CHAPTER IV

Λ_r-Connectedness and Λ_r-Compactness

Connectedness and Compactness constitute the most important classes of topological spaces. In this chapter, the concepts of Λ_r-connected and Λ_r-compact spaces are introduced and their basic properties are investigated.

Λ_r-connected spaces

Definition 4.1.

A space X is said to be Λ_r-connected if there does not exist a pair A, B of nonempty disjoint Λ_r-open subsets of X such that X = A ∪ B, otherwise X is called Λ_r-disconnected. In this case, the pair (A, B) is called a Λ_r-disconnection of X.

Theorem 4.2.

Every Λ_r-connected space is connected but the converse is not true.

Proof.

Let X be a Λ_r-connected space. If possible, let X be not connected. Then there exists nonempty disjoint open sets A and B in X such that X = A ⊔ B. By Proposition 2.5(ii), A and B are Λ_r-open sets in X. By Definition 4.1, it follows that X is Λ_r-disconnected. This contradiction proves that X is connected. Converse is not true as seen in the following example. If \( X = \{a,d\} \cup \{b,c\} \), then \( \Lambda_r \text{O}(X, \tau) = \{\emptyset, \{a\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\} \). Here \( (X, \tau) \) is connected. Since \( X = \{a,d\} \cup \{b,c\} \) where \( \{a,d\} \) and \( \{b,c\} \) are two nonempty disjoint \( \Lambda_r \)-open sets in \( (X, \tau) \), X is \( \Lambda_r \)-disconnected.

Theorem 4.3.

A space X is Λ_r-connected if and only if the only subsets of X which are both Λ_r-open and Λ_r-closed are the sets X and \( \emptyset \).
Proof.

Suppose X is $\Lambda_r$-connected. Let U be both a $\Lambda_r$-open and a $\Lambda_r$-closed subset of X. Then X \ U is both $\Lambda_r$-open and $\Lambda_r$-closed. Since X is the disjoint union of $\Lambda_r$-open sets U and X \ U, by hypothesis, one of these must be empty. Hence either U = Ø or U = X.

Conversely, suppose the only subsets of X which are both $\Lambda_r$-open and $\Lambda_r$-closed are the sets X and Ø. If X is $\Lambda_r$-disconnected, then by Definition 4.1, there exists nonempty disjoint $\Lambda_r$-open sets A and B in X such that X = A \ B. Since B = X \ A is $\Lambda_r$-open, A is both $\Lambda_r$-open and $\Lambda_r$-closed and so by hypothesis, A = Ø or X which implies that either A = Ø or B = Ø. This contradiction shows that X is $\Lambda_r$-connected. □

Theorem 4.4 characterizes $\Lambda_r$-connectedness in terms of $\Lambda_r$-frontier.

Theorem 4.4.

A space X is $\Lambda_r$-connected if and only if every nonempty proper subset of X has a nonempty $\Lambda_r$-frontier.

Proof.

Suppose X is $\Lambda_r$-connected and A is a proper nonempty subset of X. If possible, let $\Lambda_r Fr(A) = \emptyset$. Then by Theorem 2.31(v), $\Lambda_r cl(A) \cap \Lambda_r cl(X \setminus A) = \emptyset$ which implies that $\Lambda_r cl(A) \subseteq X \setminus (\Lambda_r cl(X \setminus A))$. By applying Theorem 2.22(viii), it follows that $\Lambda_r cl(A) \subseteq \Lambda_r int(A)$. Since $\Lambda_r int(A) \subseteq A \subseteq \Lambda_r cl(A)$, $\Lambda_r int(A) = A = \Lambda_r cl(A)$ and so by Theorem 2.22(vi) and Proposition 2.11(iv), A is both $\Lambda_r$-open and $\Lambda_r$-closed. By Theorem 4.3, X is $\Lambda_r$-disconnected. This contradiction proves that A has a nonempty $\Lambda_r$-frontier.

