared with known results.

Key words: Non-linear free vibrations, orthotropic circular plates, elevated temperature, critical buckling temperature.

1. Introduction

With the increased use of strong and light-weight structures, especially in aerospace engineering, and in the vibrations of machine parts, many problems of non-linear vibrations arise where complementary stresses in the middle plane of the plate must be taken into account for deriving the governing field equations of the plate.

Extensive studies on the large amplitude (non-linear) vibrations of elastic circular plates have been made by Berger's method [1-4] as well as by von Karman's method [5-7]. Berger's method has some advantages over von Karman's method since it leads to decoupled equations. However, Nowinski and Ohanabe [8] and Prathap [9] have pointed out certain inaccuracies in Berger's equations and in view of this von Karman's method should be resorted to until some alternative theory is set forth.

In the present investigation, non-linear behaviour of an axisymmetric orthotropic circular plate at elevated temperature for both clamped movable and immovable boundary conditions has been studied, using generalised dynamical field equations (in the von Karman sense) derived in terms of displacement components. Relative time-periods for linear and non-linear vibrations are seen to depend on relative amplitudes and thermal loading parameter. Critical buckling temperatures have been deduced in the limiting case and compared with known results for the isotropic plates.

2. Governing equations

Considering equilibrium equations of the non-linear theory for the case of an axisymmetric orthotropic circular plate subject to thermal stresses and with notations as in Nowinski [10], the basic governing equations for the dynamical analysis in terms of displacement components can be expressed in the forms

$$r^2 u_{,rr} + r u_{,r} - \frac{C_{22}}{C_{11}} u = 1/2\left(\frac{C_{12}}{C_{11}} - 1\right) r w_{,r}^2 -$$

$$- r^2 w_{,r} w_{,rr} + \frac{\beta_{22} - \beta_{11}}{C_{11}\, h} N_T r - \frac{\beta_{11} r^2}{C_{11}\, h} \frac{\mathrm{d} N_T}{\mathrm{d} r} \tag{1}$$

$$\frac{C_{11} h^3}{12} \left(w_{,rrrr} + \frac{2}{r} w_{,rrr}\right) - \frac{C_{22} h^3}{12}\left(\frac{1}{r^2} w_{,rr} - w_{,r}/r^3\right) - \rho h w_{,tt} =$$

$$= q - 1/r\, \mathrm{d}/\mathrm{d}r\, (r\, \mathrm{d}w/\mathrm{d}r\, N_{rr}) + (\beta_{11} - \beta_{22})\, N_T + \beta_{11}\, r\, \mathrm{d}M_T/\mathrm{d}r \tag{2}$$

3. Free vibrations

For free vibrations $q = 0$; however, it is not exactly true that $M_T = 0$; it is an assumption based on the neglect of temperature variation in depth due to compression even though Jones *et al* [11] assume $M_T = 0$. For free thermal vibrations, the temperature field should be taken to depend on the radial co-ordinate r as considered by Buckens [12]. Accordingly, M_T disappears from equation (2) and only N_T survives in equation (1).

4. Method of solution

The deflection $w(r,t)$ is expressed in the separable form

$$w(r,t) = A\left[1 + \sum_{i=2,4,\ldots}^{\infty} A_i (r/a)^i\right] F(t) \tag{3}$$

$$\approx A\left[1 + A_2 (r/a)^2 + A_4 (r/a)^4\right] F(t) \tag{3.1}$$

where A is the maximum deflection at the centre of the plate and the constants A_2 and A_4 must be determined from the boundary conditions.

Considering equations (1) and (3.1) one gets the in-plane displacement $u(r,t)$ finite at the origin, in the form

$$u(r,t) = C_0 r^k + \frac{C_1 r^3}{9-k^2} + \frac{C_2 r^5}{25-k^2} + \frac{C_3 r^7}{49-k^2} + \psi(r) \tag{4}$$

where C_1, C_2 and C_3 are known constants and C_0 is a constant of integration to be determined from in-plane boundary conditions for movable and immovable edges, $\Psi(r)$ is the particular integral for the thermal loading terms on the right-hand side of equation (1) and $k^2 = C_{22}/C_{11}$.

Assuming the temperature distribution $\theta(r,z)$ to depend on the radial co-ordinate in the form[12] given by

$$\theta(r,z) = \tau_0(r) = T_0(1 - r/a),$$

one gets the expressions for N_T and accordingly $\psi(r)$ is determined.

We now substitute the expression for $w(r,t)$ given by equation (3.1) as well as the required expression for N_{rr} given by

$$N_{rr} = \int_{-h/2}^{h/2} \tau_{rr}\,dz = C_{11}h(u_{,r} + \tfrac{1}{2} w_{,r}^2) + C_{12}h\frac{u}{r} + \beta_{11} N_T \tag{5}$$

into equation (2) and applying Galerkin procedure one arrives, after a lengthy but simple calculation, the following time-differential equation in the form

$$C_{11}hAF(t)[A_2\{2C_0a^{k-1}(\frac{1}{k+1} + \frac{A_2}{k+3} + \frac{A_4}{k+5})$$

$$(k+1)(k+\frac{C_{12}}{C_{11}}) + \frac{8C_1a^2}{9-k^2}(\frac{1}{4} + \frac{A_2}{6} + \frac{A_4}{8})(3+\frac{C_{12}}{C_{11}}) + \frac{12C_2a^4}{25-k^2}$$

$$(\frac{1}{6} + \frac{A_2}{8} + \frac{A_4}{10})(5+\frac{C_{12}}{C_{11}}) + \frac{16C_3a^6}{49-k^2}(\frac{1}{8} + \frac{A_2}{10} + \frac{A_4}{12})(7+\frac{C_{12}}{C_{11}}) +$$

$$\frac{4(\beta_{22}-\beta_{11})}{C_{11}} \times \frac{T_0}{1-k^2} \times (\frac{1}{2} + \frac{A_2}{4} + \frac{A_4}{6})(1+\frac{C_{12}}{C_{11}}) - \frac{6(\beta_{22}-2\beta_{11})}{C_{11}(4-k^2)}$$

$$T_0(\frac{1}{3} + \frac{A_2}{5} + \frac{A_4}{7})(2+\frac{C_{12}}{C_{11}})\} + A_4\{4C_0a^{k-1}(k+3)\times(k+\frac{C_{12}}{C_{11}})$$

$$(\frac{1}{k+3}+\frac{A_2}{k+5}+\frac{A_4}{k+7})+\frac{24\,C_1 a^2}{9-k^2}(\frac{1}{6}+\frac{A_2}{8}+\frac{A_4}{10})(3+\frac{C_{12}}{C_{11}})+\frac{32\,C_2 a^4}{25-k^2}$$

$$(5+\frac{C_{12}}{C_{11}})\times(\frac{1}{8}+\frac{A_2}{10}+\frac{A_4}{12})+\frac{40\,C_3 a^6}{49-k^2}(\frac{1}{10}+\frac{A_2}{12}+\frac{A_4}{14})(7+\frac{C_{12}}{C_{11}})+$$

$$\frac{16\,(\beta_{22}-\beta 11)\,T_0}{C_{11}\,(1-k^2)}(1+\frac{C_{12}}{C_{11}})\times(\frac{1}{4}+\frac{A_2}{6}+\frac{A_4}{8})-\frac{20\,(\beta_{22}-2\beta_{11})\,T_0}{C_{11}\,(4-k^2)}$$

$$(\frac{1}{5}+\frac{A_2}{7}+\frac{A_4}{9})\,(2+\frac{C_{12}}{C_{11}})\}+\frac{\beta_{11}\,T_0}{C_{11}}\,[A_2\{4(\frac{1}{2}+\frac{A_2}{4}+\frac{A_4}{6})-6(\frac{1}{3}+\frac{A_2}{5}+$$

$$+\frac{A_4}{7})\}+A_4\{16(\frac{1}{4}+\frac{A_2}{6}+\frac{A_4}{8}-20(\frac{1}{5}+\frac{A_2}{7}+\frac{A_4}{9})\}]]+$$

$$+\,C_{11}\,h\,A^3 F^3(t)[A_2^3\frac{16}{a^2}(\frac{1}{4}+\frac{A_2}{6}+\frac{A_4}{8})+\frac{A_2^2\,A_4}{a^2}\;144(\frac{1}{6}+\frac{A_2}{8}+\frac{A_4}{10})$$

$$+\frac{A_2\,A_4^2}{a^2}(\frac{1}{8}+\frac{A_2}{10}+\frac{A_4}{12})\times 384+\frac{A_4^3}{a^2}\times 320(\frac{1}{10}+\frac{A_2}{12}+\frac{A_4}{14})\,]=\frac{C_{11}\,h^3}{12\,a^2}$$

$$8\,A_4(9-\frac{C_{22}}{C_{11}})(\frac{1}{2}+\frac{A_2}{4}+\frac{A_4}{6})A\,F(t)+\rho h A F(t) a^2\,[\frac{1}{2}+\frac{A_2}{2}+\frac{A_2\,A_4}{4}$$

$$+\frac{A_2^2+2A_4}{6}+\frac{A_4^2}{10}\,] \tag{6}$$

5. Determination of the constant of integration C_0

For clamped immovable edges of the plate we have $u=0$ at $r=a$ and for movable edges of a plate we have $N_{rr}=0$ at $r=a$. Required expressions for the constant C_0 for the two cases are obtained by inserting the above boundary conditions in equations (4) and (5). With these values of C_0 inserted into equation (6) one gets, finally, the time-differential equation in the form

$$d^2F(t)/dt^2+\alpha F(t)+\beta F^3(t)=0 \tag{7}$$

6. Boundary conditions—clamped plate

For a plate clamped along the boundary

$$w=dw/dr=0 \qquad \text{at} \qquad r=a$$

and considering equation (3.1) one gets $A_2 = -2$ and $A_4 = 1$.

