CHAPTER II

Λ-\textit{r}-closed sets and Λ-\textit{r}-open sets

Maki [21] introduced the concept of Λ-sets in topological spaces. Following this, Navaneethakrishnan defined the notion of Λ-\textit{r}-sets. In this chapter, the concepts of Λ-\textit{r}-closed sets and Λ-\textit{r}-open sets are introduced and their basic properties are investigated.

Definition 2.1.

Let $A$ be a subset of a space $X$. Then $A$ is called Λ-\textit{r}-closed if $A = S \cap C$, where $S$ is a Λ-\textit{r}-set and $C$ is a closed set. $A$ is called Λ-\textit{r}-open if $X \setminus A$ is Λ-\textit{r}-closed. $A$ is called Λ-\textit{r}-clopen if $A$ is both Λ-\textit{r}-open and Λ-\textit{r}-closed.

Let \( \Lambda_r O(X, \tau) \) denote the family of all Λ-\textit{r}-open sets in \((X, \tau)\), \( \Lambda_r C(X, \tau) \) denote the family of all Λ-\textit{r}-closed sets in \((X, \tau)\) and \( \Lambda_r CO(X, \tau) \) denote the family of all Λ-\textit{r}-clopen sets in \((X, \tau)\).

Proposition 2.2.

(i) Every Λ-\textit{r}-set is Λ-\textit{r}-closed.

(ii) Every closed set is Λ-\textit{r}-closed.

Proof.

Let $A$ be a subset of a space $X$. Suppose $A$ is a Λ-\textit{r}-set. Since $A = A \cap X$, by Definition 2.1, $A$ is Λ-\textit{r}-closed. This proves (i).

Let $A$ be a closed subset of a space $X$. By Lemma 1.9(ii) and Definition 1.10, it follows that $X$ is a Λ-\textit{r}-set. Since $A = X \cap A$, by Definition 2.1, $A$ is Λ-\textit{r}-closed. This proves (ii).

Proposition 2.3.

For a subset $A$ of a space $X$, the following conditions are equivalent:

(i) $A$ is Λ-\textit{r}-closed.
(ii) \( A = L \cap \text{cl}(A) \) for some \( \Lambda_r \)-set \( L \).

(iii) \( A = A_r^\Lambda \cap \text{cl}(A) \).

**Proof.**

Suppose \( A \) is \( \Lambda_r \)-closed. Then by Definition 2.1, there exists a \( \Lambda_r \)-set \( L \) and a closed set \( C \) such that \( A = L \cap C \). Since \( A \subseteq C \), \( \text{cl}(A) \subseteq \text{cl}(C) = C \). Therefore \( A = L \cap C \supseteq L \cap \text{cl}(A) \supseteq A \). Hence \( A = L \cap \text{cl}(A) \). **This proves (i) \( \Rightarrow \) (ii).**

Suppose \( A = L \cap \text{cl}(A) \) for some \( \Lambda_r \)-set \( L \). Since \( A \subseteq L \), by Lemma 1.9(v), we have \( A_r^\Lambda \subseteq L_r^\Lambda \) and so by Definition 1.10, \( A_r^\Lambda \subseteq L \). Then by Lemma 1.9(iii), \( A = L \cap \text{cl}(A) \supseteq A_r^\Lambda \cap \text{cl}(A) \supseteq A \) and so \( A = A_r^\Lambda \cap \text{cl}(A) \). **This proves (ii) \( \Rightarrow \) (iii).**

Suppose \( A = A_r^\Lambda \cap \text{cl}(A) \). Then by Lemma 1.9(iv), \( (A_r^\Lambda)^\Lambda = A_r^\Lambda \) and so by Definition 1.10, \( A_r^\Lambda \) is a \( \Lambda_r \)-set. Thus \( A \) is an intersection of \( \Lambda_r \)-set \( A_r^\Lambda \) and the closed set \( \text{cl}(A) \) and so by Definition 2.1, \( A \) is \( \Lambda_r \)-closed. **This proves (iii) \( \Rightarrow \) (i).** \( \square \)

**Theorem 2.4.**

Let \( X \) be a space and \( A \subseteq X \). Then \( A \) is \( \Lambda_r \)-open if and only if \( A = T \cup C \) where \( T \) is a \( V_r \)-set and \( C \) is an open set.

**Proof.**

Suppose \( A \) is \( \Lambda_r \)-open. Then by Definition 2.1, \( X \setminus A \) is \( \Lambda_r \)-closed and so there exists a \( \Lambda_r \)-set \( S \) and a closed set \( D \) such that \( X \setminus A = S \cap D \). By Lemma 1.13, \( X \setminus S \) is a \( V_r \)-set. Thus \( A = X \setminus (S \cap D) = (X \setminus S) \cup (X \setminus D) \) is an union of \( V_r \)-set \( X \setminus S \) and the open set \( X \setminus D \).

For the converse, suppose \( A = T \cup C \) where \( T \) is a \( V_r \)-set and \( C \) is an open set. By Lemma 1.13, \( X \setminus T \) is a \( \Lambda_r \)-set. Thus \( X \setminus A = (X \setminus T) \cap (X \setminus C) \) is the intersection of \( \Lambda_r \)-set \( X \setminus T \) and the closed set \( X \setminus C \) and so by Definition 2.1, \( X \setminus A \) is a \( \Lambda_r \)-closed set which implies \( A \) is \( \Lambda_r \)-open. \( \square \)
Proposition 2.5.

(i) Every \( V_r \)-set is \( \Lambda_r \)-open.

(ii) Every open set is \( \Lambda_r \)-open.

Proof.

Let \( A \) be a subset of a space \( X \). Suppose \( A \) is a \( V_r \)-set. Since \( A = A \cup \emptyset \), by Theorem 2.4, \( A \) is \( \Lambda_r \)-open. This proves (i).

Let \( A \) be an open set in \( X \). Then \( X \setminus A \) is closed. By Proposition 2.2(ii), \( X \setminus A \) is \( \Lambda_r \)-closed and so \( A \) is \( \Lambda_r \)-open. This proves (ii). \( \square \)

Proposition 2.6.

For a subset \( A \) of a space \( X \), the following conditions are equivalent:

(i) \( A \) is \( \Lambda_r \)-open.

(ii) \( A = T \cup \text{int}(A) \) for some \( V_r \)-set \( T \).

(iii) \( A = A^V_r \cup \text{int}(A) \).

Proof.

Suppose \( A \) is \( \Lambda_r \)-open. Then by Theorem 2.4, there exists a \( V_r \)-set \( T \) and an open set \( C \) such that \( A = T \cup C \). Since \( C \subseteq A \), \( C = \text{int}(C) \subseteq \text{int}(A) \). Then \( A = T \cup C \subseteq T \cup \text{int}(A) \subseteq A \) which implies that \( A = T \cup \text{int}(A) \). This proves (i) \( \Rightarrow \) (ii).

Suppose \( A = T \cup \text{int}(A) \) for some \( V_r \)-set \( T \). Since \( T \subseteq A \), by Lemma 1.9(v), \( T^V_r \subseteq A^V_r \) and so by Definition 1.10, \( T \subseteq A^V_r \). Then by applying Lemma 1.9(iii), \( A = T \cup \text{int}(A) \subseteq A^V_r \cup \text{int}(A) \subseteq A \) which implies that \( A = A^V_r \cup \text{int}(A) \). This proves (ii) \( \Rightarrow \) (iii).

Suppose \( A = A^V_r \cup \text{int}(A) \). By Lemma 1.9(iv), \( (A^V_r)^V_r = A^V_r \) and so by Definition 1.10, \( A^V_r \) is a \( V_r \)-set. Thus \( A \) is an union of \( V_r \)-set \( A^V_r \) and an open set \( \text{int}(A) \) and so by Theorem 2.4, \( A \) is \( \Lambda_r \)-open. This proves (iii) \( \Rightarrow \) (i). \( \square \)
Theorem 2.7.

