CHAPTER VII

Λ_r-homeomorphisms

In this chapter, the concepts of Λ_r-homeomorphisms and Λ_r*-homeomorphisms are introduced and characterized their properties.

Λ_r-homeomorphisms

Definition 7.1.

A bijection \( f : X \to Y \) is called \( \Lambda_r \)-homeomorphism if both \( f \) and \( f^{-1} \) are \( \Lambda_r \)-continuous.

We denote the family of all \( \Lambda_r \)-homeomorphisms of a topological space \((X, \tau)\) onto itself by \( \Lambda_r \mathcal{H}(X, \tau) \).

Theorem 7.2.

Every homeomorphism is a \( \Lambda_r \)-homeomorphism but the converse is not true.

Proof.

If \( f : X \to Y \) is a homeomorphism, then \( f \) is bijective and both \( f \) and \( f^{-1} \) are continuous. By Theorem 5.2, \( f \) and \( f^{-1} \) are \( \Lambda_r \)-continuous and so by Definition 7.1, \( f \) is a \( \Lambda_r \)-homeomorphism.

The converse is not true as seen in the following example. If \( X = Y = \{a,b,c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, \{c\}, \{b,c\}, Y\} \), then \( \Lambda_r \mathcal{O}(X, \tau) = \tau \) and \( \Lambda_r \mathcal{O}(Y, \sigma) = \{\emptyset, \{b\}, \{c\}, \{b,c\}, \{a,c\}, \{a,b\}, Y\} \). Consider \( f : (X, \tau) \to (Y, \sigma) \) defined by \( f(a) = c, f(b) = b \) and \( f(c) = a \). Then \( f \) is a \( \Lambda_r \)-homeomorphism. Since \( \{b,c\} \) is open in \((X, \tau)\) but \( f(\{b,c\}) = \{a,b\} \) is not open in \((Y, \sigma)\), \( f^{-1} \) is not continuous and so \( f \) is not a homeomorphism.

\( \square \)

In general, the composition of two homeomorphisms is a homeomorphism. But the following example shows that the composition of two \( \Lambda_r \)-homeomorphisms need not be a \( \Lambda_r \)-homeomorphism.
Example 7.3.

If $X = Y = Z = \{a,b,c\}$, $\tau = \{\emptyset, \{a\}, \{a,b\}, \{b,c\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{c\}, \{b,c\}, Y\}$ and $\gamma = \{\emptyset, \{a\}, \{a,b\}, \{b,c\}, Z\}$, then $\Lambda_rO(X, \tau) = \tau$, $\Lambda_rO(Y, \sigma) = \{\emptyset, \{b\}, \{c\}, \{b,c\}, Y\}$ and $\Lambda_rO(Z, \gamma) = \gamma$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$ and define $g : (Y, \sigma) \to (Z, \gamma)$ by $g(a) = c$, $g(b) = a$ and $g(c) = b$. Then $f$ and $g$ are $\Lambda_r$-homeomorphisms. Since $\{b,c\}$ is open in $(Z, \gamma)$ but $(g \circ f)^{-1}(\{b,c\}) = \{a,c\}$ is not $\Lambda_r$-open in $(X, \tau)$, $g \circ f$ is not $\Lambda_r$-continuous and so $g \circ f$ is not $\Lambda_r$-homeomorphism.

Theorem 7.4.

Let $f : X \to Y$ be a bijective $\Lambda_r$-continuous function. Then the following are equivalent:

(i) $f$ is $\Lambda_r$-open.

(ii) $f$ is $\Lambda_r$-homeomorphism.

(iii) $f$ is $\Lambda_r$-closed.

Proof.

Suppose (i) holds. If $V$ is open in $X$, then by Definition 5.33, $f(V)$ is $\Lambda_r$-open in $Y$. But $f(V) = (f^{-1})^{-1}(V)$ and so $(f^{-1})^{-1}(V)$ is $\Lambda_r$-open in $Y$ and so by Definition 5.1, $f^{-1}$ is $\Lambda_r$-continuous. This proves (ii).