Corresponding values of A_2 and A_4 for simply-supported edges can be determined by considering the conditions

$$w = 0 = M_n \text{ (moment) at } r = a.$$

7. Solution of time-differential equation

The solution of equation (7) with initial conditions

$$F(0) = 1, \quad \mathrm{d}F(0)/\mathrm{d}t = 0 \tag{8}$$

has been given by Nash and Modeer [13] with the help of Jacobian elliptic functions and hence the ratio of the non-linear and linear time-periods T^*/T is given by

$$T^*/T = \frac{2\,\Theta}{\pi}/(1+\beta/\alpha)^{1/2} \tag{9}$$

8. Numerical results and discussion

For both movable and immovable edges of the circular plate variations of non-dimensional time-periods T^*/T for different variations of non-dimensional amplitudes A/h and non-dimensional temperature $N_T^* = -\beta_{11} T_0/C_{11}$ have been computed and presented in Tables I and II considering the set of values.

$$E_{11} = 1 \times 10^5,\ E_{22} = 0.5 \times 10^5,\ \nu_1 = 0.5,\ \nu_2 = 0.025,\ C_{22}/C_{11} = 0.5$$
$$= E_{22}/E_{11},\ C_{12}/C_{11} = 0.025,\ k^2 = 0.5,\ \beta_{22}/\beta_{11} = 0.5,\ a/h = 15.$$

From the tables, it is observed that for both the edge conditions the effect of N_T^* is to diminish the non-dimensional time-periods. The effect of temperature on non-dimensional time-periods is more for plates with immovable edges than for plates with movable edges for the corresponding variations of non-dimensional amplitudes. As it should be, the non-linear behaviour of the plates due to elevated temperature obtained here, is similar in nature as that of the plates subjected to in-plane compressive forces [14].

Table I
Circular plate with movable edge

A/h	0	0.4	0.8	1.2	1.6	2.0
$T^*/T\ (N^*_T = 0)$	1	.99250	.97099	.938052	.897102	.8515666
$T^*/T\ (N^*_T = .05)$	1	.97998	.92645	.85391	.77608	.70157
$T^*/T\ (N^*_T = .075)$	1	.87421	.67602	.52176	.41694	.34451

Table II
Circular plate with immovable edge

A/h	0	0.4	0.8	1.2	1.6	2.0
$T^*/T(N^*_T = 0)$	1	.97433	.90778	.82189	.73442	.65451
$T^*/T(N^*_T = .005)$	1	.96356	.87429	.76842	.6692	.58455
$T^*/T(N^*_T = .01)$	1	.93718	.80214	.66713	.55757	.47332

9. Buckling criterion and critical buckling temperatures

Considering the foregoing set of values of elastic constants and required expressions for α and β one gets

$$\beta/\alpha \text{ (for movable edges)} = 12(A/h)^2\,(.000624)\,/\,(.079 - N_T^*) \tag{10}$$

$$\beta/\alpha \text{ (for immovable edges)} = 12(A/h)^2\,(.0004522)\,/\,(.01626 - N_T^*) \tag{11}$$

Tables I and II have been constructed for the pre-buckling state by considering values of N_T^* sufficiently near to .079 and .01626 for movable and immovable edges respectively.

Buckling occurs when

$$N_T^* = .079 \text{ (movable edges)} \tag{12}$$

$$N_T^* = .01626 \text{ (immovable edges)} \tag{13}$$

which give the critical buckling temperature for the above two cases.

10. Results for simply-supported plates

Results for the non-linear dynamic analysis of simply-supported orthotropic circular plates at elevated temperature can be obtained by considering the values of A_2 and A_4 given below

$$A_2 = -\frac{2(3-\nu_2)}{5-\nu_2}\cdot\; A_4 = \frac{1-\nu_2}{5-\nu_2} \tag{14}$$

where ν_2 is the Poisson's ratio in the ϕ-direction.

The analysis of the preceding section may be followed by using the same equations and expressions where the values of A_2 and A_4 should be considered from equation (14).

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Authors' Note

This paper has been condensed as suggested by the Editor. As a result many equations and expressions have been omitted. Interested readers may write to the authors for additional information.

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NON-LINEAR FREE VIBRATIONS AND THERMAL BUCKLING OF CIRCULAR PLATES AT ELEVATED TEMPERATURE

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von Karman's classical non-linear differential equations extended to the dynamical case have been used for the analysis of non-linear free vibrations and thermal buckling of a circular plate at elevated temperature.

INTRODUCTION

Several studies on the non-linear free vibrations of heated elastic plates have been made by Berger's method (Pal 1970, 1973; Mazumder *et al.* 1980). The present paper employes von Karman's coupled equations extended to the dynamical case to study the thermal buckling and non-linear free vibrations of circular plates at elevated temperature. A relation connecting relative amplitudes, thermal loading parameter and relative time-periods for linear and non-linear vibrations has been obtained. Criterion for critical buckling temperature has been deduced as a limiting case. It has been noted that non-linear free vibrations of plates under thermal loading are similar to the cases of plates subjected to in-plane compressive forces.

GOVERNING FIELD EQUATIONS FOR CIRCULAR PLATES

With notations as in Timoshenko and Krieger (1959) and Nowacki, von Karman's equations extended to the dynamical case for heated circular plates are given by

$$\nabla^4 F = -Eh\,(1/r.\, W_{,rr}\, W_{,r}) + \alpha_t\, E\, \nabla^2\,(N_T) \qquad \ldots(1)$$

$$D\,\nabla^4 W + \rho h\, W_{,tt} = 1/r.\, W_{,rr}\, F_{,r} + 1/r.\, W_{,r}\, F_{,rr} + q - \alpha_t E/(1-\nu)\,\nabla^2\,(M_T). \qquad \ldots(2)$$

METHOD OF SOLUTION

For free vibrations of a circular plate of radius a one may assume the temperature field as $T(r, z) = \tau_0(r)$ and the deflection W satisfying clamped-edge boundary conditions is expressed in the form

$$W(r, t) = W_0(t)(1 - r^2/a^2)^2. \qquad ...(3)$$

Since N_T appears in the boundary conditions, one may omit, without any loss of generality, the part $\nabla^2(N_T)$ from eqn. (1).

Inserting eqn. (3) into eqn. (1), the stress function F, finite at the origin, is obtained in the form

$$F(r, t) = A + Br^2 - 16EhW_0^2(t)/a^8.\ [r^8/768 - a^2r^6/144 + a^4r^4/64]. \qquad ...(4)$$

Since the in-plane displacement vanishes on the boundary for clamped immovable edges of a plate, one gets

$$(r/Eh)[F_{,rr} - \nu F_{,r}] + r N_T/Eh = 0 \text{ on } r = a.$$

The above relation leads to

$$B = \frac{EhW_0^2(t)(5 - 3\nu)}{12(1 - \nu)a^2} - \frac{N_T}{2(1 - \nu)}. \qquad ...(5)$$

Applying Galerkin procedure in eqn. (1) one gets

$$a^4 \rho h W_{0,tt} + \frac{320}{3} D\left[1 - \frac{a^2 N_T}{16(1 - \nu)D}\right] W_0(t)$$
$$+ \frac{Eh(130 - 110\nu)}{9(1 - \nu)} W_0^3(t) = 0. \qquad ...(6)$$

Let us put $W_0(t) = A\,\tau(t)$ which gives the normalised initial conditions

$$\tau(0) = 1,\ \dot{\tau}(0) = 0. \qquad ...(7)$$

Equation (6) now leads to the well-known time-differential equation

$$\frac{d^2\tau}{dt^2} + C_1\tau + C_2\tau^3 = 0 \qquad ...(8)$$

where

$$C_1 = (320\,D/3a^4\,\rho h)[1 - a^2N_T/16D(1 - \nu)]$$
$$C_2 = 4D(1 + \nu)(130 - 110\nu)(A/h)^2/3a^4\,\rho\,h. \qquad ...(9)$$

The solution of equation (8) with the initial conditions (7) has been given by Nash and Modeer (1959) in the form

$$\tau(t) = Cn(\omega^* t, \theta) \qquad ...(10)$$

where Cn is Jacobi's elliptic function, and ω^* and θ are positive constants given by

$$\omega^{*2} = C_1 + C_2 \text{ and } \theta^2 = C_2/2(C_1 + C_2). \qquad ...(11)$$

The non-linear time period T^* is given by $T^* = 4\Theta/\omega^*$ where Θ represents the completes elliptic integral of the first kind.