(i) Every $\Lambda_r$-closed set is $\lambda$-closed.

(ii) Every $\Lambda_r$-closed set is $(\Lambda,\alpha)$-closed.

(iii) Every $\Lambda_r$-closed set is $\lambda$-semi-closed.

Proof.

Let $S$ be a $\Lambda_r$-closed set. Then by Definition 2.1, there exists a $\Lambda_r$-set $T$ and a closed set $C$ such that $S = T \cap C$. By Lemma 1.11(ii), $T$ is a $\Lambda$-set and by Definition 1.3, it follows that $S$ is $\lambda$-closed. This proves (i).

Let $S$ be a $\Lambda_r$-closed set. Then by Definition 2.1, there exists a $\Lambda_r$-set $T$ and a closed set $C$ such that $S = T \cap C$. Since every regular-open set is $\alpha$-open, by using Definitions 1.10 and 1.6, every $\Lambda_r$-set is a $\Lambda_\alpha$-set and so $T$ is a $\Lambda_\alpha$-set. Since every closed set is $\alpha$-closed, $C$ is a $\alpha$-closed set. Therefore by Definition 1.6, $S$ is $(\Lambda,\alpha)$-closed. This proves (ii).

Let $S$ be a $\Lambda_r$-closed set. Then by Definition 2.1, there exists a $\Lambda_r$-set $T$ and a closed set $C$ such that $S = T \cap C$. Since every regular-open set is semi-open, from Definitions 1.10 and 1.7, every $\Lambda_r$-set is a $\Lambda_s$-set and so $T$ is a $\Lambda_s$-set. Since every closed set is semi-closed, $C$ is a semi-closed set and so by Definition 1.7, $S$ is $\lambda$-semi-closed. This proves (iii).

□

It follows from Theorem 2.7(i) and Proposition 2.5(ii) that the class of $\Lambda_r$-open sets lies between the topology and the class of $\lambda$-open sets. However, the inclusion is proper as shown in the next example.

Example 2.8.

If $X = \{a,b,c,d\}$ and $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$, then

(i) $\{a,b\}$ is a $\lambda$-closed set in $(X,\tau)$ but it is not a $\Lambda_r$-closed in $(X,\tau)$.

(ii) $\{d\}$ is a $(\Lambda,\alpha)$-closed set in $(X,\tau)$ but it is not a $\Lambda_r$-closed in $(X,\tau)$.
(iii) \{a,d\} is a \(\lambda\)-semi-closed set in \((X, \tau)\) but it is not a \(\Lambda_r\)-closed in \((X, \tau)\).

Example 2.9 shows that the concepts of \(\Lambda_r\)-closed sets and \((\Lambda, \delta)\)-closed sets defined in Definition 1.5 are independent to each other.

**Example 2.9.**

If \(X = \{a,b,c,d\}\) and \(\tau = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}\), then \{b,c,d\} is a \(\Lambda_r\)-closed set in \((X, \tau)\) but it is not a \((\Lambda, \delta)\)-closed set in \((X, \tau)\).

If \(X = \{a,b,c,d\}\) and \(\tau = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}\), then \{a,b,c\} is a \((\Lambda, \delta)\)-closed set in \((X, \tau)\) but it is not a \(\Lambda_r\)-closed set in \((X, \tau)\).

The above discussions lead to the following implications but none of the reverse implications is true.

\[
\begin{align*}
\Lambda_r\text{-set} & \Rightarrow \Lambda\text{-set} \Rightarrow \Lambda\alpha\text{-set} \Rightarrow \Lambda_s\text{-set} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Lambda_r\text{-closed} & \Rightarrow \lambda\text{-closed} \Rightarrow (\Lambda, \alpha)\text{-closed} \Rightarrow \lambda\text{-semi-closed}
\end{align*}
\]

**Definition 2.10.**

Let \(X\) be a space and \(A \subseteq X\). Then a point \(x \in X\) is called a \(\Lambda_r\)-cluster point of \(A\) if for every \(\Lambda_r\)-open set \(U\) containing \(x\), \(A \cap U \neq \emptyset\). The collection of all \(\Lambda_r\)-cluster points of \(A\) is called the \(\Lambda_r\)-closure of \(A\) and is denoted by \(\Lambda_r-cl(A)\).

Let \(X\) be a space and \(A, B\) and \(A_k\) where \(k \in I\), subsets of \(X\). Then we have the following properties.

**Proposition 2.11.**

(i) \(A \subseteq \Lambda_r-cl(A)\).

(ii) \(\Lambda_r-cl(A) = \cap \{F: A \subseteq F\ and\ F\ is\ \Lambda_r\text{-closed}\}\).

(iii) If \(A \subseteq B\), then \(\Lambda_r-cl(A) \subseteq \Lambda_r-cl(B)\).

(iv) \(A\) is \(\Lambda_r\)-closed if and only if \(A = \Lambda_r-cl(A)\).

(v) \(\Lambda_r-cl(A)\) is \(\Lambda_r\)-closed.

(vi) \(\Lambda_r-cl(A) \subseteq cl(A)\).
Proof.

If \( x \notin \Lambda_r-cl(A) \), then by Definition 2.10, \( x \) is not a \( \Lambda_r \)-cluster point of \( A \) and so there exists a \( \Lambda_r \)-open set \( U \) containing \( x \) such that \( A \cap U = \emptyset \) which implies that \( x \notin A \). \textbf{This proves (i).}

If \( x \notin \Lambda_r-cl(A) \), then by Definition 2.10, \( x \) is not a \( \Lambda_r \)-cluster point of \( A \) and so there exists a \( \Lambda_r \)-open set \( U \) containing \( x \) such that \( A \cap U = \emptyset \). Take \( F = X \setminus U \). By Definition 2.1, \( F \) is \( \Lambda_r \)-closed. Then we obtain that \( A \subseteq F \) and \( x \notin F \) which implies that \( x \notin \cap \{ F : A \subseteq F \text{ and } F \text{ is } \Lambda_r \text{-closed} \} \). On the other hand, if \( x \notin \cap \{ F : A \subseteq F \text{ and } F \text{ is } \Lambda_r \text{-closed} \} \), then there exists a \( \Lambda_r \)-closed set \( F \supseteq A \) such that \( x \notin F \) which implies that \( x \in X \setminus F \), \( X \setminus F \) is \( \Lambda_r \)-open and \( (X \setminus F) \cap A = \emptyset \). By Definition 2.10, \( x \) is not a \( \Lambda_r \)-cluster point of \( A \) and so \( x \notin \Lambda_r-cl(A) \). \textbf{This proves (ii).}

If \( x \notin \Lambda_r-cl(B) \), then by Definition 2.10, there exists a \( \Lambda_r \)-open set \( U \) containing \( x \) such that \( B \cap U = \emptyset \). Since \( A \subseteq B \), \( A \cap U = \emptyset \) which implies that \( x \) is not a \( \Lambda_r \)-cluster point of \( A \) and so \( x \notin \Lambda_r-cl(A) \). \textbf{This proves (iii).}

Suppose \( A \) is \( \Lambda_r \)-closed. If \( x \notin A \), then \( x \in X \setminus A \) and \( X \setminus A \) is \( \Lambda_r \)-open. Take \( X \setminus A = U \). Then \( U \) is a \( \Lambda_r \)-open set containing \( x \) and \( A \cap U = \emptyset \) and hence \( x \notin \Lambda_r-cl(A) \). By using (i), it follows that \( A = \Lambda_r-cl(A) \). Conversely, suppose that \( A = \Lambda_r-cl(A) \). By using (ii), we have \( A = \cap \{ F : A \subseteq F \text{ and } F \text{ is } \Lambda_r \text{-closed} \} \) and by using Definition 1.10, it follows that \( A \) is \( \Lambda_r \)-closed. \textbf{This proves (iv).}