Conversely, suppose every nonempty proper subset of X has a nonempty $\Lambda_r$-frontier. If X is $\Lambda_r$-disconnected, then by Definition 4.1, there exists nonempty
disjoint \( \Lambda_r \)-open sets \( A \) and \( B \) in \( X \) such that \( X = A \cup B \). Since \( B = X \setminus A \) is \( \Lambda_r \)-open, \( A \) is both \( \Lambda_r \)-open and \( \Lambda_r \)-closed. By Theorem 2.22(vi), \( A = \Lambda_r \text{int}(A) \) and by Proposition 2.11(iv), \( A = \Lambda_r \text{cl}(A) \). Then by applying Theorem 2.31(v) and Theorem 2.22(v), \[ \Lambda_r \text{Fr}(A) = \Lambda_r \text{cl}(A) \cap \Lambda_r \text{cl}(X \setminus A) = A \cap (X \setminus \Lambda_r \text{int}(A)) = A \cap (X \setminus A) = \emptyset \] which implies that \( A \) has empty \( \Lambda_r \)-frontier. This contradiction shows that \( X \) is \( \Lambda_r \)-connected. \( \square \)

**Definition 4.5.**

Let \((X, \tau)\) be a space and \( A \) be a subset of \( X \). Then the class of \( \Lambda_r \)-open sets in \( A \) is defined by \( \Lambda_r O(A) = \{G \subseteq X : G = A \cap O \text{ and } O \text{ is } \Lambda_r \text{-open in } (X, \tau)\} \). That is, \( G \) is \( \Lambda_r \)-open in \( A \) if and only if \( G = A \cap O \) where \( O \) is \( \Lambda_r \)-open in \( (X, \tau) \).

**Definition 4.6.**

A subset \( B \) of a space \( X \) is \( \Lambda_r \)-connected if there does not exist a pair \( B_1, B_2 \) of nonempty disjoint \( \Lambda_r \)-open sets in \( B \) such that \( B = B_1 \cup B_2 \), otherwise \( B \) is called \( \Lambda_r \)-disconnected. In this case, the pair \( (B_1, B_2) \) is called a \( \Lambda_r \)-disconnection of \( B \).

**Theorem 4.7.**

Let \( X \) be a \( \Lambda_r \)-disconnected space and \( C \) be a \( \Lambda_r \)-connected subset of \( X \). If \( (A, B) \) is a \( \Lambda_r \)-disconnection of \( X \), then \( C \) is contained in \( A \) or in \( B \).

**Proof.**

Suppose that \( C \) is neither contained in \( A \) nor in \( B \). Since \( (A, B) \) is a \( \Lambda_r \)-disconnection of \( X \), by Definition 4.1, \( A \) and \( B \) are nonempty disjoint \( \Lambda_r \)-open sets in \( X \) such that \( X = A \cup B \). By Definition 4.5, \( C \cap A \) and \( C \cap B \) are both nonempty \( \Lambda_r \)-open sets in \( C \). Then we obtain that \( (C \cap A) \cap (C \cap B) = \emptyset \) and \( (C \cap A) \cup (C \cap B) = C \). This gives that \( (C \cap A, C \cap B) \) is a \( \Lambda_r \)-disconnection of \( C \) which implies \( C \) is \( \Lambda_r \)-disconnected. This contradiction proves the theorem. \( \square \)
**Theorem 4.8.**

Let $X$ be a space and $\{X_\alpha\}, \alpha \in I$ be a $\Lambda_r$-connected subsets of $X$ such that 

$$X = \bigcup_{\alpha \in I} \{X_\alpha\} \text{ and } \bigcap_{\alpha \in I} \{X_\alpha\} \neq \emptyset.$$ 

Then $X$ is $\Lambda_r$-connected.

**Proof.**

Suppose $X$ is $\Lambda_r$-disconnected and $(A, B)$ is a $\Lambda_r$-disconnection of $X$. Since each $X_\alpha$ is $\Lambda_r$-connected, by Theorem 4.7, $X_\alpha \subseteq A$ or $X_\alpha \subseteq B$. Since $\bigcap_{\alpha \in I} \{X_\alpha\} \neq \emptyset$, all $X_\alpha$ are contained in $A$ or in $B$. This gives that $X \subseteq A$ or $X \subseteq B$. If $X \subseteq A$, then $B = \emptyset$ or if $X \subseteq B$, then $A = \emptyset$. This contradiction proves that $X$ is $\Lambda_r$-connected. \qed

**Theorem 4.9.**

A space $X$ is $\Lambda_r$-connected if and only if for every pair of points $x, y$ in $X$, there is a $\Lambda_r$-connected subset of $X$ which contains both $x$ and $y$.