The usual linear time-period T is given by $T = 2\pi/\Omega_0$ where Ω_0 is found from equation (8) by dropping the non-linear term $\tau^3(t)$ so that $\Omega_0 = \sqrt{C_1}$.

Relative time-period T^*/T is given by $T^*/T = 2\theta/\pi\,(1 + C_2/C_1)$ where

$$C_2/C_1 = \frac{(1 + \nu)\,(130 - 110\nu)\,(A/h)^2}{80\,[1 - a^2 N_T/16\,D\,(1 - \nu)]} \qquad \ldots(12)$$

Buckling Criterion and Critical Temperature

For the pre-buckling state relative time-periods can be obtained from (12) by taking values of

$a^2 N_T/16\,(1 - \nu)\,D = N_T^*$ (say) sufficiently near to unity. Buckling occurs when $N_T^* = 1$, and critical buckling temperature $(N_r)_{cr}$ is given by

$$(N_T)_{cr} = 16\,(1 - \nu)\,D/a^2. \qquad \ldots(13)$$

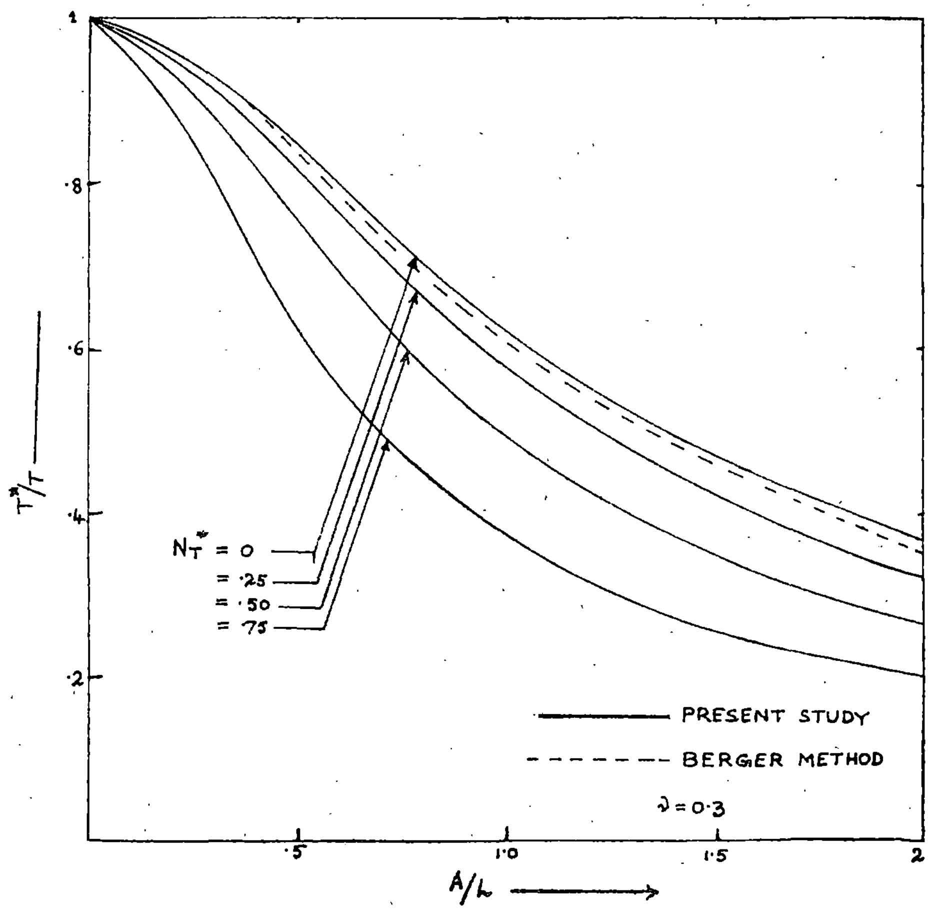

Fig. 1. The variations of relative time-periods versus relative amplitudes and non-dimensional thermal loading parameter.

Numerical Results and Discussion

Figure 1 shows the variations of relative time-periods T^*/T for different values of relative amplitudes A/h and temperature parameter N_T^*. It is seen that the effect of N_T^* is to diminish the relative time-periods. Also it is seen that circular frequency diminishes due to the presence of N_T^*. Moreover, it is interesting to note that the effect of N_T^* on the relative time-periods is the same in the cases of plates subjected to in-plane compressive forces which has been treated by Biswas (1981). It is mentioned also that the result (13) will lead to more accurate values as obtained by Biswas (1976) by considering two-term solution for $W(r, t)$.

It would not be inappropriate to discuss when free thermal vibrations occur. For free vibrations $M_T = 0$ is not exactly true; it is an assumption based on the neglect of temperature variation in depth due to compression. It does not follow from Mazumder (1980) who considered $M_T = 0$. For free thermal vibrations the temperature field should be taken to depend on the angular co-ordinate r as considered by Buckens (1979) for vibrations in a thermally stressed plates.

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Non-linear free vibrations of traiangular plates at elevated temperature

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Abstract

In this paper, non-linear free vibrations of triangular plates at elevated temperature have been analysed using Berger's approximation. Criterion for critical buckling temperature has also been deduced as a limiting case.

Key words: Non-linear, amplitude, time-period, elevated temperature, thermal buckling, critical temperature.

1. Introduction

In a recent publication, Jones *et al*[1], have studied the vibrations of both elastic and visco-elastic plates at elevated temperature. Although they have tried to establish the accuracy of Berger's method, Nowinski and Ohnabe[2] raised some points regarding its inapplicability for movable edges of a plate. Prathap[3] and Prathap and Varadan[4] also criticised the method regarding inaccuracies of Berger's method in certain cases. Yet it was claimed by Banerjee and Sarkar[5] that its applicability may be restricted to the cases of clamped square plates and circular plates with immovable edges, and to some extent, to the simply-supported circular and rectangular plates having smaller aspect ratios.

The present paper deals with the dynamic behaviour of a right-angled isosceles triangular plate and an equilateral triangular plate at elevated temperature and having simply-supported boundary conditions with immovable edges. The analysis is based on Berger's method as the plates considered have immovable inplane edge conditions. Moreover, Berger's equations are in decoupled form and have wide advantage in the analysis in comparison to the classical von Karman equations which are in the coupled forms and lead to mathematical complexities.

Some numerical computations for the variations of non-dimensional periods *vs* non-dimensional amplitudes and temperature parameters have been presented graphically. Criterion for thermal buckling has been established.

2. Governing equations for heated plates

Berger's approximate quasi-linear uncoupled differential equations governing the motion of heated elastic plates are given by[6]:

$$D \nabla^4 w + K^2 \nabla^2 w + \rho h\, w_{,tt} + \nabla^2 M_T/(1 - v) = 0 \tag{1}$$

$$N_T/(1 - v) - 12\, De_1/h^2 = K^2 \tag{2}$$

where

$$e_1 = u_{,x} + v_{,y} + 1/2\, w_{,x}^2 + 1/2\, w_{,y}^2 \tag{3}$$

$$N_T = \alpha_t E \int_{-h/2}^{h/2} T(x,y,z)\, dz, \qquad M_T = \alpha_t E \int_{-h/2}^{h/2} z\, T(x,y,z)\, dz \tag{4}$$

and u, v, w are displacement components, α_t, co-efficient of thermal expansion, ρ, density per unit mass, v, Poisson's ratio, ∇^2, Laplacian operator, D, flexural rigidity, h, plate thickness, E, Young's modulus, $T(x,y,z)$, temperature distribution within the plate given by

$$T(x,y,z) = \tau_0(x,y) + z\, \tau(x,y) \tag{5}$$

in which $\tau_0(x,y)$ and $\tau(x,y)$ satisfy certain temperature distribution differential equations[7] and K^2 is independent of x and y but involves the time t.