By (i) and (iii), we have \( \Lambda_r-cl(A) \subseteq \Lambda_r-cl (\Lambda_r-cl(A)) \). If \( x \in \Lambda_r-cl(\Lambda_r-cl(A)) \), then by Definition 2.10, \( x \) is a \( \Lambda_r \)-cluster point of \( \Lambda_r-cl(A) \) which implies that for every \( \Lambda_r \)-open set \( U \) containing \( x \), \( (\Lambda_r-cl(A)) \cap U \neq \emptyset \). Let \( y \in \Lambda_r-cl(A) \cap U \). Then \( y \) is a \( \Lambda_r \)-cluster point of \( A \). Therefore for every \( \Lambda_r \)-open set \( G \) containing \( y \), \( A \cap G \neq \emptyset \). Now \( U \) is \( \Lambda_r \)-open and \( y \in U \) which implies that \( A \cap U \neq \emptyset \) and so
Thus we have \( \Lambda r-cl(A) = \Lambda r-cl(\Lambda r-cl(A)) \). By (iv), \( \Lambda r-cl(A) \) is \( \Lambda r \)-closed. **This proves (v).**

If \( x \not\in cl(A) \), then there exists a closed set \( F \supseteq A \) such that \( x \not\in F \). By Proposition 2.2(ii), \( F \) is \( \Lambda r \)-closed and so by (ii), \( x \not\in \Lambda r-cl(A) \). **This proves (vi).** □

The following example shows that the reverse inclusion of Proposition 2.11(i) and Proposition 2.11(vi) need not be true.

**Example 2.12.**

If \( X = \{a,b,c\} \) and \( \tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\} \), then we have

(i) \( \Lambda r-cl(\{a,b\}) = X \not\subset \{a,b\} \).

(ii) \( cl(\{a\}) = \{a,c\} \) and \( \Lambda r-cl(\{a\}) = \{a\} \) and so \( cl(\{a\}) \not\subset \Lambda r-cl(\{a\}) \).

**Remark 2.13.**

(i) \( X \) and \( \emptyset \) are both \( \Lambda r \)-open and \( \Lambda r \)-closed.

(ii) By Proposition 2.11(ii) and 2.11(v), \( \Lambda r-cl(A) \) is the smallest \( \Lambda r \)-closed set containing \( A \).

**Proposition 2.14.**

(i) If \( A_k \) is \( \Lambda r \)-closed for each \( k \in I \), then \( \bigcap_{k \in I} A_k \) is \( \Lambda r \)-closed.

(ii) If \( A_k \) is \( \Lambda r \)-open for each \( k \in I \), then \( \bigcup_{k \in I} A_k \) is \( \Lambda r \)-open.

**Proof.**

Suppose that \( A = \bigcap_{k \in I} A_k \) and \( x \in \Lambda r-cl(A) \). Then by Definition 2.10, for every \( \Lambda r \)-open set \( U \) containing \( x \), \( A \cap U \neq \emptyset \) which implies that \( A_k \cap U \neq \emptyset \) for each \( k \in I \). If \( x \not\in A \), then \( x \not\in A_i \) for some \( i \in I \). Since \( A_i \) is \( \Lambda r \)-closed, by Proposition 2.11(iv), \( A_i = \Lambda r-cl(A_i) \) and hence \( x \not\in \Lambda r-cl(A_i) \) which implies that there exists a \( \Lambda r \)-open set \( V \) containing \( x \) such that \( A_i \cap V = \emptyset \). This contradiction shows that \( x \in A \) and hence \( \Lambda r-cl(A) \subseteq A \). By using Proposition 2.11(i), it follows that \( A = \Lambda r-cl(A) \).
and so by Proposition 2.11(iv), A is \( \Lambda_r \)-closed, that is, \( \bigcap_{k \in I} A_k \) is \( \Lambda_r \)-closed. This proves (i).

Suppose \( A_k \) is \( \Lambda_r \)-open for each \( k \in I \). Then by Definition 2.1, \( X \setminus A_k \) is \( \Lambda_r \)-closed for each \( k \in I \). By (i), \( \bigcap_{k \in I} (X \setminus A_k) \) is \( \Lambda_r \)-closed which implies that \( X \setminus (\bigcup_{k \in I} A_k) \) is \( \Lambda_r \)-closed and so \( \bigcup_{k \in I} A_k \) is \( \Lambda_r \)-open. This proves (ii). □

**Remark 2.15.**

Consider a topological space \((X, \tau)\) where \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \). Then the \( \Lambda_r \)-open sets in \((X, \tau)\) are \( \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, X \) and the \( \Lambda_r \)-closed sets in \((X, \tau)\) are \( \{a, c, d\}, \{c, d\}, \{a\}, \{b\}, \emptyset \). This example shows that union of \( \Lambda_r \)-closed sets is not \( \Lambda_r \)-closed and intersection of \( \Lambda_r \)-open sets is not \( \Lambda_r \)-open.

**Definition 2.16.**

Let \( X \) be a space and \( A \subseteq X \). Then \( \Lambda_r \)-kernal of \( A \), denoted by \( \Lambda_r \)-ker\( (A) \) is defined by \( \Lambda_r \)-ker\( (A) = \bigcap \{G : G \in \Lambda_r O(X, \tau) \text{ and } A \subseteq G\} \).

Let \( X \) be a space, \( A \) and \( B \) be subsets of \( X \) and \( x, y \in X \). Then we have the following properties.

**Proposition 2.17.**

(i) \( A \subseteq \Lambda_r \)-ker\( (A) \).

(ii) If \( A \subseteq B \), then \( \Lambda_r \)-ker\( (A) \subseteq \Lambda_r \text{ker}(B) \).

(iii) \( \Lambda_r \)-ker\( (A) = \Lambda_r \text{ker}(\Lambda_r \text{ker}(A)) \).

(iv) \( y \in \Lambda_r \text{ker}(\{x\}) \) if and only if \( x \in \Lambda_r \text{cl}(\{y\}) \).

(v) \( \Lambda_r \text{ker}(A) = \{x : \Lambda_r \text{cl}(\{x\}) \cap A \neq \emptyset\} \).

**Proof.**

If \( x \notin \Lambda_r \text{ker}(A) \), then by Definition 2.16, there exists \( V \in \Lambda_r O(X, \tau) \) such that \( A \subseteq V \) and \( x \notin V \) and so \( x \notin A \). This proves (i).
If \( x \notin \Lambda_r\text{-ker}(B) \), then there exists \( G \in \Lambda_rO(X, \tau) \) such that \( B \subseteq G \) and \( x \notin G \). Since \( A \subseteq B \), \( A \subseteq G \) and hence \( x \notin \Lambda_r\text{-ker}(A) \). This proves (ii).

If \( x \in \Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A)) \), then for every \( \Lambda_r \)-open set \( G \supseteq \Lambda_r\text{-ker}(A) \), \( x \in G \). By (i), \( A \subseteq \Lambda_r\text{-ker}(A) \). Thus for every \( \Lambda_r \)-open set \( G \supseteq A \), \( x \in G \) which implies that \( x \in \Lambda_r\text{-ker}(A) \) and so \( \Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A)) \subseteq \Lambda_r\text{-ker}(A) \). By (i), we have \( \Lambda_r\text{-ker}(A) \subseteq \Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A)) \). This proves (iii).

