CHAPTER I

Basic Concepts

In this chapter, we give the definitions and results which are essentially needed for the development of the thesis. By a space $X$, we always mean a topological space $(X, \tau)$ in which no separation axioms are assumed unless otherwise specified. If $A \subseteq X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of $A$ in $(X, \tau)$.

In any topological space, the arbitrary intersection of open sets is not open and the arbitrary union of closed sets is not closed. These properties motivated Maki to introduce the notions of $\Lambda$-sets and $V$-sets in topological spaces. The following definition and the subsequent lemma are due to Maki.

**Definition 1.1.**

Let $X$ be a space and $A \subseteq X$. Then $A^V = \bigcup \{F : F \subseteq A$ and $F$ is closed\}$ and $A^\Lambda = \bigcap \{U : A \subseteq U$ and $U$ is open\}. Moreover, $A$ is said to be a $V$-set if $A = A^V$ and $A$ is said to be a $\Lambda$-set if $A = A^\Lambda$. [21]

**Lemma 1.2.**

Let $X$ be a space and $A$ and $B$ be subsets of $X$. Then the following hold:

(i) $A^V \subseteq A$.

(ii) $A \subseteq A^\Lambda$.

(iii) $A \subseteq B$ implies $A^V \subseteq B^V$ and $A^\Lambda \subseteq B^\Lambda$.

(iv) $(X \setminus A)^\Lambda = X \setminus A^V$.

(v) $(A^\Lambda)^\Lambda = A^\Lambda$ for every subset $A$ of $X$. [21]

Bourbaki [3] introduced the concept of locally closed sets in topological spaces. Following this, Arenas introduced $\lambda$-closed set which is an intersection of...
Λ-set and closed set. By using λ-closed sets, many separation axioms and functions have been introduced by the several mathematicians.

**Definition 1.3.**

A subset A of a space X is called λ-closed if $A = L \cap D$ where L is a Λ-set and D is a closed set. The complement of a λ-closed set is called λ-open. [1]

The collection of all λ-open sets in $(X, \tau)$ is denoted by $\lambda O(X, \tau)$ and the collection of all λ-closed sets in $(X, \tau)$ is denoted by $\lambda C(X, \tau)$.

Stone introduced regular-open sets, which are stronger than open sets. The following definition is due to Stone.

**Definition 1.4.**

A subset A of a space X is called regular-open if $A = int(cl(A))$ and regular-closed if $A = cl(int(A))$. [35]

The collection of all regular-open sets in $(X, \tau)$ is denoted by $RO(X, \tau)$ and the collection of all regular-closed sets in $(X, \tau)$ is denoted by $RC(X, \tau)$. Every regular-open set is open. It is easy to see that the closure of every open set is regular-closed. If $U$ is open in X, then $int(cl(U))$ is regular-open in X. Every clopen set is both regular-closed and regular-open.

Veličko introduced the concept of $\delta$-closed sets and Georgiou continued his work to define $\Lambda_\delta$-set and $(\Lambda, \delta)$-closed set. The formal definition of these sets is as follows:

**Definition 1.5.**

Let $S$ be any subset of a space X. A point $x \in X$ is called a $\delta$-cluster point of $S$ if $S \cap U \neq \emptyset$ for each regular-open set $U$ of X containing $x$. The set of all $\delta$-cluster points of $S$ is called the $\delta$-closure of $S$ and is denoted by $\delta cl(S)$. A subset $S$ is called $\delta$-closed if $\delta cl(S) = S$. The complement of a $\delta$-closed set is called $\delta$-open. [36]
The family of all \( \delta \)-open sets of a space \( X \) is denoted by \( \delta O(X, \tau) \) and the family of all \( \delta \)-closed sets of a space \( X \) is denoted by \( \delta C(X, \tau) \).

A subset \( B \) of a space \( X \) is said to be a \( \Lambda_\delta \)-set if \( B = \Lambda_\delta(B) \) where 
\[
\Lambda_\delta(B) = \cap \{ P : P \supseteq B, P \in \delta O(X, \tau) \}. \quad [13]
\]

A subset \( A \) of a space \( X \) is called \( (\Lambda, \delta) \)-closed if \( A = T \cap C \) where \( T \) is a \( \Lambda_\delta \)-set and \( C \) is a \( \delta \)-closed set. The complement of a \( (\Lambda, \delta) \)-closed set is called \( (\Lambda, \delta) \)-open. \([13,24]\)

We denote the collection of all \( (\Lambda, \delta) \)-open sets in \( (X, \tau) \) by \( \Lambda_\delta O(X, \tau) \) and the collection of all \( (\Lambda, \delta) \)-closed sets in \( (X, \tau) \) by \( \Lambda_\delta C(X, \tau) \).

Njåstad introduced and studied the notion of \( \alpha \)-open sets in topological space.