In the present analysis for free flexural vibrations of heated plates, equation (1) reduces to the form

$$D \nabla^4 w + K^2 \nabla^2 w + \rho h w_{,tt} = 0 \tag{6}$$

as $M_T = 0$.

3. Method of solution

3.1 Right-angled isosceles triangular plate

The origin of a simply-supported right-angled isosceles triangular plate is chosen at the vortex containing the right angle with the equal sides of length 'a' along the co-ordinate axes (fig. 1).

For such a plate the inplane and transverse boundary conditions are[8]

$$
\begin{aligned}
&u = w = w_{,xx} = 0 \quad \text{at } x = 0\\
&v = w = w_{,yy} = 0 \quad \text{at } y = 0\\
&w = w_{,\eta\eta} = 0 \quad \text{at} \quad x + y = a
\end{aligned}
\tag{7}
$$

where

$$\partial/\partial\eta = 1/\sqrt{2}\,(\partial/\partial x + \partial/\partial y) \tag{8}$$

Compatible with the above boundary conditions u, v and w are chosen in the forms[8]

$$u = \sum_{k=1,3,\ldots}^{\infty} B_k \sin k\pi x/a\,(\cos k\pi y/a + \sin k\pi x/a - k\pi/4)\,H(t) \tag{9}$$

$$v = \sum_{k=1,3,\ldots}^{\infty} B_k \sin k\pi y/a\,(\cos k\pi x/a - \sin k\pi y/a + k\pi/4)\,G(t) \tag{10}$$

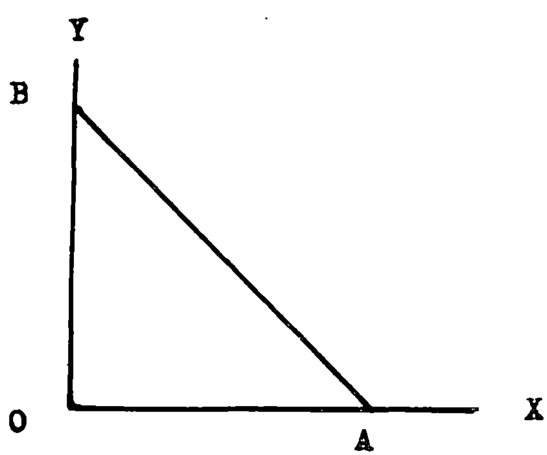

FIG. 1 Right-angled isosceles triangular plate of sides $OA = OB = a$; equation of the side AB is $X + Y = a$.

$$w = \sum_{m=1,3,\dots}^{\infty} A_m (\sin 2m\pi x/a \sin m\pi y/a + \sin m\pi x/a \sin 2m\pi y/a)\ F(t) \quad (11)$$

Combining equations (6) and (11) one gets

$$\frac{25\,D\,m^4\,\pi^4}{a^4}\,F(t) - 5\,m^2\,\pi^2\,K^2\,F(t)\,/\,a^2 = -\,\rho h\,F_{,tt} \quad (12)$$

Integrating now equation (2) over the area of the plate and eliminating K^2 with the help of equation (12) one gets the non-linear time-differential equation as

$$F_{,tt}(t) + C_1\,F(t) + C_2\,F^3(t) = 0 \quad (13)$$

where

$$C_1 = 25\,m^4\,\pi^4\,D\,(1 - \frac{a^2\,N_T^*}{5\,(1-v)\,D\,m^2\,\pi^2})\,/\,a^4\,\rho h \quad (14)$$

$$C_2 = \frac{75\,m^2\,\pi^4\,D}{a^4\,\rho\,h} \sum_{m=1,3,\dots}^{\infty} m^2\,(A_m/h)^2 \quad (15)$$

and

$$N_T^* = 1/A \iint N_T\,dx\,dy$$

is the mean value of N_T over the area A of the plate.

The solution of equation (13) with the initial conditions

$$F(0) = 1, \qquad dF(0)/dt = 0 \quad (16)$$

has been given by Nash and Modeer[9] in terms of Jacobian elliptic function of cosine type and obtained the ratio of the time-periods for linear and non-linear vibrations of elastic plates. In the present case such ratio is given by

$$T^*/T = (\frac{2\Theta}{\pi}) \cdot (1 + \frac{C_2}{C_1})^{-1/2} \quad (17)$$

where

$$C_2 / C_1 = \frac{3 \sum_{m=1,3,\dots}^{\infty} m^2 (A_m / h)^2}{m^2 [1 - a^2 N_T^* / 5(1-v) D m^2 \pi^2]} \tag{18}$$

and T and T^* denote the periods for linear and non-linear vibrations.

For free fundamental mode of vibrations without thermal loading equation (17) reduces to the form

$$T^*/T = \frac{2\,\Theta}{\pi} \frac{1}{\sqrt{1 + 3\,(A/h)^2}} \tag{19}$$

as obtained by Banerjee[8].

3.2 Buckling criterion

For the pre-buckling state non-dimensional time-period T^*/T can be obtained from equation (17) by taking values of

$$a^2 N_T^* / 5\, \pi^2 (1 - v)\, D = \lambda \text{ (say)}$$

sufficiently near to unity. Buckling occurs when $\lambda = 1$, and the critical buckling temperature $(N_T^*)_{cr}$ is obtained as

$$(N_T^*)_{cr} = 5\, \pi^2 D (1 - v) / a^2$$

which is in agreeement with the result obtained by Banerjee[10]

3.3 Simply-supported equilateral triangular plate

Analysis of this section shall be carried out with the help of trilinear co-ordinates[11]. Let ABC be an equilateral triangle of sides '$2a$'. The centroid O on the undeflected middle surface is taken as the origin and the x and y axes are taken perpendicular and parallel to the side BC. If p_1, p_2, p_3 be the lengths of perpendiculars from any point (x, y) within the triangle on the sides CA, AB and BC respectively and r, the radius of the inscribed circle (fig. 2), then

$$p_1 = r + x/2 - \sqrt{3}/2\, y, \quad p_2 = r + x/2 + \sqrt{3}/2\, y, \quad p_3 = r - x \tag{21}$$

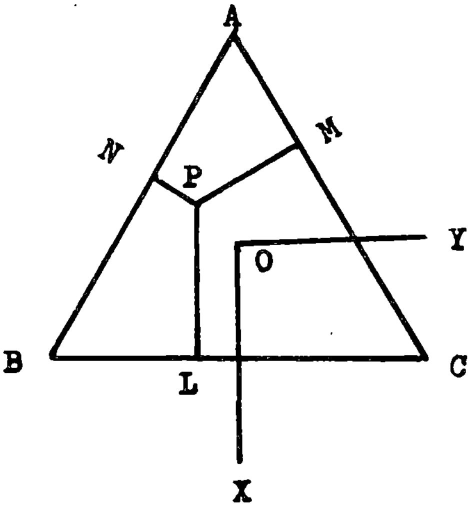

FIG. 2. Equilateral triangular plate of sides $2a$.

Hence $p_1 + p_2 + p_3 = 3r = \sqrt{3}\,a = k$ (say) (22)

Two-dimensional Laplacian operator shall be obtained as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{\partial^2}{\partial p_3^2} - \frac{\partial^2}{\partial p_2\,\partial p_3} - \frac{\partial^2}{\partial p_3\,\partial p_1} - \frac{\partial^2}{\partial p_1\,\partial p_2} \quad (23)$$

The transverse displacement w satisfying the simply-supported boundary conditions

$$w = \nabla^2 w = 0 \text{ at } p_1 = p_2 = p_3 = 0$$

is assumed in the form

$$w = \sum_{m=1,2,\ldots}^{\infty} A_m \left(\sin \frac{2m\pi p_1}{k} + \sin \frac{2m\pi p_2}{k} + \sin \frac{2m\pi p_3}{k} \right) F(t) \quad (24)$$

Also the following forms of u and v

$$u = \sum_{m=1}^{\infty} \sqrt{3}\, B_m \left[\sin \frac{2m\pi (p_2 + p_3)}{k} + \sin \frac{2m\pi (p_1 + p_3)}{k} \right] H(t) \quad (25)$$

$$v = \sum_{m=1}^{\infty} B_m \left[\sin \frac{2m\pi(p_1+p_3)}{k} - \sin \frac{2m\pi(p_2+p_3)}{k} \right] G(t) \tag{26}$$

can be chosen in conformity with the boundary conditions

$$\begin{aligned} u &= 0 \quad \text{at} \quad p_3 = 0 \\ \sqrt{3}\, v + u &= 0 \quad \text{at} \quad p_2 = 0 \\ \sqrt{3}\, v - u &= 0 \quad \text{at} \quad p_1 = 0 \end{aligned} \tag{27}$$

Proceeding in the same way as laid down in the preceding section one arrives at the same type of differential equation (13) where

$$C_1 = \frac{D}{a^4 \rho h} \frac{16 m^4 \pi^4}{9 a^4} \left[1 - \frac{3 a^2 N_T^*}{4(1-v) D m^2 \pi^2} \right] \tag{28}$$

$$C_2 = \frac{D}{a^4 \rho h} \frac{16 m^2 \pi^4}{a^4} \sum_{m=1}^{\infty} m^2 (A_m / h)^2 \tag{29}$$

Non-dimensional time-periods T^* / T is given by the same equation (17) where C_1 and C_2 are to be replaced by equations (28) and (29)

For free fundamental mode of vibrations without thermal loading one gets

$$T^* / T = \frac{2\Theta}{\pi} \frac{1}{\sqrt{1 + 9(A/h)^2}} \tag{30}$$

as obtained by Kármakar[12]

As in the previous case critical buckling temperature $(N_T^*)_{cr}$ is obtained in the form

$$(N_T^*)_{cr} = \frac{4(1-v) D \pi^2}{3 a^2} \tag{31}$$

as obtained by Datta[13].