If \( y \notin \Lambda_r\text{-ker}(\{x\}) \), then by Definition 2.16, there exists a \( \Lambda_r \)-open set \( V \supseteq \{x\} \) such that \( y \notin V \) which implies that \( V \) is a \( \Lambda_r \)-open set containing \( x \) such that \( \{y\} \cap V = \emptyset \). By Definition 2.10, \( x \) is not a \( \Lambda_r \)-cluster point of \( \{y\} \) and so \( x \notin \Lambda_r\text{-cl}(\{y\}) \). On the other hand, suppose \( x \notin \Lambda_r\text{-cl}(\{y\}) \). By Definition 2.10, \( x \) is not a \( \Lambda_r \)-cluster point of \( \{y\} \) and so there exists a \( \Lambda_r \)-open set \( U \) containing \( x \) such that \( \{y\} \cap U = \emptyset \) which implies that \( U \) is a \( \Lambda_r \)-open set, \( U \supseteq \{x\} \) and \( y \notin U \). By Definition 2.16, \( y \notin \Lambda_r\text{-ker}(\{x\}) \). This proves (iv).

If \( x \in \Lambda_r\text{-ker}(A) \), then for every \( \Lambda_r \)-open set \( G \supseteq A \), \( x \in G \). If possible, let \( \Lambda_r\text{-cl}(\{x\}) \cap A = \emptyset \), then \( A \subseteq X \setminus (\Lambda_r\text{-cl}(\{x\})) \). Take \( V = X \setminus (\Lambda_r\text{-cl}(\{x\})) \). Then by using Proposition 2.11(v), \( V \) is a \( \Lambda_r \)-open set containing \( A \) and \( x \notin V \). By this contradiction, we have \( \Lambda_r\text{-cl}(\{x\}) \cap A \neq \emptyset \). Conversely, let \( x \in X \) such that \( \Lambda_r\text{-cl}(\{x\}) \cap A \neq \emptyset \). If \( y \in \Lambda_r\text{-cl}(\{x\}) \cap A \), then by Definition 2.10, \( y \) is a \( \Lambda_r \)-cluster point of \( \{x\} \) and so for every \( \Lambda_r \)-open set \( U \) containing \( y \), \( \{x\} \neq \emptyset \), that is, \( x \in U \). If \( x \notin \Lambda_r\text{-ker}(A) \), then by Definition 2.16, there exists a \( \Lambda_r \)-open set \( V \supseteq A \) such that \( x \notin V \). Since \( y \in A \), we have \( V \) is a \( \Lambda_r \)-open set containing \( y \) but \( x \notin V \). By this contradiction, we have \( x \in \Lambda_r\text{-ker}(A) \). This proves (v).

\[ \square \]

**Theorem 2.18.**

For any points \( x \) and \( y \) in a space \( X \), \( \Lambda_r\text{-ker}(\{x\}) \neq \Lambda_r\text{-ker}(\{y\}) \) if and only if \( \Lambda_r\text{-cl}(\{x\}) \neq \Lambda_r\text{-cl}(\{y\}) \).
Proof.

Suppose $\Lambda_r\ker(\{x\}) \neq \Lambda_r\ker(\{y\})$. Then there exists a point $z$ in $X$ such that $z \in \Lambda_r\ker(\{x\})$ and $z \notin \Lambda_r\ker(\{y\})$. By Proposition 2.17(iv), $x \in \Lambda_r\cl(\{z\})$ and $y \notin \Lambda_r\cl(\{z\})$. By Remark 2.13(ii), $\Lambda_r\cl(\{x\}) \subseteq \Lambda_r\cl(\{z\})$ and $y \notin \Lambda_r\cl(\{z\})$ which implies that $y \notin \Lambda_r\cl(\{x\})$. This shows that $\Lambda_r\cl(\{x\}) \neq \Lambda_r\cl(\{y\})$.

For the converse, suppose $\Lambda_r\cl(\{x\}) \neq \Lambda_r\cl(\{y\})$. Then there exists a point $z$ in $X$ such that $z \in \Lambda_r\cl(\{x\})$ and $z \notin \Lambda_r\cl(\{y\})$ which implies that by Definition 2.10, there exists a $\Lambda_r$-open set $V$ containing $z$ such that $x \in V$ and $y \notin V$. Thus $V$ is a $\Lambda_r$-open set containing $x$ but not $y$. If $y \in \Lambda_r\ker(\{x\})$, then by Proposition 2.17(iv), $x \in \Lambda_r\cl(\{y\})$ and so by Definition 2.10, for every $\Lambda_r$-open set $G$ containing $x$, $G \cap \{y\} \neq \emptyset$, that is, $y \in G$, a contradiction. Hence $y \notin \Lambda_r\ker(\{x\})$ and hence $\Lambda_r\ker(\{x\}) \neq \Lambda_r\ker(\{y\})$. □

Definition 2.19.

Let $X$ be a space and $x \in X$. Then we define a subset $\Lambda_r\langle x \rangle$ of $X$ as follows:

$$\Lambda_r\langle x \rangle = \Lambda_r\cl(\{x\}) \cap \Lambda_r\ker(\{x\}).$$

Proposition 2.20.

Let $X$ be a space. Then the following properties hold:

(i) For each $x \in X$, $\Lambda_r\ker(\Lambda_r\langle x \rangle) = \Lambda_r\ker(\{x\})$.

(ii) For each $x \in X$, $\Lambda_r\cl(\Lambda_r\langle x \rangle) = \Lambda_r\cl(\{x\})$.

(iii) If $U$ is a $\Lambda_r$-open set of $X$ and $x \in U$, then $\Lambda_r\langle x \rangle \subseteq U$.

(iv) If $F$ is a $\Lambda_r$-closed set of $X$ and $x \in F$, then $\Lambda_r\langle x \rangle \subseteq F$.

Proof.

Let $x \in X$. By Proposition 2.11(i) and Proposition 2.17(i), $\{x\} \subseteq \Lambda_r\cl(\{x\})$ and $\{x\} \subseteq \Lambda_r\ker(\{x\})$ and so by Definition 2.19, it follows that $\{x\} \subseteq \Lambda_r\langle x \rangle$. By
Proposition 2.17(ii), \( \Lambda_r\ker(\langle x \rangle) \subseteq \Lambda_r\ker(\Lambda_r\langle x \rangle) \). For the reverse inclusion, if \( y \not\in \Lambda_r\ker(\langle x \rangle) \), then by Definition 2.16, there exists a \( \Lambda_r \)-open set \( V \) such that \( x \in V \) and \( y \not\in V \). By Definition 2.19, Definition 2.16 and Proposition 2.17, it follows that 
\[ \Lambda_r\langle x \rangle \subseteq \Lambda_r\ker(\langle x \rangle) \subseteq \Lambda_r\ker(V) = V \]
and so \( \Lambda_r\ker(\Lambda_r\langle x \rangle) \subseteq \Lambda_r\ker(V) = V \).

Since \( y \not\in V \), \( y \not\in \Lambda_r\ker(\Lambda_r\langle x \rangle) \). Consequently, \( \Lambda_r\ker(\Lambda_r\langle x \rangle) \subseteq \Lambda_r\ker(\langle x \rangle) \).

This proves (i).

By applying Proposition 2.11(i), Proposition 2.17(i) and Definition 2.19, we have \( \{x\} \subseteq \Lambda_r\langle x \rangle \). Then by Proposition 2.11(iii), \( \Lambda_r\cl(\{x\}) \subseteq \Lambda_r\cl(\Lambda_r\langle x \rangle) \). On the other hand, by Definition 2.19, \( \Lambda_r\langle x \rangle \subseteq \Lambda_r\cl(\{x\}) \) and so by Proposition 2.11(v) and Proposition 2.11(iv), \( \Lambda_r\cl(\Lambda_r\langle x \rangle) \subseteq \Lambda_r\cl(\Lambda_r\cl(\{x\})) = \Lambda_r\cl(\{x\}) \).

This proves (ii).