Suppose (ii) holds. Let $F$ be closed in $X$. Then by Definition 7.1, $f^{-1}$ is $\Lambda_r$-continuous and so by Theorem 5.5, $(f^{-1})^{-1}(F) = f(F)$ is $\Lambda_r$-closed in $Y$. By Definition 5.45, $f$ is $\Lambda_r$-closed. This proves (iii).

Suppose (iii) holds. If $V$ is open in $X$, then $X \setminus V$ is closed in $X$. By Definition 5.45, $f(X \setminus V)$ is $\Lambda_r$-closed in $Y$. But $f(X \setminus V) = Y \setminus f(V)$ which implies that $Y \setminus f(V)$ is $\Lambda_r$-closed in $Y$ and so $f(V)$ is $\Lambda_r$-open in $Y$. By Definition 5.33, $f$ is $\Lambda_r$-open. This proves (i).
\[\Lambda^*_r\text{-homeomorphisms}\]

**Definition 7.5.**

A bijection \( f : X \to Y \) is said to be \( \Lambda^*_r \)-homeomorphism if both \( f \) and \( f^{-1} \) are \( \Lambda^*_r \)-irresolute.

The spaces \( X \) and \( Y \) are said to be \( \Lambda^*_r \)-homeomorphic if there exists a \( \Lambda^*_r \)-homeomorphism from \( X \) onto \( Y \).

The family of all \( \Lambda^*_r \)-homeomorphisms of a topological space \((X, \tau)\) onto itself is denoted by \( \Lambda^*_r H(X, \tau) \).

**Theorem 7.6.**

Every \( \Lambda^*_r \)-homeomorphism is a \( \Lambda_r \)-homeomorphism but the converse is not true.

**Proof.**

If \( f : X \to Y \) is a \( \Lambda^*_r \)-homeomorphism, then by Definition 7.5, \( f \) is bijective, \( \Lambda^*_r \)-irresolute and \( f^{-1} \) is \( \Lambda^*_r \)-irresolute. By Remark 5.17, \( f \) and \( f^{-1} \) are \( \Lambda_r \)-continuous and so by Definition 7.1, \( f \) is a \( \Lambda_r \)-homeomorphism.

The converse is not true as seen in the following example. Consider the topological spaces \((X, \tau)\) and \((Y, \sigma)\) where \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, Y\}. \) Then \( \Lambda^*_r O(X, \tau) = \tau \) and \( \Lambda^*_r O(Y, \sigma) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}, Y\}. \) Consider \( f : (X, \tau) \to (Y, \sigma) \) defined by \( f(a) = c, f(b) = b \) and \( f(c) = a. \) Then \( f \) is a \( \Lambda_r \)-homeomorphism. Since \( \{a, c\} \) is \( \Lambda_r \)-open in \((Y, \sigma)\) but \( f^{-1}(\{a, c\}) = \{a, c\} \) is not \( \Lambda_r \)-open in \((X, \tau)\), \( f \) is not \( \Lambda_r \)-irresolute and so \( f \) is not a \( \Lambda^*_r \)-homeomorphism. \( \square \)

**Theorem 7.7.**

Let \( f : X \to Y \) be a \( \Lambda^*_r \)-homeomorphism. Then the following are true:

(i) \( \Lambda^*_r cl(f^{-1}(B)) = f^{-1}(\Lambda^*_r cl(B)) \) for every \( B \subseteq Y. \)

(ii) \( \Lambda^*_r cl(f(B)) = f(\Lambda^*_r cl(B)) \) for every \( B \subseteq X. \)
(iii) \( f(\Lambda_r\text{int}(B)) = \Lambda_r\text{int}(f(B)) \) for every \( B \subseteq X \).

(iv) \( f^{-1}(\Lambda_r\text{int}(B)) = \Lambda_r\text{int}(f^{-1}(B)) \) for every \( B \subseteq Y \).