4. *Numerical results and discussion*

Figure 3 shows the variations of non-dimensional time-periods T^* / T for different values of non-dimensional amplitudes A_1 / h and temperature parameter λ. It is seen that the effect of

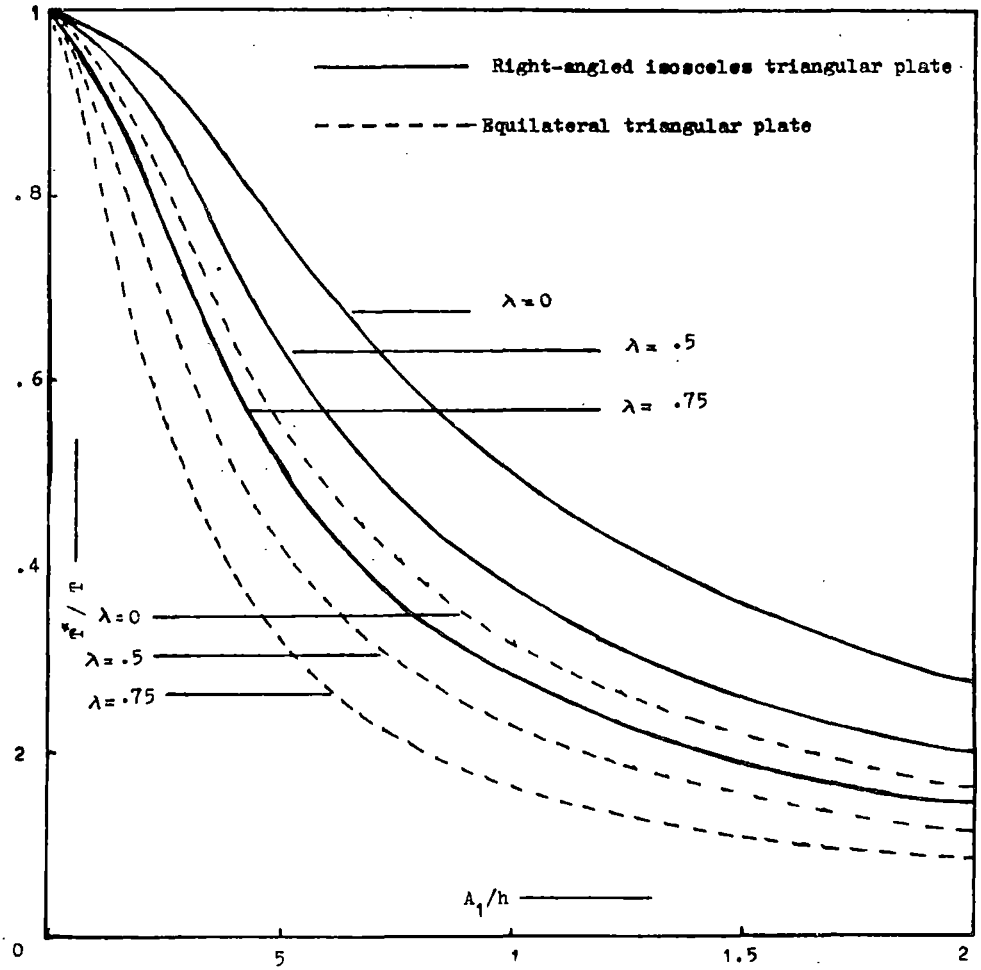

FIG. 3 Variations of non-dimensional time-periods T^* / T *vs* non-dimensional amplitudes A_1 / h for the fundamental mode of vibrations ($m = 1$) for different values of temperature parameter λ.

N_T^* is to diminish the non-dimensional time-periods. Also the circular frequency is given by the expression $\omega_0 = \sqrt{C_1}$ and equations (14) and (31) show that the circular frequency in each case diminishes due to the presence of (N_T^*). It is seen from fig. 3 that the non-dimensional time-periods are less for corresponding non-dimensional amplitudes in the cases of plates of more regular shapes. As it should be, the non-linear behaviour of the plates due to elevated temperature obtained here, is similar in nature as that of plates subjected to in-plane forces given in Biswas[14].

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NONLINEAR FREE VIBRATIONS AND THERMAL BUCKLING OF POLYGONAL PLATES AT ELEVATED TEMPERATURE BY CONFORMAL TRANSFORMATION

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ABSTRACT

von Karman's coupled nonlinear equations extended to thermal loading for the dynamic case have been transformed into the complex domain and with the help of conformal mapping and Galerkin procedure the vibrational characteristics of different regular polygonal plates have been investigated for clamped immovable edges. Critical buckling temperatures for such plates have been deduced as limiting cases and compared with available results.

INTRODUCTION

PROBLEMS of nonlinear vibrations of heated elastic plates frequently occur in different engineering fields—particularly in aeronautics and high-speed space vehicles. Several authors[1-4] have investigated nonlinear vibration problems of specific plate-shapes. The present paper is devoted to the analysis of nonlinear free vibrations of regular polygonal plates of different shapes using the von Karman's coupled equations in the dynamical case under thermal loading and transformed into complex co-ordinates. Conformal mapping and Galerkin procedures have been adopted throughout the analysis.

BASIC GOVERNING EQUATIONS

With notations as in Chia[5] von Kàrman's dynamical equations including the thermal effect can be expressed as

$$D\nabla^4 W+\rho h W_{,tt}+\frac{\alpha_t E}{1-\nu}\nabla^2 M_T= = W_{,xx}\phi_{,yy}-2W_{,xy}\phi_{,xy}+W_{,yy}\phi_{,xx}, \quad (1)$$

$$\nabla^4\phi = Eh[W_{,xy}^2 - W_{,xx}W_{,yy}] - \alpha_t E\nabla^2 N_T. \quad (2)$$

TRANSFORMATION INTO COMPLEX CO-ORDINATES

The time variable is separated with the substitution $W = w(x, y)F(t)$ in (1) and (2) which are transformed into complex co-ordinates $(z, \bar{z})$ where $z = x + iy$, and then the domain is mapped onto a unit circle by the mapping function $z = f(\xi)$. The above two equations finally reduce to the forms

$$\begin{aligned}
&16\,D\,F(t)\Bigg[\frac{\partial^4 w}{\partial\xi^2\partial\bar{\xi}^2}\frac{dz}{d\xi}\frac{dz}{d\bar{\xi}}-\frac{\partial^3\omega}{\partial\xi^2\partial\bar{\xi}}\cdot\frac{dz}{d\xi}\frac{d^2\bar{z}}{d\bar{\xi}^2}\\
&-\frac{\partial^3\omega}{\partial\xi\partial\bar{\xi}^2}\cdot\frac{d\bar{z}}{d\bar{\xi}}\frac{d^2 z}{d\xi^2}+\frac{\partial^2\omega}{\partial\xi\,\partial\bar{\xi}}\frac{d^2 z}{d\xi^2}\frac{d^2\bar{z}}{d\bar{\xi}^2}\Bigg]\\
&+\rho h\omega(\xi,\bar{\xi})\left(\frac{dz}{d\xi}\frac{d\bar{z}}{d\bar{\xi}}\right)^3\ddot{F}(t)=4F(t)\Bigg[\Bigg(2\frac{\partial^2\omega}{\partial\xi\,\partial\bar{\xi}}\frac{\partial^2\phi}{\partial\xi\,\partial\bar{\xi}}\\
&-\frac{\partial^2\omega}{\partial\xi^2}\frac{\partial^2\phi}{\partial\bar{\xi}^2}-\frac{\partial^2\omega}{\partial\bar{\xi}^2}\frac{\partial^2\phi}{\partial\xi^2}\Bigg)\frac{dz}{d\xi}\frac{d\bar{z}}{d\bar{\xi}}\\
&+\left(\frac{\partial^2\omega}{\partial\bar{\xi}^2}\frac{\partial\phi}{\partial\xi}+\frac{\partial^2\phi}{\partial\bar{\xi}^2}\frac{\partial\omega}{\partial\xi}\right)\frac{d^2 z}{d\xi^2}\frac{d\bar{z}}{d\bar{\xi}}\\
&+\left(\frac{\partial^2\omega}{\partial\xi^2}\frac{\partial\phi}{\partial\xi}+\frac{\partial^2\phi}{\partial\xi^2}\frac{\partial\omega}{\partial\bar{\xi}}\right)\frac{d^2\bar{z}}{d\bar{\xi}^2}\frac{dz}{d\xi}\\
&-\left(\frac{\partial\omega}{\partial\xi}\frac{\partial\phi}{\partial\bar{\xi}}+\frac{\partial\omega}{\partial\bar{\xi}}\frac{\partial\phi}{\partial\xi}\right)\frac{d^2 z}{d\xi^2}\frac{d^2\bar{z}}{d\bar{\xi}^2}\Bigg]\\
&-\frac{\alpha_t E}{(1-\nu)}\frac{\partial^2 M_T}{\partial\xi\,\partial\bar{\xi}}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2
\end{aligned} \quad (3)$$