Suppose \( U \) is a \( \Lambda_r \)-open set and \( x \in U \). Then by Proposition 2.17(ii) and Definition 2.16, we have \( \Lambda_r\ker(\{x\}) \subseteq \Lambda_r\ker(U) = U \) and so \( \Lambda_r\langle x \rangle \subseteq U \). This proves (iii).

Suppose \( F \) is \( \Lambda_r \)-closed and \( x \in F \). By Remark 2.13(ii), \( x \in \Lambda_r\cl(\{x\}) \subseteq F \). By Definition 2.19, we have \( x \in \Lambda_r\langle x \rangle \) and \( \Lambda_r\langle x \rangle \subseteq \Lambda_r\cl(\{x\}) \) which implies that \( \Lambda_r\langle x \rangle \subseteq F \). This proves (iv).

\( \square \)

Definition 2.21.

Let \( X \) be a space. A point \( x \in X \) is said to be a \( \Lambda_r \)-interior point of \( A \) if there exists a \( \Lambda_r \)-open set \( U \) containing \( x \) such that \( U \subseteq A \).

The collection of all \( \Lambda_r \)-interior points of \( A \) is called \( \Lambda_r \)-interior of \( A \) and is denoted by \( \Lambda_r\text{-int}(A) \).
Theorem 2.22.

For subsets $A, B$ of a space $X$, the following statements are true:

(i) $\text{int}(A) \subseteq \Lambda_r\text{-int}(A)$ and $\text{int}(A) = \Lambda_r\text{-int}(A)$ if $A$ is open.

(ii) If $A \subseteq B$, then $\Lambda_r\text{-int}(A) \subseteq \Lambda_r\text{-int}(B)$.

(iii) $\Lambda_r\text{-int}(A) = \bigcup \{G : G \in \Lambda_rO(X, \tau) \text{ and } G \subseteq A\}$.

(iv) $\Lambda_r\text{-int}(\Lambda_r\text{-int}(A)) = \Lambda_r\text{-int}(A)$.

(v) $\Lambda_r\text{-cl}(X \setminus A) = X \setminus \Lambda_r\text{-int}(A)$.

(vi) $A$ is $\Lambda_r$-open if and only if $A = \Lambda_r\text{-int}(A)$.

(vii) $\Lambda_r\text{-int}(A)$ is the largest $\Lambda_r$-open set contained in $A$.

(viii) $\Lambda_r\text{-int}(X \setminus A) = X \setminus \Lambda_r\text{-cl}(A)$.

Proof.

If $x \in \text{int}(A)$, then there exists a open set $G \subseteq A$ such that $x \in G$. By Proposition 2.5(ii), $G$ is $\Lambda_r$-open and hence by Definition 2.21, $x \in \Lambda_r\text{-int}(A)$. This shows that $\text{int}(A) \subseteq \Lambda_r\text{-int}(A)$. If $A$ is open, then $A = \text{int}(A)$. Since $\Lambda_r\text{-int}(A) \subseteq A$, it follows that $\Lambda_r\text{-int}(A) = \text{int}(A)$. This proves (i).

Suppose $x \in \Lambda_r\text{-int}(A)$. Then by Definition 2.21, there exists a $\Lambda_r$-open set $U$ containing $x$ such that $U \subseteq A$. Since $A \subseteq B$, $U \subseteq B$ and hence $x \in \Lambda_r\text{-int}(B)$. This proves (ii).

If $x \in \Lambda_r\text{-int}(A)$, then by Definition 2.21, there exists a $\Lambda_r$-open set $U$ containing $x$ such that $U \subseteq A$ and so $x \in \bigcup \{G : G \in \Lambda_rO(X, \tau) \text{ and } G \subseteq A\}$. The reverse inclusion can be obtained similarly. This proves (iii).

By Definition 2.21 itself, we have $\Lambda_r\text{-int}(A) \subseteq A$ and so by (ii), we have $\Lambda_r\text{-int}(\Lambda_r\text{-int}(A)) \subseteq \Lambda_r\text{-int}(A)$. On the other hand, let $x \in \Lambda_r\text{-int}(A)$. Then there exists a $\Lambda_r$-open set $U$ containing $x$ such that $U \subseteq A$. If $U \not\subseteq \Lambda_r\text{-int}(A)$, then there exists $y \in X$ such that $y \in U$ but $y \not\in \Lambda_r\text{-int}(A)$ which implies that for every $\Lambda_r$-open
set $G \subseteq A$, $y \notin G$. But $U$ is a $\Lambda_r$-open set containing $y$ such that $U \subseteq A$, a contradiction. Hence $U \subseteq \Lambda_r\text{-int}(A)$. Thus we have $U$ is a $\Lambda_r$-open set containing $x$ such that $U \subseteq \Lambda_r\text{-int}(A)$ which implies that $x \in \Lambda_r\text{-int}(\Lambda_r\text{-int}(A))$. This proves (iv).

Suppose $x \notin \Lambda_r\text{-cl}(X \setminus A)$. Then by Definition 2.10, $x$ is not a $\Lambda_r$-cluster point of $X \setminus A$ and so there exists a $\Lambda_r$-open set $U$ containing $x$ such that $U \cap (X \setminus A) = \emptyset$, that is, $U \subseteq A$. Hence by Definition 2.21, $x \in \Lambda_r\text{-int}(A)$ and so $x \notin X \setminus \Lambda_r\text{-int}(A)$. The reverse inclusion can be similarly obtained. This proves (v).

Suppose $A$ is $\Lambda_r$-open. Then by (iii), $A = \Lambda_r\text{-int}(A)$. For the converse, suppose $A = \Lambda_r\text{-int}(A)$. Then $X \setminus A = X \setminus \Lambda_r\text{-int}(A) = \Lambda_r\text{-cl}(X \setminus A)$ by (v). By Proposition 2.11(iv), $X \setminus A$ is $\Lambda_r$-closed and so $A$ is $\Lambda_r$-open. This proves (vi).

By (iv) and (vi), $\Lambda_r\text{-int}(A)$ is $\Lambda_r$-open. If $U$ is a $\Lambda_r$-open set such that $\Lambda_r\text{-int}(A) \subseteq U \subseteq A$, then by (iii), $\Lambda_r\text{-int}(A) = U$. This proves (vii).

Suppose $x \in \Lambda_r\text{-int}(X \setminus A)$. Then by Definition 2.21, there exists a $\Lambda_r$-open set $U$ containing $x$ such that $U \subseteq X \setminus A$ which implies that $U \cap A = \emptyset$. By Definition 2.10, $x \notin \Lambda_r\text{-cl}(A)$ and so $x \in X \setminus \Lambda_r\text{-cl}(A)$. The reverse inclusion can be obtained similarly. This proves (viii).

Definition 2.23.

Let $A$ be a subset of a space $X$. A point $x \in X$ is said to be $\Lambda_r$-limit point of $A$ if for every $\Lambda_r$-open set $U$ containing $x$, $U \cap (A \setminus \{x\}) \neq \emptyset$. The set of all $\Lambda_r$-limit points of $A$ is called a $\Lambda_r$-derived set of $A$ and is denoted by $\Lambda_rD(A)$.

Theorem 2.24.

For subsets $A$, $B$ of a space $X$, the following statements hold:

(i) $\Lambda_rD(A) \subseteq D(A)$ where $D(A)$ is the derived set of $A$.

(ii) If $A \subseteq B$, then $\Lambda_rD(A) \subseteq \Lambda_rD(B)$.

(iii) $\Lambda_rD(A) \cup \Lambda_rD(B) \subseteq \Lambda_rD(A \cup B)$ and $\Lambda_rD(A \cap B) \subseteq \Lambda_rD(A) \cap \Lambda_rD(B)$
(iv) \( \Lambda_r D(\Lambda_r D(A)) \setminus A \subseteq \Lambda_r D(A) \).