**Proof.**

Suppose $X$ is a $\Lambda_r$-connected space. Since $X$ itself contains these two points, it is obvious.

Conversely, suppose that for any two points $x$ and $y$ of $X$, there is a $\Lambda_r$-connected subset $C_{xy}$ of $X$ such that $x, y \in C_{xy}$. Let $a \in X$ be a fixed point and $\{C_{a,x} : x \in X\}$ be a class of all $\Lambda_r$-connected subsets of $X$ which contain the points $a$ and $x$. Then $X = \bigcup_{x \in X} \{C_{a,x}\}$ and $\bigcap_{x \in X} \{C_{a,x}\} \neq \emptyset$. By Theorem 4.8, $X$ is $\Lambda_r$-connected. \qed

**Definition 4.10.**

Let $X$ be a space and $A$ be a $\Lambda_r$-connected subset of $X$. Then $A$ is said to be maximal $\Lambda_r$-connected if there exists no $\Lambda_r$-connected subset $B$ of $X$ such that $A \subseteq B$. A maximal $\Lambda_r$-connected subset of a space $X$ is called a $\Lambda_r$-component of $X$. 
Remark 4.11.

If a space $X$ itself $\Lambda_r$-connected, then $X$ is the only $\Lambda_r$-component of $X$.

Theorem 4.12.

Each $\Lambda_r$-connected subset of a space $X$ is contained in exactly one $\Lambda_r$-component of $X$.

Proof.

Let $A$ be a $\Lambda_r$-connected subset of a space $X$ which is not a $\Lambda_r$-component of $X$. Suppose $C_1$ and $C_2$ are the two $\Lambda_r$-components of $X$ such that $A \subseteq C_1$ and $A \subseteq C_2$. Then $C_1 \cap C_2 \neq \emptyset$. Put $C = C_1 \cup C_2$. By Theorem 4.8, $C$ is a $\Lambda_r$-connected set which contains $C_1$ as well as $C_2$, a contradiction to the fact that $C_1$ and $C_2$ are $\Lambda_r$-components. This contradiction shows that $A$ is contained in exactly one $\Lambda_r$-component of $X$. \qed

Theorem 4.13.

Let $X$ be a space. Then for each $x \in X$, there is exactly one $\Lambda_r$-component of $X$ containing $x$.

Proof.

Let $x \in X$ and $\{C_\alpha : \alpha \in I\}$ a class of all $\Lambda_r$-connected subsets of $X$ containing $x$. Put $C = \bigcup_{\alpha \in I} C_\alpha$. Then by Theorem 4.8, $C$ is $\Lambda_r$-connected and $x \in C$. By Theorem 4.12, $C$ is contained in exactly one $\Lambda_r$-component of $X$. Let $C^*$ be the $\Lambda_r$-component of $X$ such that $C \subseteq C^*$. Then $x \in C^*$ and hence $C^*$ is one of the $C_\alpha$’s which implies $C^* \subseteq C$. Consequently, $C = C^*$. This proves that $C$ is the only $\Lambda_r$-component of $X$ which contains $x$. \qed

Next we give some examples of $\Lambda_r$-connected spaces which are independent with the separation axioms $\Lambda_r$-$T_0$, $\Lambda_r$-$T_1$, $\Lambda_r$-$T_2$, $\Lambda_r$-$R_0$, $\Lambda_r$-$R_1$, $\Lambda_r$-$D_0$, $\Lambda_r$-$D_1$, $\Lambda_r$-$D_2$, $\Lambda_r$-regular and $\Lambda_r$-normal.
Example 4.14.

If $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, then $\Lambda_rO(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Here $(X, \tau)$ is $\Lambda_rT_0$ but $\Lambda_r$-disconnected. If $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{c\}, X\}$, then $\Lambda_rO(X, \tau) = \tau$. Here $(X, \tau)$ is $\Lambda_r$-connected but not $\Lambda_rT_0$. This shows that the concepts of $\Lambda_rT_0$ and $\Lambda_r$-connected spaces are independent.

Example 4.15.

If $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$, then $\Lambda_rO(X, \tau) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}, X\}$. Here $(X, \tau)$ is $\Lambda_rT_1$ but $\Lambda_r$-disconnected. If $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, X\}$, then $\Lambda_rO(X, \tau) = \tau$. Here $(X, \tau)$ is $\Lambda_r$-connected but not $\Lambda_rT_1$. This shows that the concepts of $\Lambda_rT_1$ and $\Lambda_r$-connected spaces are independent.