$$\begin{aligned}
&4\Bigg[\frac{\partial^4\phi}{\partial\xi^2\partial\bar{\xi}^2}\frac{dz}{d\xi}\frac{d\bar{z}}{d\bar{\xi}}-\frac{\partial^3\phi}{\partial\xi^2\partial\bar{\xi}}\frac{d^2 z}{d\xi^2}\frac{d\bar{z}}{d\bar{\xi}}\\
&-\frac{\partial^3\phi}{\partial\xi\,\partial\bar{\xi}^2}\frac{d^2\bar{z}}{d\bar{\xi}^2}\frac{dz}{d\xi}+\frac{\partial^2\phi}{\partial\xi\,\partial\bar{\xi}}\frac{d^2 z}{d\xi^2}\frac{d^2\bar{z}}{d\bar{\xi}^2}\Bigg]\\
&=EhF^2(t)\Bigg[\left(\frac{\partial^2\omega}{\partial\xi^2}\frac{\partial^2\omega}{\partial\bar{\xi}^2}-\left(\frac{\partial^2\omega}{\partial\xi\,\partial\bar{\xi}}\right)^2\right)\frac{dz}{d\xi}\frac{d\bar{z}}{d\bar{\xi}}\\
&-\frac{\partial^2\omega}{\partial\xi^2}\frac{\partial\omega}{\partial\bar{\xi}}\frac{d^2\bar{z}}{d\bar{\xi}^2}\frac{dz}{d\xi}-\frac{\partial^2\omega}{\partial\bar{\xi}^2}\frac{\partial\omega}{\partial\xi}\frac{d^2 z}{d\xi^2}\frac{d\bar{z}}{d\xi}\Bigg]\\
&-\alpha_t E\frac{\partial^2 N_T}{\partial\xi\,\partial\bar{\xi}}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2
\end{aligned} \quad (4)$$

FREE VIBRATIONS UNDER THERMAL LOADING

For free vibrations it is not exactly true that $M_{T=0}$; it is an assumption based on the neglect of temperature variation in depth due to compression – nor does it follow from Majumder *et al*[6] who considered $M_T = 0$. For free thermal vibrations the temperature field should be taken to depend on the radial co-ordinate as considered by Buckens[7] and Biswas[8–9] for vibrations of thermally-stressed plates.

METHOD OF SOLUTION AND BOUNDARY CONDITIONS

For plates with immovable edges and clamped along the boundary the appropriate form of $w(\xi, \bar{\xi})$ should be

$$w(\xi, \bar{\xi}) = w_0 (1 - \xi\bar{\xi})^2, \ \xi = r \exp(i\theta) \tag{5}$$

Since N_T is constant and appears in the boundary condition for in-plane displacement we can take $\nabla^2 N_T = 0$, and considering only one term of the mapping function, namely $z = a\delta\xi$ where δ is the mapping function co-efficient and a is the characteristic dimension of the plate, the solution of (4) is expressed in the form

$$\phi = A\xi\bar{\xi} + Eh\, w_0^2 F^2(t)\left[-\frac{\xi\bar{\xi}}{4} + \frac{(\xi\bar{\xi})^3}{9} - \frac{(\xi\bar{\xi})^4}{48}\right], \tag{6}$$

in which A is a constant determined from the condition for inplane displacement for immovable edge of the plate in the form

$$A = \frac{Eh\, w_0^2 F^2(t)(5-3\nu)}{12(1-\nu)} - \frac{E\alpha_t a^2 N_T}{2(1-\nu)}. \tag{7}$$

Again retaining only one term of the mapping function and inserting the expressions for ϕ and A into (3) one gets the error function. Applying Galerkin procedure one gets

$$\frac{d^2F(t)}{dt^2} + C_1 F(t) + C_2 F^3(t) = 0 \tag{8}$$

where

$$C_1 = \frac{D}{a^4 \rho h \delta^4}\left[\frac{320}{3} - \frac{20}{3}\cdot\frac{\alpha_t E a^2 \delta^2 N_T}{(1-\nu)D}\right] \tag{9}$$

$$C_2 = \frac{D}{a^4 \rho h \delta^4}(53.62)(w_0/h)^2. \tag{10}$$

The solution of (8) is given by Nash and Moderer[11] and one gets

$$T^*/T = \frac{2\Theta}{\pi}(1 + C_2/C_1)^{-1/2} \tag{11}$$

where

$$C_2/C_1 = 0.5026875\,(w_0/h)^2 / \left(1 - \frac{\delta^2 N_N^*}{16}\right) \tag{12}$$

$$N_T^* = \alpha_t E a^2 N_T / D(1-\nu) \tag{13}$$

Table 1 *Critical buckling temperature for polygonal plates.*

Plate shape	Value of δ	Critical buckling temperature
Equilateral triangle	1.353	8.74
Square	1.08	13.42
Pentagon	1.0526	14.44
Hexagon	1.0376	14.86
Circle	1.000	16.00

Table 2 *Variations of non-dimensional time-periods for different values of non-dimensional amplitudes and thermal loading parameter.*

	w_0/h	0	0.5	1	1.5	2
T^*/T	Equilateral triangular plate $N_T^* = 0.5$	1	0.93935	0.8076	0.6742	0.5600
T^*/T	Square plate $N_T^* = 0.5$	1	0.9404	0.81065	0.6802	0.5648
T^*/T	Circular plate $N_T^* = 0.5$	1	0.9412	0.8124	0.6801	0.5703
T^*/T	Circular plate $N_T^* = 0$	1	0.9430	0.8161	0.6852	0.5763 [P.S.]
		1	0.9436	0.8165	0.68599	0.5773 [Ref 13]

P.S.—Present study.

CRITICAL BUCKLING TEMPERATURE, RESULTS AND DISCUSSION

For the pre-buckling state non-dimensional time-periods T^*/T can be obtained from (11) and (12) by taking the values of $(\delta^2/16)N_T^*$ sufficiently near to unity. Buckling occurs when $\delta^2 N_T^*/16$ equals to unity and the critical buckling temperature $(N_T^*)_{cr}$ for polygonal plates can be expressed as

$$(N_T^*)_{cr} = 16/\delta^2. \quad (14)$$

It is observed from table 1 that critical buckling temperature of polygonal plates increases as the number of sides increases.

Variations of non-dimensional time-periods T^*/T for variations of non-dimensional amplitudes (w_0/h) and temperature parameter N_T^* have been presented in table 2. It is observed that the values of T^*/T are less for plates with thermal effect than for those without thermal effect, *i.e.*, the effect of N_T^* is to diminish the relative time-periods. Moreover, the nature of the effect of N_T^* on the relative time-periods is similar to that of plates subjected to in-plane compressive stresses discussed by Biswas[12].

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SHORT COMMUNICATIONS

NONLINEAR VIBRATIONS OF PARABOLIC PLATES AT ELEVATED TEMPERATURE

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STUDIES on nonlinear vibrations of thermally-stressed elastic plates are very few when compared with those without thermal effect. In aerospace engineering and in vibrations of machine parts, the problems have to be treated with nonlinear theory, when complementary stresses in the middle plane of the plate are taken into account. As a follow-up of an earlier paper[1] on nonlinear vibration analysis of triangular plates at elevated temperature, the present study is analyzed with Berger's approximation[2].