**Proof.**

Suppose \( f : X \rightarrow Y \) is a \( \Lambda_r^* \)-homeomorphism. Then by Definition 7.5, \( f \) is bijective and both \( f \) and \( f^{-1} \) are \( \Lambda_r \)-irresolute. If \( B \subseteq Y \), then by Proposition 2.11(v), \( \Lambda_r\text{cl}(B) \) is a \( \Lambda_r \)-closed set in \( Y \) and so by Theorem 5.16, \( f^{-1}(\Lambda_r\text{cl}(B)) \) is \( \Lambda_r \)-closed in \( X \). Using Proposition 2.11(i), it follows that \( f^{-1}(B) \subseteq f^{-1}(\Lambda_r\text{cl}(B)) \). By Remark 2.13(ii), \( \Lambda_r\text{cl}(f^{-1}(B)) \) is the smallest \( \Lambda_r \)-closed set containing \( f^{-1}(B) \) and so \( \Lambda_r\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\Lambda_r\text{cl}(B)) \). Again, by Proposition 2.11(v), \( \Lambda_r\text{cl}(f^{-1}(B)) \) is \( \Lambda_r \)-closed in \( Y \) and then by Theorem 5.16, \( f(\Lambda_r\text{cl}(f^{-1}(B))) \) is \( \Lambda_r \)-closed in \( Y \). Moreover, \( B = f(f^{-1}(B)) \subseteq f(\Lambda_r\text{cl}(f^{-1}(B))) \). By Remark 2.13(ii), \( B \subseteq \Lambda_r\text{cl}(B) \subseteq f(\Lambda_r\text{cl}(f^{-1}(B))) \) and so \( f^{-1}(\Lambda_r\text{cl}(B)) \subseteq f^{-1}(f(\Lambda_r\text{cl}(f^{-1}(B)))) = \Lambda_r\text{cl}(f^{-1}(B)) \) which implies that \( f^{-1}(\Lambda_r\text{cl}(B)) \subseteq \Lambda_r\text{cl}(f^{-1}(B)) \). From this, it follows that \( \Lambda_r\text{cl}(f^{-1}(B)) = f^{-1}(\Lambda_r\text{cl}(B)) \). This proves (i).

Let \( f : X \rightarrow Y \) be a \( \Lambda_r^* \)-homeomorphism. Then by Definition 7.5 itself, \( f^{-1} \) is also a \( \Lambda_r^* \)-homeomorphism. By (i), it follows that \( \Lambda_r\text{cl}(f(B)) = f(\Lambda_r\text{cl}(B)) \) for every \( B \subseteq X \). This proves (ii).

Suppose \( f : X \rightarrow Y \) is a \( \Lambda_r^* \)-homeomorphism. Then by Theorem 2.22(v), for any subset \( B \) of \( X \), \( \Lambda_r\text{int}(B) = X \setminus \Lambda_r\text{cl}(X \setminus B) \) which implies that \( f(\Lambda_r\text{int}(B)) = f(X \setminus \Lambda_r\text{cl}(X \setminus B)) = Y \setminus f(\Lambda_r\text{cl}(X \setminus B)) \). Then by (ii), we see that \( f(\Lambda_r\text{int}(B)) = Y \setminus \Lambda_r\text{cl}(f(X \setminus B)) = \Lambda_r\text{int}(f(B)) \). This proves (iii).

Let \( f : X \rightarrow Y \) be a \( \Lambda_r^* \)-homeomorphism. Then by Definition 7.5, \( f^{-1} \) is also a \( \Lambda_r^* \)-homeomorphism. By (iii), \( f^{-1}(\Lambda_r\text{int}(B)) = \Lambda_r\text{int}(f^{-1}(B)) \) for every \( B \subseteq Y \). This proves (iv). \( \square \)
Theorem 7.8.

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are $\Lambda^*_r$-homeomorphisms, then the composition $g \circ f : X \rightarrow Z$ is also $\Lambda^*_r$-homeomorphism.

**Proof.**

Since $f$ and $g$ are $\Lambda^*_r$-homeomorphisms, by Definition 7.5, $f$ and $g$ are $\Lambda_r$-irresolute functions and so by Theorem 5.19, $g \circ f$ is $\Lambda_r$-irresolute. Again by Definition 7.5, $f^{-1}$ and $g^{-1}$ are $\Lambda_r$-irresolute and by Theorem 5.19, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is $\Lambda_r$-irresolute. By Definition 7.5, $f$ and $g$ are bijective and so $g \circ f$ is bijective. This completes the proof. □

Theorem 7.9.

The set $\Lambda^*_r H(X, \tau)$ is a group under composition of functions.