Using this, Caldas et. al. introduced \( \Lambda_\alpha \)-sets and \( (\Lambda, \alpha) \)-closed sets as follows:

**Definition 1.6.**

Let \( X \) be a space and \( A \) be a subset of \( X \). Then \( A \) is called \( \alpha \)-open if 
\[
A \subseteq int(cl(int(A))).
\]

The complement of an \( \alpha \)-open set is called \( \alpha \)-closed. \([28]\)

By \( \alpha(X, \tau) \), we denote the family of all \( \alpha \)-open sets of \( X \) and by \( \alpha C(X, \tau) \), we denote the family of all \( \alpha \)-closed sets of \( X \).

A subset \( A \) of a space \( X \) is said to be a \( \Lambda_\alpha \)-set if \( A = \Lambda_\alpha(A) \) where 
\[
\Lambda_\alpha(A) = \cap \{ O : O \in \alpha(X, \tau) \text{ and } A \subseteq O \}. \quad [6]
\]

The collection of all \( \Lambda_\alpha \)-sets in \( (X, \tau) \) is denoted by \( \tau^{\Lambda_\alpha} \).

A subset \( A \) of a space \( X \) is called \( (\Lambda, \alpha) \)-closed if \( A = T \cap C \) where \( T \) is a \( \Lambda_\alpha \)-set and \( C \) is an \( \alpha \)-closed set. \([6]\)

The concept of semi-open sets in topological space was introduced by Levine. Using this, Caldas and Dontchev extended Maki’s work by introducing and studying \( \Lambda_\alpha \)-sets and \( \lambda \)-semi-closed sets. The next definition gives the formal definition of these sets.
Definition 1.7.

Let \( A \) be a subset of a topological space \((X, \tau)\). Then \( A \) is called semi-open if \( A \subseteq \text{cl}(\text{int}(A)) \). The complement of a semi-open set is called semi-closed. [19]

The family of all semi-open sets in \((X, \tau)\) is denoted by \( \text{SO}(X, \tau) \).

A subset \( B \) of a space \( X \) is said to be a \( \Lambda_s \)-set if \( B = B^{\Lambda_s} \) where \( B^{\Lambda_s} = \cap \{ O : O \supseteq B, O \in \text{SO}(X, \tau) \} \). [4]

A subset \( A \) of a space \( X \) is called \( \lambda \)-semi-closed if \( A = L \cap F \) where \( L \) is a \( \Lambda_s \)-set and \( F \) is semi-closed. [22]

Navaneethakrishnan used regular-open sets to define and investigate the notions of \( \mathcal{V}_r \)-sets and \( \Lambda_r \)-sets in topological spaces. The following definitions and lemmas are due to Navaneethakrishnan.

Definition 1.8.

Let \((X, \tau)\) be a topological space. If \( B \subseteq X \), then define \( B^V = \bigcup \{ F : F \subseteq B \text{ and } F \text{ is regular-closed} \} \) and \( B^\Lambda = \bigcap \{ U : B \subseteq U \text{ and } U \text{ is regular-open} \} \). [27]

Lemma 1.9.

Let \((X, \tau)\) be a topological space. If \( A \) and \( B \) are subsets of \( X \), then the following hold:

(i) \( \emptyset^V = \emptyset \) and \( \emptyset^\Lambda = \emptyset \).

(ii) \( X^V = X \) and \( X^\Lambda = X \).

(iii) \( A^V \subseteq A \text{ and } A \subseteq A^\Lambda \).

(iv) \( (A^V)^V = A^V \) and \( (A^\Lambda)^\Lambda = A^\Lambda \).

(v) \( A \subseteq B \) implies \( A^V \subseteq B^V \) and \( A^\Lambda \subseteq B^\Lambda \).

(vi) \( A^V \cup B^V \subseteq (A \cup B)^V \).

(vii) \( A^\Lambda \cup B^\Lambda \subseteq (A \cup B)^\Lambda \).
(viii) \((A \cap B)^V_r \subseteq A^V_r \cap B^V_r\).

(ix) \((A \cap B)^\Lambda_r \subseteq A^\Lambda_r \cap B^\Lambda_r\).

(x) \(A^V_r \subseteq A^V\) and \(A^\Lambda_r \supseteq A^\Lambda\). [27]

**Definition 1.10.**

A subset \(B\) of a topological space \((X, \tau)\) is said to be \(V_r\)-set if \(B = B^V_r\).

A subset \(B\) of a topological space \((X, \tau)\) is said to be \(\Lambda_r\)-set if \(B = B^\Lambda_r\). [27]

Every regular-closed set is a \(V_r\)-set and every regular-open set is a \(\Lambda_r\)-set.

The arbitrary union of \(V_r\)-sets is a \(V_r\)-set and the arbitrary intersection of \(\Lambda_r\)-sets is a \(\Lambda_r\)-set.