Example 4.16.

If $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, then $\Lambda_rO(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b\}, X\}$. Here $(X, \tau)$ is $\Lambda_rT_2$ but $\Lambda_r$-disconnected. If $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$, then $\Lambda_rO(X, \tau) = \tau$. Here $(X, \tau)$ is $\Lambda_r$-connected but not $\Lambda_rT_2$. This shows that the concepts of $\Lambda_rT_2$ and $\Lambda_r$-connected spaces are independent.

Example 4.17.

If $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, then $\Lambda_rO(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, X\}$. Here $(X, \tau)$ is $\Lambda_rR_0$ but $\Lambda_r$-disconnected. If $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, X\}$, then $\Lambda_rO(X, \tau) = \tau$. Here $(X, \tau)$ is $\Lambda_r$-connected but not $\Lambda_rR_0$. This shows that the concepts of $\Lambda_rR_0$ and $\Lambda_r$-connected spaces are independent.
Example 4.18.

If \( X = \{a,b,c\} \) and \( \tau = \{\emptyset,\{b\},\{a,c\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-\(R_1\) but \( \Lambda_r\)-disconnected. If \( X = \{a,b,c\} \) and \( \tau = \{\emptyset,\{c\},\{b,c\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-connected but not \( \Lambda_r\)-\(R_1\). This shows that the concepts of \( \Lambda_r\)-\(R_1\) and \( \Lambda_r\)-connected spaces are independent.

Example 4.19.

If \( X = \{a,b,c\} \) and \( \tau = \{\emptyset,\{a\},\{a,b\},\{a,c\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-\(D_0\) but \( \Lambda_r\)-disconnected. If \( X = \{a,b,c,d,e\} \) and \( \tau = \{\emptyset,\{a\},\{a,b\},\{a,c,d\},\{a,b,d\},\{a,b,c,d\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-connected but not \( \Lambda_r\)-\(D_0\). This shows that the concepts of \( \Lambda_r\)-\(D_0\) and \( \Lambda_r\)-connected spaces are independent.

Example 4.20.

If \( X = \{a,b,c\} \) and \( \tau = \{\emptyset,\{a\},\{a,b\},\{b,c\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-\(D_1\) but \( \Lambda_r\)-disconnected. If \( X = \{a,b,c,d\} \) and \( \tau = \{\emptyset,\{a\},\{a,c\},\{a,b,d\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-connected but not \( \Lambda_r\)-\(D_1\). This shows that the concepts of \( \Lambda_r\)-\(D_1\) and \( \Lambda_r\)-connected spaces are independent.

Example 4.21.

If \( X = \{a,b,c\} \) and \( \tau = \{\emptyset,\{a\},\{c\},\{a,c\},\{a,b\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-\(D_2\) but \( \Lambda_r\)-disconnected. If \( X = \{a,b\} \) and \( \tau = \{\emptyset,\{a\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-connected but not \( \Lambda_r\)-\(D_2\). This shows that the concepts of \( \Lambda_r\)-\(D_2\) and \( \Lambda_r\)-connected spaces are independent.

Example 4.22.

If \( X = \{a,b,c,d\} \) and \( \tau = \{\emptyset,\{a\},\{b,c,d\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-regular but \( \Lambda_r\)-disconnected. If \( X = \{a,b,c,d\} \) and \( \tau = \{\emptyset,\{b,d\},\{b,c,d\},\{a,b,d\},X\} \), then \( \Lambda_r O(X,\tau) = \tau \). Here \((X,\tau)\) is \( \Lambda_r\)-connected but not
Λₜ-regular. This shows that the concepts of Λₜ-regular and Λₜ-connected spaces are independent.

**Example 4.23.**

If \( X = \{a,b,c,d\} \) and \( \tau = \{\emptyset,\{a\},\{b,c\},\{a,b,c\},X\} \), then \( \Lambda_{o}(X, \tau) = \{\emptyset, \{a\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\} \). Here \((X, \tau)\) is Λₜ-normal but Λₜ-disconnected.