**Proof.**

Let $f, g \in \Lambda^*_r H(X, \tau)$. Then by Theorem 7.8, $f \circ g \in \Lambda^*_r H(X, \tau)$. Since $f$ is bijective, $f^{-1} \in \Lambda^*_r H(X, \tau)$. This completes the proof. □

We hereby call $\Lambda^*_r H(X, \tau)$ as a group of $\Lambda^*_r$-homeomorphisms from $(X, \tau)$ to itself. The next theorem states that the group of $\Lambda^*_r$-homeomorphisms have an isomorphism.

Theorem 7.10.

If $f : X \rightarrow X$ is a $\Lambda^*_r$-homeomorphism, then $f$ induces an isomorphism from the group $\Lambda^*_r H(X, \tau)$ onto the group $\Lambda^*_r H(X, \tau)$.

**Proof.**

Let $f \in \Lambda^*_r H(X, \tau)$. Then define a function $\psi_f : \Lambda^*_r H(X, \tau) \rightarrow \Lambda^*_r H(X, \tau)$ by $\psi_f (h) = f \circ h \circ f^{-1}$ for every $h \in \Lambda^*_r H(X, \tau)$. Let $h_1, h_2 \in \Lambda^*_r H(X, \tau)$. Then we have

\[\psi_f (h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = f \circ (h_1 \circ f^{-1} \circ f \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \psi_f (h_1) \circ \psi_f (h_2).\]

Since $\psi_f (f^{-1} \circ h \circ f) = h$, $\psi_f$ is onto. Now $\psi_f (h) = I$ implies...
That implies \( h = I \). Hence \( \psi_f \) is one-one. This shows that \( \psi_f \) is a isomorphism. \( \square \)

Some preservation theorems

**Theorem 7.11.**

Let \( f : X \to Y \) be bijective. Then the following hold:

- (i) If \( f \) is \( \Lambda_r^* \)-open and \( X \) is \( \Lambda_r \)-T₂, then \( Y \) is \( \Lambda_r \)-T₂.
- (ii) If \( f \) is continuous and \( \Lambda_r^* \)-open and \( X \) is \( \Lambda_r \)-regular, then \( Y \) is \( \Lambda_r \)-regular.
- (iii) If \( f \) is continuous and \( \Lambda_r^* \)-open and \( X \) is \( \Lambda_r \)-normal, then \( Y \) is \( \Lambda_r \)-normal.
- (iv) If \( f \) is \( \Lambda_r^* \)-open and \( Y \) is \( \Lambda_r \)-connected, then \( X \) is \( \Lambda_r \)-connected.
- (v) If \( f \) is \( \Lambda_r \)-open and \( Y \) is \( \Lambda_r \)-connected, then \( X \) is connected.
- (vi) If \( f \) is \( \Lambda_r^* \)-open, \( \Lambda_r \)-irresolute and \( Y \) is \( \Lambda_r \)-compact, then \( X \) is \( \Lambda_r \)-compact.

**Proof.**

If \( y_1, y_2 \in Y \) such that \( y_1 \neq y_2 \), then there exists \( x_1, x_2 \in X \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \) and so \( x_1 \neq x_2 \). Since \( X \) is \( \Lambda_r \)-T₂, by Definition 3.11, there exists \( \Lambda_r \)-open sets \( U \) and \( V \) in \( X \) such that \( x_1 \in U, x_2 \in V \) and \( U \cap V = \emptyset \). Since \( f \) is \( \Lambda_r^* \)-open, by Definition 5.54, \( f(U) \) and \( f(V) \) are \( \Lambda_r \)-open in \( Y \). Also we have \( y_1 = f(x_1) \in f(U), y_2 = f(x_2) \in f(V) \) and \( f(U) \cap f(V) = f(U \cap V) = \emptyset \). This shows that \( Y \) is \( \Lambda_r \)-T₂. This proves (i).