(v) \( \Lambda_r D(A \cup \Lambda_r D(A)) \subseteq A \cup \Lambda_r D(A) \).

**Proof.**

If \( x \notin D(A) \), then there exists a open set \( U \) containing \( x \) such that \( U \cap (A \setminus \{x\}) = \emptyset \). By Proposition 2.5(ii), \( U \) is a \( \Lambda_r \)-open set containing \( x \) such that \( U \cap (A \setminus \{x\}) = \emptyset \) and so by Definition 2.23, \( x \notin \Lambda_r D(A) \). This proves (i).

If \( x \in \Lambda_r D(A) \), then for every \( \Lambda_r \)-open set \( U \) containing \( x \), \( U \cap (A \setminus \{x\}) \neq \emptyset \). Since \( A \subseteq B \), \( U \cap (B \setminus \{x\}) \neq \emptyset \) and so \( x \in \Lambda_r D(B) \). This proves (ii).

By (ii), \( \Lambda_r D(A) \subseteq \Lambda_r D(A \cup B) \) and \( \Lambda_r D(B) \subseteq \Lambda_r D(A \cup B) \) which implies that \( \Lambda_r D(A) \cup \Lambda_r D(B) \subseteq \Lambda_r D(A \cup B) \). Again by (ii), \( \Lambda_r D(A \cap B) \subseteq \Lambda_r D(A) \) and \( \Lambda_r D(A \cap B) \subseteq \Lambda_r D(B) \) which implies \( \Lambda_r D(A \cap B) \subseteq \Lambda_r D(A) \cap \Lambda_r D(B) \). This proves (iii).

If \( x \in \Lambda_r D(\Lambda_r D(A)) \setminus A \), then by Definition 2.23, for every \( \Lambda_r \)-open set \( U \) containing \( x \), \( U \cap (\Lambda_r D(A) \setminus \{x\}) \neq \emptyset \) and \( x \notin A \). Let \( y \in U \cap (\Lambda_r D(A) \setminus \{x\}) \). Then \( y \in U \), \( y \in \Lambda_r D(A) \) and \( y \neq x \). Thus we have \( U \) is a \( \Lambda_r \)-open set containing \( y \) and \( U \cap (A \setminus \{y\}) \neq \emptyset \). Take \( z \in U \cap (A \setminus \{y\}) \). Then \( z \in U \), \( z \in A \) and \( z \neq y \). Since \( x \notin A \), we have \( z \neq x \) and \( U \cap (A \setminus \{x\}) \neq \emptyset \). This implies that \( x \notin \Lambda_r D(A) \). This proves (iv).

Suppose \( x \in \Lambda_r D(A \cup \Lambda_r D(A)) \). If \( x \in A \), then \( x \in A \cup \Lambda_r D(A) \). If \( x \notin A \), then \( x \in \Lambda_r D(A \cup \Lambda_r D(A)) \setminus A \), which implies that for every \( \Lambda_r \)-open set \( U \) containing \( x \), \( U \cap ((A \cup \Lambda_r D(A)) \setminus \{x\}) \neq \emptyset \). Then we obtain that \( U \cap (A \setminus \{x\}) \neq \emptyset \) or \( U \cap (\Lambda_r D(A) \setminus \{x\}) \neq \emptyset \) which implies that \( x \in \Lambda_r D(A) \) or \( x \in \Lambda_r D(\Lambda_r D(A)) \).

If \( x \in \Lambda_r D(A) \), then clearly it follows that \( x \in A \cup \Lambda_r D(A) \). If \( x \in \Lambda_r D(\Lambda_r D(A)) \), then \( x \in \Lambda_r D(\Lambda_r D(A)) \setminus A \) since \( x \notin A \). Hence by (iv), \( x \in \Lambda_r D(A) \) and so \( x \in A \cup \Lambda_r D(A) \). This proves (v). \( \square \)
The reverse inclusions in Theorem 2.24 need not be true which is shown in the following example.

**Example 2.25.**

Consider a topological space \((X, \tau)\) where \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}\). Then the \(\Lambda_r\)-open sets in \((X, \tau)\) are \(\emptyset, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\) and \(X\).

\(\text{(i)}\) If \(A = \{b, c, d\}\), then \(D(A) = \{b, c, d\}\) and \(\Lambda_rD(A) = \{b, c\}\). and so
\[D(A) \not\subseteq \Lambda_rD(A).\]

\(\text{(ii)}\) If \(A = \{a\}\) and \(B = \{b\}\), then \(\Lambda_rD(A) = \emptyset\), \(\Lambda_rD(B) = \{c\}\) and \(\Lambda_rD(A \cup B) = \{c, d\}\) and so \(\Lambda_rD(A \cup B) \not\subseteq \Lambda_rD(A) \cup \Lambda_rD(B)\).

\(\text{(iii)}\) If \(A = \{a, b, d\}\) and \(B = \{a, c, d\}\), then \(\Lambda_rD(A) = \{c, d\}\), \(\Lambda_rD(B) = \{b, d\}\) and \(\Lambda_rD(A \cap B) = \emptyset\) and so \(\Lambda_rD(A) \cap \Lambda_rD(B) \not\subseteq \Lambda_rD(A \cap B)\).

\(\text{(iv)}\) If \(A = \{a, b, c\}\), then \(\Lambda_rD(\Lambda_rD(A)) \setminus A = \emptyset\) and \(\Lambda_rD(A) = \{b, c, d\}\) and so
\[\Lambda_rD(A) \not\subseteq \Lambda_rD(\Lambda_rD(A)) \setminus A.\]

\(\text{(v)}\) If \(A = \{c, d\}\), then \(\Lambda_rD(A \cup \Lambda_rD(A)) = \{b, c\}\) and \(A \cup \Lambda_rD(A) = \{b, c, d\}\) and so \(A \cup \Lambda_rD(A) \not\subseteq \Lambda_rD(A \cup \Lambda_rD(A))\).

**Theorem 2.26.**

For any subset \(A\) of a space \(X\), the following are true:

\(\text{(i)}\) \(\Lambda_r-cl(A) = A \cup \Lambda_rD(A)\).

\(\text{(ii)}\) \(\Lambda_r-int(A) = A \setminus \Lambda_rD(X \setminus A)\).

**Proof.**

By Definition 2.10 and Definition 2.23 itself, we have \(\Lambda_rD(A) \subseteq \Lambda_r-cl(A)\). Then we obtain \(A \cup \Lambda_rD(A) \subseteq A \cup \Lambda_r-cl(A) = \Lambda_r-cl(A)\) by Proposition 2.11(i). On the other hand, let \(x \in \Lambda_r-cl(A)\). Then by Definition 2.10, for every \(\Lambda_r\)-open set
U containing x, \( U \cap A \neq \emptyset \). If \( x \in A \), then clearly \( x \in A \cup \Lambda_{r^{-}}D(A) \). If \( x \notin A \), then \( U \cap A \setminus \{x\} \neq \emptyset \) and so \( x \in \Lambda_{r^{-}}D(A) \) and so \( x \in A \cup \Lambda_{r^{-}}D(A) \). This proves (i).

If \( x \in A \setminus \Lambda_{r^{-}}D(X \setminus A) \), then \( x \notin \Lambda_{r^{-}}D(X \setminus A) \) and so by Definition 2.23, there exists a \( \Lambda_{r^-} \)-open set \( U \) containing \( x \) such that \( U \cap ((X \setminus A) \setminus \{x\}) = \emptyset \). Since \( x \in A \), \( U \cap (X \setminus A) = \emptyset \) which implies that \( U \subseteq A \) and so by Definition 2.21, \( x \in \Lambda_{r^-}\text{int}(A) \).