Governing equations:
Free thermal vibrations of heated elastic plates are governed by the following equations[1]

$$D\nabla^4 w + K^2\nabla^2 w + \rho h w_{,tt} = 0, \tag{1}$$

$$\frac{N_T}{1-\nu} - \frac{12De_1}{h^2} = K^2, \tag{2}$$

where

$$e_1 = u_{,x} + v_{,y} + \tfrac{1}{2}w_{,x}^2 + \tfrac{1}{2}w_{,y}^2 \tag{3}$$

$$N_T = \alpha_t E \int_{-h/2}^{+h/2} T(x,y,z)\,dz, \tag{4}$$

and u, v, w are displacement components, α_t the coefficient of thermal expansion, ρ the density per unit mass, ν the Poisson's ratio, E the Young's modulus and $T(x,y,z)$ is the temperature distribution within the plate given by[3]

$$T(x,y,z) = \tau_0(x,y) + z\tau(x,y), \tag{5}$$

in which $\tau_0(x,y)$ and $\tau(x,y)$ satisfy certain temperature distribution differential equations[3] and K^2 is independent of x and y but involves time t.

Method of solution for a parabolic plate

We consider a parabolic plate with boundary given by

$$x^2 = \frac{a}{2}(2a-y), \quad y = 0. \tag{6}$$

For this plate-shape clamped along the boundary the deflection w is expressed in the form

$$w = \frac{Ay^2}{a^6}\left[\frac{a}{2}(2a-y) - x^2\right]^2 F(t). \tag{7}$$

Combining (1) and (4) and applying Galerkin procedure one gets

$$\int_{y=0}^{2a}\int_{x=-[(a/2)(2a-y)]^{1/2}}^{[(a/2)(2a-y)]^{1/2}} \left[D\left\{\frac{24}{a^6}y^2 AF(t) + \frac{2A}{a^6}(24x^2 - 5a^2 + 12ay)F(t)\right\} + K^2(2a^4 - 6a^3y - a^2y^2 + 2ay^3 + 12x^2y^2 - 4a^2x^2 + 6ax^2y + 2x^4) \frac{AF(t)}{a^6} + \frac{A}{a^6}\rho h y^2\left\{\frac{a}{2}(2a-y) - x^2\right\}^2 \ddot{F}(t)\right] \times \left\{y^2\left[\frac{a}{2}(2a-y) - x^2\right]^2\right\} dx\,dy = 0. \tag{8}$$

Performing the necessary integrations we arrive at the equation

$$F(t)[6.694\,D - 2.0231\,a^2K^2] + 0.0288\,a^4\rho h A\ddot{F}(t) = 0, \tag{9}$$

in which K^2 is still unknown which is obtained by integrating (2) over the area of the plate leading to

$$\iint \frac{N_T}{1-\nu}\,dx\,dy - \frac{6D}{h^2}\iint\left\{(\partial\omega/\partial x)^2 + (\partial\omega/\partial y)^2\right\}dx\,dy = K^2\iint dx\,dy \tag{10}$$

with limits of integrations as in (8). Since u and v vanish on the boundary of the plate clamped along the immovable edges, (10) ultimately leads to

$$\frac{1}{a^2}\iint\frac{N_T}{(1-\nu)D}\,dx\,dy - \frac{7.13358\,A^2F^2(t)}{h^2} = 2.66\,K^2/D. \tag{11}$$

Eliminating K^2 with the help of (9) and (11) one gets the well-known cubic equation in the form

$$\ddot{F}(t) + C_1F(t) + C_2F^3(t) = 0, \tag{12}$$

where

$$C_1 \doteq D(232{\cdot}2 - 26{\cdot}29\, N_T^*)/a^4 \rho h, \tag{13}$$

$$C_2 = 187{\cdot}57\, D(A/h)^2/a^4 \rho h, \tag{14}$$

$$N_T^* = \frac{1}{a^2} \iint \frac{N_T}{D(1-\nu)}\, \mathrm{d}x\, \mathrm{d}y. \tag{15}$$

The solution of (12) with the initial conditions $F(O) = 1$, $\mathrm{d}F(0)/\mathrm{d}t = 0$ has been given by Nash and Modeer[4] in terms of Jacobian elliptic functions of cosine type and obtained the ratio of the time-periods for nonlinear and linear vibrations of elastic plates. In the present case such ratio is given by

$$T^*/T = \frac{2\Theta}{\pi}(1 + C_2/C_1)^{-1/2} \tag{16}$$

in which T and T^* denote the time-periods for linear and nonlinear vibrations.

Numerical results and discussion: Variations of non-dimensional time-periods T^*/T for different values of non-dimensional amplitudes A/h and temperature parameter N_T^* have been computed and presented graphically. It is seen that the effect of increasing N_T^* is to diminish the relative time-periods. As expected, the nonlinear behaviour of plates due to elevated temperature, obtained here, is similar in nature to that of the plates subjected to in-plane compressive forces investigated by Biswas[5].

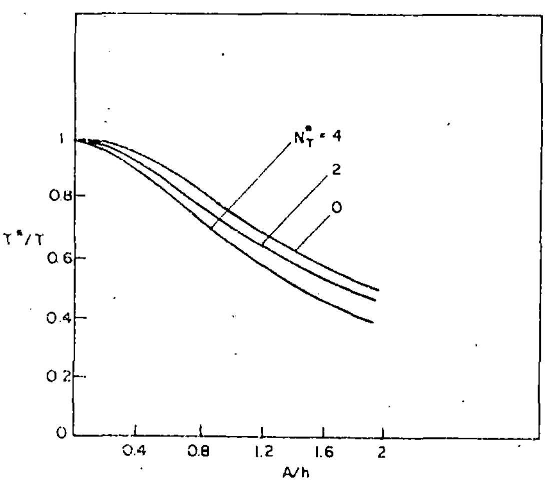

Figure 1. Variations of non-dimensional time-periods T^*/T for different values of non-dimensional amplitudes A/h and thermal loading parameter N_T^*.

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TECHNICAL NOTE

NON-LINEAR VIBRATIONS AND THERMAL BUCKLING OF ELASTIC PLATES AT ELEVATED TEMPERATURES

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Introduction

Thin plates of different shapes frquently occur in many structures and the study of the bending properties of plates is imperative to a design engineer. With the increased use of strong and lightweight structures, especially in aero-space engineering, and in the study of vibrations of machine parts many problems of non-linear deformations naturally arise where supplementary stresses in the middle plane of the plate must be taken into account in deriving the governing field equations of the plates.

Conventional governing field equations for such large elastic deformations of plates are non-linear as well as coupled and hence they are not easily solvable. It is, therefore, not inappropriate to seek another set of governing field equations for dealing with non-linear problems. Although Berger's approximate method[1] serves such purpose, as the equations are in decoupled forms, yet due to some inaccuracies and inconsistencies of this method pointed out by several authors[2-4], some other alternative method is required.

Datta and Banerjee[5] suggested a modified energy expression by bringing directly the expression for σ_{rr} in the total potential energy of the system and derived a new set of non-linear differential equations, also in the decoupled form, for the analysis of static deflections of elastic plates. They have observed accuracy of results for a circular plate with both movable and immovable edge conditions. Banerjee[6] has generalised the method for orthotropic plates and obtained satisfactory results.

In this paper generalised field equations in the dynamical case for a heated elastic plate have been derived by applying Euler's variational equations on the modified energy expression for bending and stretching of the plate under normal thermal loading. The solution of these equations have been obtained for a circular plate with both clamped movable and immovable boundary conditions. Critical buckling temperatures for the two cases have been deduced as limiting cases. Some numerical results have been presented showing variations of the ratio of time-periods for non-linear and linear vibrations for different variations of non-dimensional amplitude and temperature parameter.

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Derivation of Field Equations

For a circular plate of radius a and thickness h the sum of the membrane and bending energies undergoing large deflections in the absence of any external load may be expressed in the form

$$V = \frac{D}{2}\int_0^a \left[w_{,rr}^2 + \frac{2\sigma}{r} w_{,r}\, w_{,rr} + \frac{1}{r^2} w_{,r}^2 + \frac{12}{h^2}\{e_1^2 + 2(\sigma-1)e_2\}\right] r\, dr \qquad (1)$$

where D is the flexural rigidity of the plate given by $D=Eh^3/12(1-\sigma^2)$, w, the central deflection, σ, the Poisson's ratio, E, the Young's modulus, h, plate thickness and e_1, e_2 are the first and second strain invariants of the middle surface respectively given by

$$e_1 = u_{,r} + \frac{1}{2} w_{,r}^2 + \frac{u}{r}, \quad e_2 = \frac{u}{r}(u_{,r} + \frac{1}{2} w_{,r}^2) \qquad (2)$$

where u is the in-plane displacement.