Let \( y \in Y \) and \( F \) be any closed subset of \( Y \) with \( y \notin F \). Since \( f \) is continuous, \( f^{-1}(F) \) is closed in \( X \). Since \( f \) is onto, let \( y = f(x) \), where \( x \in X \). Then \( x \notin f^{-1}(F) \). Since \( X \) is \( \Lambda_r \)-regular, by Definition 3.46, there exists \( \Lambda_r \)-open sets \( U \) and \( V \) in \( X \) such that \( x \in U, f^{-1}(F) \subseteq V \) and \( U \cap V = \emptyset \). Since \( f \) is \( \Lambda_r^* \)-open, by Definition 5.54, \( f(U) \) and \( f(V) \) are \( \Lambda_r \)-open in \( Y \). Also we have \( y \in f(U), F \subseteq f(V) \) and \( f(U) \cap f(V) = f(U \cap V) = \emptyset \). This shows that \( Y \) is \( \Lambda_r \)-regular. This proves (ii).
Let $A$ and $B$ be any two closed sets in $Y$ such that $A \cap B = \emptyset$. Since $f$ is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are closed in $X$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Since $X$ is $\Lambda_r$-normal, by Definition 3.51, there exists $\Lambda_r$-open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since $f$ is $\Lambda_r$*-open, by Definition 5.54, $f(U)$ and $f(V)$ are $\Lambda_r$-open in $Y$. Since $f$ is bijective, $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. This shows that $Y$ is $\Lambda_r$-normal. This proves (iii).

Suppose that $X$ is $\Lambda_r$-disconnected. Then by Definition 4.1, there exists a pair $A$, $B$ of nonempty disjoint $\Lambda_r$-open sets in $X$ such that $X = A \cup B$. Since $f$ is $\Lambda_r$*-open, by Definition 5.54, $f(A)$ and $f(B)$ are $\Lambda_r$-open in $Y$. Also $f(A)$ and $f(B)$ are nonempty subsets of $Y$. Since $f$ is injective and onto, $f(A) \cap f(B) = \emptyset$ and $f(X) = Y$. Thus we have $Y = f(X) = f(A \cup B) = f(A) \cup f(B)$ where $f(A)$ and $f(B)$ are nonempty disjoint $\Lambda_r$-open sets in $Y$ and so $Y$ is $\Lambda_r$-disconnected. This contradiction shows that $X$ is $\Lambda_r$-connected. This proves (iv).

Suppose that $X$ is not connected. Then $X = A \cup B$ where $A$ and $B$ are disjoint nonempty open sets in $X$. Since $f$ is $\Lambda_r$-open, $f(A)$ and $f(B)$ are $\Lambda_r$-open in $Y$. Since $f$ is bijective, $f(A) \cap f(B) = \emptyset$ and $f(X) = Y$. Therefore $Y = f(A) \cup f(B)$ where $f(A)$ and $f(B)$ are disjoint nonempty $\Lambda_r$-open sets in $Y$ and so $Y$ is $\Lambda_r$-disconnected. This contradiction shows that $X$ is connected. This proves (v).

Let \{ $A_i : i \in I$ \} be a $\Lambda_r$-open cover of $X$. Since $f$ is $\Lambda_r$*-open, by Definition 5.54, \{ $f(A_i) : i \in I$ \} is a $\Lambda_r$-open cover of $Y$. Since $Y$ is $\Lambda_r$-compact, by Definition 4.24, $Y$ has a finite subcover, say \{ $f(A_1)$, $f(A_2)$,..., $f(A_n)$ \}. Since $f$ is $\Lambda_r$-irresolute, by Definition 5.15, \{ $f^{-1}(f(A_1))$, $f^{-1}(f(A_2))$,..., $f^{-1}(f(A_n))$ \} is a finite subcover of $X$ which implies that \{ $A_1$,..., $A_n$ \} is a finite subcover of $X$ and so $X$ is $\Lambda_r$-compact. This proves (vi). \[\Box\]
Theorem 7.12.

Let \( f : X \to Y \) be injective. Then the following hold:

(i) If \( f \) is \( \Lambda_r \)-irresolute and \( Y \) is \( \Lambda_r \)-\( T_2 \), then \( X \) is \( \Lambda_r \)-\( T_2 \).

(ii) If \( f \) is \( \Lambda_r \)-irresolute and closed and \( Y \) is \( \Lambda_r \)-regular, then \( X \) is \( \Lambda_r \)-regular.