On the other hand, let \( x \in \Lambda_{r^-}\text{int}(A) \). Then by Definition 2.21, there exists a \( \Lambda_{r^-} \)-open set \( U \) containing \( x \) such that \( U \subseteq A \) and so \( U \cap (X \setminus A) = \emptyset \). Since \( x \in A \), we obtain that \( U \cap ((X \setminus A) \setminus \{x\}) = \emptyset \) which implies that \( x \notin \Lambda_{r^-}D(X \setminus A) \) and so \( x \in A \setminus \Lambda_{r^-}D(X \setminus A) \).

This proves (ii). \( \square \)

**Definition 2.27.**

Let \( A \) be a subset of a space \( X \). Then the \( \Lambda_{r^-} \)-border of \( A \) denoted by \( \Lambda_{r^-}b(A) \), is defined as \( \Lambda_{r^-}b(A) = A \setminus \Lambda_{r^-}\text{int}(A) \).

**Theorem 2.28.**

For a subset \( A \) of a space \( X \), the following statements hold:

(i) \( \Lambda_{r^-}b(A) \subseteq b(A) \) where \( b(A) = A \setminus \text{int}(A) \), is the border of \( A \).

(ii) \( A = \Lambda_{r^-}\text{int}(A) \cup \Lambda_{r^-}b(A) \).

(iii) \( \Lambda_{r^-}\text{int}(A) \cap \Lambda_{r^-}b(A) = \emptyset \).

(iv) \( A \) is \( \Lambda_{r^-} \)-open if and only if \( \Lambda_{r^-}b(A) = \emptyset \).

(v) \( \Lambda_{r^-}b(\Lambda_{r^-}\text{int}(A)) = \emptyset \).

(vi) \( \Lambda_{r^-}\text{int}(\Lambda_{r^-}b(A)) = \emptyset \).

(vii) \( \Lambda_{r^-}b(\Lambda_{r^-}b(A)) = \Lambda_{r^-}b(A) \).

(viii) \( \Lambda_{r^-}b(A) = A \cap \Lambda_{r^-}\text{cl}(X \setminus A) \).

(ix) \( \Lambda_{r^-}b(A) = A \cap \Lambda_{r^-}D(X \setminus A) \).
Proof.

Suppose \( x \not\in b(A) \). Then \( x \not\in A \) or \( x \in \text{int}(A) \). If \( x \not\in A \), then \( x \not\in A \setminus \Lambda_r\text{-int}(A) \) and so \( x \not\in \Lambda_r\text{-}b(A) \). If \( x \in \text{int}(A) \), then there exists a open set \( V \subseteq A \) such that \( x \in V \). By using Proposition 2.5(ii), \( V \) is a \( \Lambda_r \)-open set containing \( x \) such that \( V \subseteq A \) and so by Definition 2.21, \( x \in \Lambda_r\text{-int}(A) \) which implies that \( x \not\in A \setminus \Lambda_r\text{-int}(A) \). This proves \((i)\).

Suppose \( x \in \Lambda_r\text{-int}(A) \cup \Lambda_r\text{-}b(A) \). If \( x \in \Lambda_r\text{-int}(A) \), then by Definition 2.21, there exists a \( \Lambda_r \)-open set \( U \) containing \( x \) such that \( U \subseteq A \) and so \( x \in A \). If \( x \in \Lambda_r\text{-}b(A) \), then by Definition 2.27 itself, \( x \in A \). For the reverse inclusion, suppose \( x \not\in \Lambda_r\text{-int}(A) \cup \Lambda_r\text{-}b(A) \). Then \( x \not\in \Lambda_r\text{-}b(A) \) and \( x \not\in \Lambda_r\text{-int}(A) \) which implies that \( x \not\in A \). This proves \((ii)\).

If \( x \in \Lambda_r\text{-int}(A) \cap \Lambda_r\text{-}b(A) \), then \( x \in \Lambda_r\text{-int}(A) \) and \( x \in \Lambda_r\text{-}b(A) \). Since \( \Lambda_r\text{-int}(A) \subseteq A \), \( x \in A \). Since \( x \in \Lambda_r\text{-}b(A) \), by Definition 2.27, \( x \not\in \Lambda_r\text{-int}(A) \). Thus we obtain that \( x \in \Lambda_r\text{-int}(A) \) and \( x \not\in \Lambda_r\text{-int}(A) \), a contradiction. This proves \((iii)\).

Suppose \( A \) is \( \Lambda_r \)-open. Then by Theorem 2.22(vi), \( A = \Lambda_r\text{-int}(A) \). If \( x \in \Lambda_r\text{-}b(A) \), then by Definition 2.27, \( x \in A \) and \( x \not\in \Lambda_r\text{-int}(A) \) = \( A \). This contradiction shows that \( \Lambda_r\text{-}b(A) = \varnothing \). For the converse, suppose that \( \Lambda_r\text{-}b(A) = \varnothing \). Then \( A = \Lambda_r\text{-int}(A) \) and so by Theorem 2.22(vi), \( A \) is \( \Lambda_r \)-open. This proves \((iv)\).

Suppose \( \Lambda_r\text{-}b(\Lambda_r\text{-int}(A)) \neq \varnothing \) and let \( x \in \Lambda_r\text{-}b(\Lambda_r\text{-int}(A)) \). Then we obtain that \( x \in \Lambda_r\text{-int}(A) \) and \( x \not\in \Lambda_r\text{-int}(\Lambda_r\text{-int}(A)) \) which implies that \( x \not\in \Lambda_r\text{-int}(A) \) and \( x \not\in \Lambda_r\text{-int}(A) \) by Theorem 2.22(iv), a contradiction. This proves \((v)\).

If \( x \in \Lambda_r\text{-int}(\Lambda_r\text{-}b(A)) \), then by Definition 2.21, we have \( x \in \Lambda_r\text{-}b(A) \). By Definition 2.27 itself, \( \Lambda_r\text{-}b(A) \subseteq A \) and so by Theorem 2.22(ii), \( \Lambda_r\text{-int}(\Lambda_r\text{-}b(A)) \subseteq \Lambda_r\text{-int}(A) \) and so \( x \in \Lambda_r\text{-int}(A) \). Thus we obtain that \( \Lambda_r\text{-}b(A) \cap \Lambda_r\text{-int}(A) \neq \varnothing \), which contradicts \((iii)\). Hence \( \Lambda_r\text{-int}(\Lambda_r\text{-}b(A)) \) must be empty. This proves \((vi)\).
If \( x \in \Lambda_r-b(\Lambda_r-b(A)) \), then by Definition 2.27, \( x \in \Lambda_r-b(A)\setminus\Lambda_r-int(\Lambda_r-b(A)) \) and so by \((vi)\), \( x \in \Lambda_r-b(A) \). For the converse, let \( x \not\in \Lambda_r-b(\Lambda_r-b(A)) \). Then by Definition 2.27, \( x \not\in \Lambda_r-b(A)\setminus\Lambda_r-int(\Lambda_r-b(A)) \). By using \((vi)\), \( x \not\in \Lambda_r-b(A) \). This proves \((vii)\).

By Definition 2.27, \( \Lambda_r-b(A) = A \setminus \Lambda_r-int(A) = A \cap (X \setminus \Lambda_r-int(A)) \) and so by Theorem 2.22\((v)\), \( \Lambda_r-b(A) = A \cap \Lambda_r-cl(X\setminus A) \). This proves \((vii)\).

By \((viii)\), \( \Lambda_r-b(A) = A \cap \Lambda_r-cl(X\setminus A) \). Then by Theorem 2.26\((i)\), we have \( \Lambda_r-cl(X\setminus A) = (X\setminus A) \cup \Lambda_r-D(X\setminus A) \) and so \( \Lambda_r-b(A) = A \cap ((X\setminus A) \cup \Lambda_r-D(X\setminus A)) \) which implies \( \Lambda_r-b(A) = A \cap \Lambda_r-D(X\setminus A) \). This proves \((ix)\). \(\square\)

The next example shows that the reverse inclusion of Theorem 2.28\((i)\) need not be true.