**Lemma 1.11.**

Let \(A\) be a subset of a topological space \((X, \tau)\). Then the following hold:

(i) If \(A\) is a \(V_r\)-set, then it is a \(V\)-set.

(ii) If \(A\) is a \(\Lambda_r\)-set, then it is a \(\Lambda\)-set. [27]

**Lemma 1.12.**

Let \((X, \tau)\) be a topological space. Then

(i) \((X \setminus B)^\Lambda_r = X \setminus B^\Lambda_r\) for every subset \(B\) of \(X\).

(ii) \((X \setminus B)^V_r = X \setminus B^V_r\) for every subset \(B\) of \(X\). [27]

**Lemma 1.13.**

Let \((X, \tau)\) be a topological space. Then a subset \(B\) of a space \(X\) is a \(V_r\)-set if and only if \(X \setminus B\) is a \(\Lambda_r\)-set. [27]

It is easy to see that a subset \(B\) of a space \(X\) is a \(\Lambda_r\)-set if and only if \(X \setminus B\) is a \(V_r\)-set.

Now we furnish the definitions of various functions which are defined by Singal, Noiri, Khwdr and Dontchev.
Definition 1.14.

A function \( f : X \to Y \) is called

(i) contra-continuous if \( f^{-1}(V) \) is closed in \( X \) for each open set \( V \) of \( Y \),
    equivalently if \( f^{-1}(V) \) is open in \( X \) for each closed set \( V \) of \( Y \) [8],
(ii) regular set-connected if \( f^{-1}(V) \) is clopen in \( X \) for every regular-open set \( V \) in \( Y \) [9],
(iii) perfectly continuous if \( f^{-1}(V) \) is clopen in \( X \) for every open set \( V \) of \( Y \) [29],
(iv) almost continuous if \( f^{-1}(V) \) is open in \( X \) for every regular-open set \( V \) of \( Y \) [31],
(v) R-map if \( f^{-1}(V) \) is regular-open in \( X \) for every regular-open set \( V \) of \( Y \) [17],
(vi) super continuous if \( f^{-1}(V) \) is regular-open in \( X \) for every open set \( V \) in \( Y \). [29]

Now we furnish the definitions of various spaces.

Definition 1.15.

Let \((X, \tau)\) be a topological space. Then \((X, \tau)\) is said to be

(i) \(R_0\) space if for each open set \( G \), \( x \in G \Rightarrow \text{cl} \{x\} \subseteq G \) [26],
(ii) \(R_1\) space if for each \( x, y \in X \) with \( \text{cl} \{x\} \neq \text{cl} \{y\} \), there exists disjoint open sets \( U \) and \( V \) such that \( \text{cl} \{x\} \subseteq U \) and \( \text{cl} \{y\} \subseteq V \). [25]

Definition 1.16.

A space \( X \) is said to be ultra normal if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets. [34]

Definition 1.17.

A space \( X \) is said to be weakly Hausdorff if each element of \( X \) is an intersection of regular-closed sets. [33]
It is easy to see that if a space $X$ is weakly Hausdorff, then for any pair of distinct points $x$ and $y$ in $X$, there exists regular-closed sets $U$ and $V$ in $X$ such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

**Definition 1.18.**

A space $X$ is said to be semi-regular if any open set $U$ of $X$ and each point $x \in U$, there exists a regular-open set $V$ of $X$ such that $x \in V \subseteq U$. [32]

**Definition 1.19.**

A space $X$ is said to be almost-regular if for each regular-closed set $F$ of $X$ and each point $x \in X \setminus F$, there exists disjoint open sets $U$ and $V$ such that $F \subseteq V$ and $x \in U$. [32]

It is easy to see that a topological space $(X, \tau)$ is almost-regular if and only if for each $x \in X$ and each regular-open set $V$ containing $x$, there exists a regular-open set $U$ such that $x \in U \subseteq \text{cl}(U) \subseteq V$.

**Definition 1.20.**

A space $X$ is said to be

(i) extremally disconnected (briefly, E.D.) if the closure of every open subsets of $X$ is open in $X$ [37],

(ii) submaximal if every dense subset is open [3],

(iii) locally indiscrete if every open subset of $X$ is closed. [16]

**Lemma 1.21.**

In a E.D. space, $\text{RO}(X, \tau) = \text{RC}(X, \tau)$. [10]

Levine introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. Recently, the study of modifications of generalized closed sets has found considerable interest among general topologists. The following is the original definition.
Definition 1.22.

A subset $A$ of a space $X$ is called generalized closed (briefly, g-closed) in $X$ if $\text{cl}(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $X$. A subset $A$ is called generalized open (briefly, g-open) in $X$ if $X \setminus A$ is g-closed. [18]

Equivalently, $A$ is g-open if and only if $F \subseteq \text{int}(A)$ whenever $F \subseteq A$ and $F$ is closed in $X$. Every open set is g-open.

For basic concepts in topology, Willard [37] is referred.