(iii) If \( f \) is \( \Lambda_r \)-irresolute and closed and \( Y \) is \( \Lambda_r \)-normal, then \( X \) is \( \Lambda_r \)-normal.

Proof.

Since \( f \) is injective, \( f(x) \neq f(y) \) for \( x, y \in X \) and \( x \neq y \). Since \( Y \) is \( \Lambda_r \)-\( T_2 \), by Definition 3.11, there exists disjoint \( \Lambda_r \)-open sets \( G \) and \( H \) in \( Y \) such that \( f(x) \in G, f(y) \in H \). Let \( U = f^{-1}(G) \) and \( V = f^{-1}(H) \). By Definition 5.15, \( U \) and \( V \) are \( \Lambda_r \)-open in \( X \). Then we obtain \( x \in f^{-1}(G) = U, y \in f^{-1}(H) = V \) and \( U \cap V = f^{-1}(G \cap H) = \emptyset \) and so by Definition 3.11, \( X \) is \( \Lambda_r \)-\( T_2 \). This proves (i).

Let \( F \) be any closed set in \( X \) and \( x \notin F \). Since \( f \) is a closed map, \( f(F) \) is closed in \( Y \) and \( f(x) \notin f(F) \). Since \( Y \) is \( \Lambda_r \)-regular, by Definition 3.46, there exists \( \Lambda_r \)-open sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U, f(F) \subseteq V \) and \( U \cap V = \emptyset \). Since \( f \) is \( \Lambda_r \)-irresolute, by Definition 5.15, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \Lambda_r \)-open in \( X \). Also we have \( x \in f^{-1}(U), F \subseteq f^{-1}(V) \) and \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). This shows that \( X \) is \( \Lambda_r \)-regular. This proves (ii).

Let \( A \) and \( B \) be any two disjoint closed subsets of \( X \). Since \( f \) is closed, \( f(A) \) and \( f(B) \) are closed in \( Y \). Since \( f \) is injective, \( f(A) \cap f(B) = f(A \cap B) = \emptyset \). Since \( Y \) is \( \Lambda_r \)-normal, by Definition 3.51, there exists \( \Lambda_r \)-open sets \( U \) and \( V \) in \( Y \) such that \( f(A) \subseteq U, f(B) \subseteq V \) and \( U \cap V = \emptyset \) which implies that \( A \subseteq f^{-1}(U), B \subseteq f^{-1}(V) \) and \( f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset \). Since \( f \) is \( \Lambda_r \)-irresolute, by Definition 5.15, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \Lambda_r \)-open in \( X \). This proves \( X \) is \( \Lambda_r \)-normal. This proves (iii). \( \square \)
Theorem 7.13.

Let \( f : X \to Y \) be surjective. Then the following hold:

(i) If \( f \) is \( \Lambda_r \)-continuous and \( X \) is \( \Lambda_r \)-connected, then \( Y \) is connected.

(ii) If \( f \) is \( \Lambda_r \)-irresolute and \( X \) is \( \Lambda_r \)-connected, then \( Y \) is \( \Lambda_r \)-connected.

(iii) If \( f \) is \( \Lambda_r \)-irresolute and \( X \) is \( \Lambda_r \)-compact, then \( Y \) is \( \Lambda_r \)-compact.

Proof.

Suppose that \( Y \) is not connected. Then there exists nonempty disjoint open sets \( A \) and \( B \) in \( Y \) such that \( Y = A \cup B \). Since \( f \) is \( \Lambda_r \)-continuous, by Definition 5.1, \( f^{-1}(A) \) and \( f^{-1}(B) \) are \( \Lambda_r \)-open in \( X \). Since \( f \) is onto, \( X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \). Thus \( X = f^{-1}(A) \cup f^{-1}(B) \) where \( f^{-1}(A) \) and \( f^{-1}(B) \) are nonempty \( \Lambda_r \)-open sets in \( X \) such that \( f^{-1}(A) \cap f^{-1}(B) = \emptyset \). By Definition 4.1, \( X \) is \( \Lambda_r \)-disconnected. This contradiction shows that \( Y \) is connected. This proves (i).