**Example 2.29.**

Let \( X = \{a,b,c,d\} \) having the topology \( \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\} \). Then the \( \Lambda_r \)-open sets in \( (X, \tau) \) are \( \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c,d\}, \{b,c,d\} \) and \( X \). Take \( A = \{a,c,d\} \).

Then \( b(A) = \{c,d\} \) and \( \Lambda_r-b(A) = \emptyset \). Hence \( b(A) \not\subseteq \Lambda_r-b(A) \).

**Definition 2.30.**

Let \( A \) be a subset of a space \( X \). Then the \( \Lambda_r \)-frontier of \( A \) denoted by \( \Lambda_r-Fr(A) \), is defined as \( \Lambda_r-Fr(A) = \Lambda_r-cl(A) \setminus \Lambda_r-int(A) \).

**Theorem 2.31.**

For a subset \( A \) of a space \( X \), the following statements hold:

\[(i) \quad \Lambda_r-Fr(A) \subseteq Fr(A) \text{ where } Fr(A) = cl(A) \setminus int(A), \text{ is the frontier of } A.\]

\[(ii) \quad \Lambda_r-cl(A) = \Lambda_r-int(A) \cup \Lambda_r-Fr(A).\]

\[(iii) \quad \Lambda_r-int(A) \cap \Lambda_r-Fr(A) = \emptyset.\]

\[(iv) \quad \Lambda_r-b(A) \subseteq \Lambda_r-Fr(A).\]

\[(v) \quad \Lambda_r-Fr(A) = \Lambda_r-cl(A) \cap \Lambda_r-cl(X\setminus A).\]
(vi) $\Lambda_r Fr(A) = \Lambda_r Fr(X \setminus A)$.

(vii) $\Lambda_r b(A) = \Lambda_r Fr(A)$ if $A$ is $\Lambda_r$-closed.

(viii) $\Lambda_r Fr(A)$ is $\Lambda_r$-closed.

(ix) $\Lambda_r int(A) = A \setminus \Lambda_r Fr(A)$.

**Proof.**

If $x \in \Lambda_r Fr(A)$, then by Definition 2.30, $x \in \Lambda_r cl(A)$ and $x \notin \Lambda_r int(A)$. By Proposition 2.11(vi), $\Lambda_r cl(A) \subseteq cl(A)$ and by Theorem 2.22(i), $int(A) \subseteq \Lambda_r int(A)$ and so $x \in cl(A) \setminus int(A) = Fr(A)$. **This proves (i).**

Suppose that $x \in \Lambda_r int(A) \cup \Lambda_r Fr(A)$. Then $x \in \Lambda_r int(A)$ or $x \in \Lambda_r Fr(A)$. If $x \in \Lambda_r int(A)$, by Definition 2.21 itself, $\Lambda_r int(A) \subseteq A$ and by Proposition 2.11(i), $A \subseteq \Lambda_r cl(A)$ and so $\Lambda_r int(A) \subseteq \Lambda_r cl(A)$ and so $x \in \Lambda_r cl(A)$. If $x \in \Lambda_r Fr(A)$, from Definition 2.30, $x \in \Lambda_r cl(A)$. On the other hand, let $x \notin \Lambda_r int(A) \cup \Lambda_r Fr(A)$. Then $x \notin \Lambda_r int(A)$ and $x \notin \Lambda_r Fr(A)$ and so $x \notin \Lambda_r cl(A)$. **This proves (ii).**

By Definition 2.30, we have $\Lambda_r int(A) \cap \Lambda_r Fr(A) = \Lambda_r int(A) \cap (\Lambda_r cl(A) \setminus \Lambda_r int(A)) = \Lambda_r int(A) \cap \Lambda_r cl(A) \cap (X \setminus \Lambda_r int(A)) = \emptyset$. **This proves (iii).**

By using Proposition 2.11(i), $A \setminus \Lambda_r int(A) \subseteq \Lambda_r cl(A) \setminus \Lambda_r int(A)$ and so $\Lambda_r b(A) \subseteq \Lambda_r Fr(A)$. **This proves (iv).**

By Definition 2.30, we have $\Lambda_r Fr(A) = \Lambda_r cl(A) \setminus \Lambda_r int(A) = \Lambda_r cl(A) \cap (X \setminus \Lambda_r int(A)) = \Lambda_r cl(A) \cap \Lambda_r cl(X \setminus A)$ by Theorem 2.22(v). **This proves (v).**

By (v), $\Lambda_r Fr(A) = \Lambda_r cl(A) \cap \Lambda_r cl(X \setminus A) = \Lambda_r cl(X \setminus A) \cap \Lambda_r cl(A) = \Lambda_r Fr(X \setminus A)$. **This proves (vi).**

Suppose $A$ is $\Lambda_r$-closed. By Proposition 2.11(iv), $A = \Lambda_r cl(A)$ and hence $A \setminus \Lambda_r int(A) = \Lambda_r cl(A) \setminus \Lambda_r int(A)$ which implies that $\Lambda_r b(A) = \Lambda_r Fr(A)$. **This proves (vii).**
\[ \Lambda r-cl(\Lambda r-cl(A) \cap \Lambda r-cl(X \setminus A)) \subseteq \Lambda r-cl(\Lambda r-cl(A)) \cap \Lambda r-cl(\Lambda r-cl(X \setminus A)) \] by using Proposition 2.11(iii). Using Proposition 2.11(iv) and Proposition 2.11(v), 
\[ \Lambda r-cl(\Lambda r-cl(A)) = \Lambda r-cl(A) \text{ and } \Lambda r-cl(\Lambda r-cl(X \setminus A)) = \Lambda r-cl(X \setminus A) \] and so by (v), 
\[ \Lambda r-cl(\Lambda r-Fr(A)) \subseteq \Lambda r-Fr(A). \] By Proposition 2.11(i), \( \Lambda r-Fr(A) \subseteq \Lambda r-cl(\Lambda r-Fr(A)) \) and so by Proposition 2.11(i), \( \Lambda r-Fr(A) \) is \( \Lambda r \)-closed. This proves (viii).

\[ A \setminus \Lambda r-Fr(A) = A \setminus (\Lambda r-cl(A) \setminus \Lambda r-int(A)) = (A \setminus \Lambda r-cl(A)) \cup (A \cap \Lambda r-int(A)) \] by Definition 2.30. By using Proposition 2.11(i) and Definition 2.21, \( A \setminus \Lambda r-cl(A) = \emptyset \) and \( A \cap \Lambda r-int(A) = \Lambda r-int(A) \) which implies that \( A \setminus \Lambda r-Fr(A) = \Lambda r-int(A) \).

This proves (ix). \( \square \)

The reverse inclusions in Theorem 2.31 need not be true as seen in the next example.

**Example 2.32.**

Consider a topological space \((X, \tau)\) where \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}\). Then we have

(i) If \(A = \{b, c\}\), then \(Fr(A) = \{d\}\) and \(\Lambda r-Fr(A) = \emptyset\) and so \(Fr(A) \not\subseteq \Lambda r-Fr(A)\).

(ii) If \(A = \{a, c, d\}\), then \(\Lambda r-Fr(A) = \{b, c\}\) and \(\Lambda r-b(A) = \{c\}\) and so

\[ \Lambda r-Fr(A) \not\subseteq \Lambda r-b(A) \].

**Conclusion**

The concepts of \(\Lambda r\)-closed sets and \(\Lambda r\)-open sets are used to characterize the \(\lambda\)-closed, \(\lambda\)-semi-closed, \((\Lambda, \alpha)\)-closed and \((\Lambda, \delta)\)-closed sets. Also the basic operators namely closure and interior operators are studied using the above sets.