The foregoing expression for V can be rearranged as

$$V = \int_0^a \frac{D}{2}\left[w_{,rr}^2 + \frac{2\sigma}{r} w_{,r}\, w_{,rr} + (\frac{1}{r} w_{,r})^2 + \frac{12}{h^2}\{\bar{e}_1^2 + (1-\sigma^2)\frac{u^2}{r^2}\}\right] r\, dr \qquad (3)$$

where

$$\bar{e}_1 = u_{,r} + \frac{1}{2} w_{,r}^2 + \frac{u}{r} = \frac{1-\sigma^2}{E}\sigma_{rr} \qquad (4)$$

and σ_{rr} is the meridian stress which has been included in (3). The kinetic energy T of the plate is given by

$$T = \frac{\rho_h}{2}\iint (u_{,t}^2 + w_{,t}^2)\, r\, dr\, dt \qquad (5)$$

where ρ is the density of the plate material.

The strain energy SE_T due to heating is given by[7]

$$SE_T = \int_0^a \int_{-h/2}^{h/2} \frac{E\alpha_t\theta}{1-\sigma} (\bar{e}_1 - z\nabla^2 w)\, r\, dr\, dz \qquad (6)$$

where α_t is the co-efficient of thermal expansion, $\theta(r,z)$, temperature distribution in the plate and ∇^2, Laplacian operator.

If the term $(1-\sigma^2)u^2/r^2$ in equation (3) is replaced $(\lambda/4)(w,_r)^4$ [Datta and Banerjee, Ref.5] where λ is a factor depending on the Poisson's ratio of the plate material, then decoupling will be possibie. By doing so and applying Hamilton's principle to the Lagragian $L=(T-SE_T-V)$ and using Euler's variational equations one gets the following equations

$$D\left[\nabla^4 w - \frac{6\lambda}{h^2} w,_r^2 (\nabla^2 w + 2w,_{rr})\right] -$$

$$Cf(t)r^{\sigma-1}(w,_{rr} + \frac{\sigma}{r} w,_r) + \frac{\nabla^2 M_T}{1-\sigma} + \rho h\, w,_{tt} = 0 \qquad (7)$$

$$\frac{12\,D}{h^2}\bar{e}_1 = Cf(t)\, r^{\sigma-1} + \frac{N_T}{1-\sigma} \qquad (8)$$

where C is a constant and f(t) is some function of time, and

$$M_T = \int_{-h/2}^{-h/2} \alpha_t E \cdot z\theta(r,z)\, dz,$$

$$N_T = \int_{-h/2}^{h/2} \alpha_t E \cdot \theta(r,z)\, dz \qquad (9)$$

Method of Solution

In the present analysis the membrane stress due to thermal loading is considered and so the temperature distribution is a function of the radial co-ordinate r only[9]

$$\theta(r,z) = \tau_0(r) \quad \text{so that } M_T = 0$$

For clamped and simply-supported boundary conditions for a circular plate

$$w = 0 = w,_r \qquad \text{at } r = a \qquad (10)$$

$$w = 0 = \frac{\sigma}{r} w,_r + w,_{rr} \qquad \text{at } r = a \qquad (11)$$

For movable edge condition we have C=0 and for immovable edges the condition is u=0 at r=a.

If now the problem is restricted to finding the fundamental mode of vibration only, then the form of w satisfying the above conditions can be taken as

$$w = A\left[1 - 2P\frac{r^2}{a^2} + Q\frac{r^4}{a^4}\right] F(t) \qquad (12)$$

For clamped plates P=Q=1, and for simply-supported edges

$$P = \frac{3+\sigma}{5+\sigma}, \qquad Q = \frac{1+\sigma}{5+\sigma}$$

Substituting the above expression for w in equation (7) one gets the error function $\varepsilon(r,t)$ which does not vanish in general, since the expression for w is not the exact solution for equation (7). Galerkin procedure requires that the error function be orthogonal over the domain of the plate, i.e.

$$\iint_S \varepsilon(r,t)\, ds = 0 \qquad (13)$$

where S is the surface area of the plate. Equation (13) can be integrated after a lengthy but simple calculation.

The term C f(t) involving the constant C can be determined from equation (8) by integrating over the area of the plate after inserting the expression for w. Terms involving the in-plane displacement u can be eliminated by considering suitable expression compatible with the boundaryconditions. Equation (13) finally leads to the non-linear time-equation

$$C_1 F(t),_{tt} + C_2 F(t) + C_3 F^3(t) = 0 \qquad (14)$$

which can be solved in terms of Jacobian elliptic functions[9].

Clamped Circular Plate – Immovable Edge

For a clamped circular plate with immovable edge equation (13) simplifies into

$$\frac{1}{10} a^4 \rho h F(t),_{tt} + D\left[\frac{32}{3} F(t) + 7.314\left(\frac{A}{h}\right)^2 F^3(t)\right] +$$

$$+ 1.08\, C f(t)\, a^{\sigma-1} F(t) = 0 \qquad (15)$$

Equation (8) can be integrated giving the term C f(t) from the equation

$$4D(A/h)^2 F^2(t) = C f(t)\, a^{\sigma+1}/\sigma+1 + a^2 N_T/2(1-\sigma) \qquad (16)$$

Eliminating Cf(t) from equations (15) and (16) one gets

$$F(t)_{,tt} + \alpha F(t) + \beta F^3(t) = 0 \tag{17}$$

where

$$\alpha = \frac{10D}{a^4\rho h}\left\{\frac{32}{3} - 0.54(1+\sigma)a^2N_T/D(1-\sigma)\right\}$$

$$\beta = [4.32(1+\sigma) + 7.314\lambda](A/h)^2 \tag{18}$$

The solution of equation (17) with the normalised conditions

$$F(0) = 1, \quad F_{,t}(0) = 0$$

is given by

$$F(t) = Cn(\omega^* t, R) \tag{19}$$

where

$$\omega^{*2} = \alpha+\beta, \quad R^2 = \beta/2(\alpha+\beta)$$

Cn being the cosine type of Jacobian elliptic function.

The non-linear time-period T^* is given by $T^* = 4K/\omega^*$, where K is the complete elliptic integral of the first kind. The linear time-period T is given by $T = 2\pi/\Omega_0$ where Ω_0 is the circular frequency obtained from equation (17) by dropping the non-linear term so that

$$\Omega_0 = \sqrt{\alpha} \tag{20}$$

Ratio of the time-periods T^*/T is given by

$$T^*/T = \frac{2K}{\pi}\left(1 + \frac{\beta}{\alpha}\right)^{-1/2}$$

where

$$\frac{\beta}{\alpha} = 0.64(A/h)^2/(1-N_T^*)$$

$$N_T^* = 0.0658\, a^2N_T/(1-\sigma)D \tag{21}$$

and

$$\lambda = \frac{1-\sigma^2}{5} \text{ for clamped plates}^5.$$

Buckling Criterion and Critical Buckling Temperature

For the pre-buckling state ratio of the time-periods can be obtained by considering different values of N_T^* $(0 \leq N_T^* < 1)$. Critical buckling temperature $(N_T)_{cr}$ is obtained when $N_T^* = 1$ and is given by

$$(N_T)_{cr} = 1.52$$

which is in agreement with the result obtained in ref.10.

Plates with Movable Edges

For a circular plate with clamped movable edges C=0 and T^*/T can be obtained in the form given by

$$T^*/T = \frac{2K}{\pi}\left(1 + \frac{\beta}{\alpha}\right)^{-1/2} \tag{22}$$

where

$$\beta/\alpha = 0.1248(A/h)^2 \tag{23}$$

Critical Buckling Temperture for Movable Edge

With C=0 in equation (16) one gets

$$F^2(t) = a^2N_T/8D(1-\sigma)(A/h)^2 \tag{24}$$

Since N_T is time-independent so $F(t)$ must be a constant as seen in equation (24) and equation (15) leads to

$$(N_T)_{cr} = 64.105\, D(1-\sigma)/a^2 \tag{25}$$

Numerical Results and Observations

Table I shows the variations of the ratio of time-periods for non-linear and linear vibrations for different values of non-dimensional amplitudes (A/h) and temperature parameter. It is seen that the effect of temperature is to diminish the relative time-periods. Also it is seen from equation (20) that circular frequency diminishes due to the presence of temperature. Moreover T^*/T is seen to be independent of temperature parameter for plates with movable edge whereas critical buckling temperature for plates with movable edge exceeds four times that for plates with immovable edge.

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Table I

Dependence of T*/T on A/h and N_T^*

A/h →	0	.2	.4	.6	.8	1	
T*/T (N_T^* = 0) →	1	.99	.97	.94	.90	.88	[Ref.9]
Present case:	1	.99	.968	.937	.892	.876	
T*/T (N_T^* = .4)	1	.98	.937	.904	.846	.808	
T*/T (N_T^* = .8)	1	.952	.864	.764	.681	.562	

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