Suppose that \( Y \) is \( \Lambda_r \)-disconnected. Then by Definition 4.1, there exists a pair \( A, B \) of nonempty disjoint \( \Lambda_r \)-open sets in \( Y \) such that \( Y = A \cup B \). Since \( f \) is \( \Lambda_r \)-irresolute, by Definition 5.15, \( f^{-1}(A) \) and \( f^{-1}(B) \) are \( \Lambda_r \)-open in \( X \). Since \( f \) is onto, \( f^{-1}(A) \) and \( f^{-1}(B) \) are nonempty subsets of \( X \) and \( X = f^{-1}(Y) \). Thus we have \( X = f^{-1}(A) \cup f^{-1}(B) \) where \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint nonempty \( \Lambda_r \)-open sets in \( X \). Hence \( X \) is \( \Lambda_r \)-disconnected, which is a contradiction. Therefore \( Y \) is \( \Lambda_r \)-connected. This proves (ii).

Let \( \{A_i : i \in I\} \) be a \( \Lambda_r \)-open cover of \( Y \). Then by Definition 5.15, we have \( \{f^{-1}(A_i) : i \in I\} \) is a \( \Lambda_r \)-open cover of \( X \). Since \( X \) is \( \Lambda_r \)-compact, by Definition 4.24, \( X \) has a finite subcover, say \( \{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\} \). Since \( f \) is onto, \( \{A_1, A_2, \ldots, A_n\} \) is a finite subcover of \( Y \) and so \( Y \) is \( \Lambda_r \)-compact. This proves (iii). \( \Box \)
Theorem 7.14.

If \( f : X \rightarrow Y \) is surjective, \( \Lambda_r \)-irresolute and \( E \) is a \( \Lambda_r \)-difference set in \( Y \), then \( f^{-1}(E) \) is a \( \Lambda_r \)-difference set in \( X \).

Proof.

If \( E \) is a \( \Lambda_r \)-difference set in \( Y \), then by Definition 3.31, there exists \( \Lambda_r \)-open sets \( U \) and \( V \) in \( Y \) such that \( E = U \setminus V \) and \( U \neq Y \). By Definition 5.15, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \Lambda_r \)-open in \( X \). Since \( f \) is surjective and \( U \neq Y \), \( f^{-1}(U) \neq X \) and so \( f^{-1}(E) = f^{-1}(U) \setminus f^{-1}(V) \) is a \( \Lambda_r \)-difference set in \( X \). \( \square \)

Theorem 7.15.

If \( f : X \rightarrow Y \) is bijective, \( \Lambda_r \)-irresolute and \( Y \) is \( \Lambda_r \)-\( D_1 \), then \( X \) is \( \Lambda_r \)-\( D_1 \).

Proof.

Let \( x, y \in X \) with \( x \neq y \). Since \( f \) is injective, \( f(x) \neq f(y) \) in \( Y \). Since \( Y \) is \( \Lambda_r \)-\( D_1 \), by Definition 3.35(ii), there exists \( \Lambda_r \)-difference sets \( G_x \) and \( G_y \) in \( Y \) such that \( f(x) \in G_x \), \( f(y) \notin G_x \) and \( f(y) \in G_y \), \( f(x) \notin G_y \). By Theorem 7.14, \( f^{-1}(G_x) \) and \( f^{-1}(G_y) \) are \( \Lambda_r \)-difference sets in \( X \). Also we have \( x \in f^{-1}(G_x) \), \( y \notin f^{-1}(G_x) \) and \( y \in f^{-1}(G_y) \), \( x \notin f^{-1}(G_y) \). This shows that \( X \) is \( \Lambda_r \)-\( D_1 \). \( \square \)

Theorem 7.16.

A space \( X \) is \( \Lambda_r \)-\( D_1 \) if and only if for each pair of distinct points \( x, y \) in \( X \), there exists a \( \Lambda_r \)-irresolute surjective function \( f : X \rightarrow Y \) where \( Y \) is \( \Lambda_r \)-\( D_1 \) such that \( f(x) \neq f(y) \).

Proof.

Suppose \( X \) is a \( \Lambda_r \)-\( D_1 \) space. Let \( x, y \in X \) such that \( x \neq y \). Define \( f : X \rightarrow X \) by \( f(x) = x \), for every \( x \in X \). Then \( f \) is \( \Lambda_r \)-irresolute, surjective and \( f(x) \neq f(y) \). Conversely, suppose the condition holds. Let \( x \) and \( y \) be any pair of distinct points in \( X \). Then by hypothesis, there exists a \( \Lambda_r \)-irresolute, surjective function \( f : X \rightarrow Y \)
where $Y$ is $\Lambda_r\text{-}D_1$ such that $f(x) \neq f(y)$ which implies that $f$ is $\Lambda_r$-irresolute, bijective and $Y$ is $\Lambda_r\text{-}D_1$ and so by Theorem 7.15, $X$ is $\Lambda_r\text{-}D_1$.

\[ \square \]

\textbf{Theorem 7.17.}

If $f : X \to Y$ is a $\Lambda_r$-continuous function where $X$ is $\Lambda_r$-connected and $Y$ is a discrete space with atleast two points, then $f$ is a constant function.

\textbf{Proof.}

Let $f : X \to Y$ be a $\Lambda_r$-continuous function from a $\Lambda_r$-connected space $X$ into a discrete topological space $Y$. Then for each $y \in Y$, $\{y\}$ is both open and closed in $Y$. Since $f$ is $\Lambda_r$-continuous, by Definition 5.1 and Theorem 5.5, $f^{-1}(y)$ is both $\Lambda_r$-open and $\Lambda_r$-closed in $X$ which implies that $X$ is covered by $\Lambda_r$-open and $\Lambda_r$-closed covering $\{f^{-1}(y) : y \in Y\}$. Since $X$ is $\Lambda_r$-connected, by Theorem 4.3, $f^{-1}(y) = \emptyset$ or $X$ for each $y \in Y$. If $f^{-1}(y) = \emptyset$ for each $y \in Y$, then $f$ fails to be a map. Hence there exists only one point $y \in Y$ such that $f^{-1}(y) = X$ which shows that $f$ is a constant function.

\[ \square \]

\textbf{Theorem 7.18.}

If $f : X \to Y$ is $\Lambda_r$-irresolute and a subset $B$ of $X$ is $\Lambda_r$-compact relative to $X$, then the image $f(B)$ is $\Lambda_r$-compact relative to $Y$.

\textbf{Proof.}

Let $\{A_i : i \in I\}$ be a $\Lambda_r$-open cover of $f(B)$. Then $f(B) \subseteq \cup \{A_i : i \in I\}$ and hence $B \subseteq \cup \{f^{-1}(A_i) : i \in I\}$ where each $f^{-1}(A_i)$ is $\Lambda_r$-open in $X$ since $f$ is $\Lambda_r$-irresolute and so $\{f^{-1}(A_i) : i \in I\}$ is a $\Lambda_r$-open cover of $B$. By Definition 4.25, there exists a finite subset $I_0$ of $I$ such that $B \subseteq \cup \{f^{-1}(A_i) : i \in I_0\}$. Then we have $f(B) \subseteq \cup \{A_i : i \in I_0\}$. This shows that $f(B)$ is $\Lambda_r$-compact relative to $Y$.

\[ \square \]
Theorem 7.19.

Let \( f : X \to Y \) be a \( \Lambda_r \)-continuous function from a \( \Lambda_r \)-compact space \( X \) onto a space \( Y \). Then \( Y \) is compact.

Proof.

If \( \{ A_i : i \in I \} \) is an open cover of \( Y \), then by Definition 5.1, \( \{ f^{-1}(A_i) : i \in I \} \) is a \( \Lambda_r \)-open cover of \( X \). Since \( X \) is \( \Lambda_r \)-compact, it has a finite subcover, say \( \{ f^{-1}(A_1), f^{-1}(A_2),..., f^{-1}(A_n) \} \). Since \( f \) is onto, \( \{ A_1, A_2, ..., A_n \} \) is an open cover of \( Y \) and so \( Y \) is compact. \( \Box \)

Conclusion

The properties of \( \Lambda_r \)-homeomorphisms and \( \Lambda_r^* \)-homeomorphisms are investigated and it has been proved that the set of all \( \Lambda_r^* \)-homeomorphisms from a topological space \( X \) to itself, is a group under composition of functions. Some preservation theorems are also established in this chapter.