If \( X = \{a,b,c,d\} \) and \( \tau = \{\emptyset,\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X\} \), then \( \Lambda_{o}(X, \tau) = \tau \). Here \((X, \tau)\) is Λₜ-connected but not Λₜ-normal. This shows that the concepts of Λₜ-normal and Λₜ-connected spaces are independent.

From the above examples, we get the following diagram

\[
\begin{array}{c}
\Lambda_{t}\text{-regular} \\
\Lambda_{t}\text{-normal} \\
\Lambda_{t}\text{-R}_{1} \\
\Lambda_{t}\text{-T}_{0} \\
\Lambda_{t}\text{-connected} \\
\Lambda_{t}\text{-R}_{0} \\
\Lambda_{t}\text{-T}_{1} \\
\Lambda_{t}\text{-R}_{2} \\
\Lambda_{t}\text{-D}_{0} \\
\Lambda_{t}\text{-D}_{1} \\
\Lambda_{t}\text{-D}_{2}
\end{array}
\]

In this diagram, \( A \leftrightarrow B \) means \( A \) and \( B \) are independent to each other.

**Λₜ-compact spaces**

**Definition 4.24.**

A space \( X \) is said to be Λₜ-compact if every Λₜ-open cover of \( X \) has a finite subcover.

**Definition 4.25.**

A subset \( B \) of a space \( X \) is said to be Λₜ-compact relative to \( X \) if for every cover \( \{A_{i} : i \in I\} \) of \( B \) by Λₜ-open subsets of \( X \), there exists a finite subset \( I_0 \) of \( I \) such that \( B \subseteq \bigcup \{A_{i} : i \in I_0\} \).

A space $X$ is $\Lambda_r$-compact if and only if for every family $\{F_\alpha : \alpha \in I\}$ of $\Lambda_r$-closed sets with finite intersection property, $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.

Proof.

Suppose $X$ is a $\Lambda_r$-compact space. Let $\{F_\alpha : \alpha \in I\}$ be a family of $\Lambda_r$-closed subsets of $X$ with finite intersection property such that $\bigcap_{\alpha \in I} F_\alpha = \emptyset$. Let us consider the $\Lambda_r$-open sets $U_\alpha = X \setminus F_\alpha$. Then $\bigcup \{U_\alpha : \alpha \in I\} = \bigcup \{X \setminus F_\alpha : \alpha \in I\} = X \setminus \bigcap \{F_\alpha : \alpha \in I\} = X \setminus \emptyset = X$. Hence $\{U_\alpha : \alpha \in I\}$ is a $\Lambda_r$-open cover of $X$.

Since $X$ is $\Lambda_r$-compact, by Definition 4.24, it has a finite subcover $\{U_{\alpha_i} : \alpha_i \in I_0\}$. Then we have $X = \bigcup \{U_{\alpha_i} : \alpha_i \in I_0\} = \bigcup \{X \setminus F_{\alpha_i} : \alpha_i \in I_0\} = X \setminus \bigcap \{F_{\alpha_i} : \alpha_i \in I_0\}$ which implies $\bigcap \{F_{\alpha_i} : \alpha_i \in I_0\} = \emptyset$, a contradiction to the fact that $\{F_\alpha : \alpha \in I\}$ satisfies finite intersection property. Thus, if the family $\{F_\alpha : \alpha \in I\}$ of $\Lambda_r$-closed sets with finite intersection property, then $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.

Conversely, suppose for every family $\{F_\alpha : \alpha \in I\}$ of $\Lambda_r$-closed sets with finite intersection property, $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$. Let $\{U_\alpha : \alpha \in I\}$ be a $\Lambda_r$-open cover of $X$. Then $X = \bigcup \{U_\alpha : \alpha \in I\}$ and $\emptyset = X \setminus \bigcup \{U_\alpha : \alpha \in I\} = X \setminus \bigcap \{X \setminus U_\alpha : \alpha \in I\}$ which implies that $\{X \setminus U_\alpha : \alpha \in I\}$ is a family of $\Lambda_r$-closed sets with an empty intersection. By hypothesis, there exists a finite subfamily $\{X \setminus U_{\alpha_i} : \alpha_i \in I_0\}$ such that $\bigcap \{X \setminus U_{\alpha_i} : \alpha_i \in I_0\} = \emptyset$. Then we have $X = X \setminus \bigcup \{X \setminus U_{\alpha_i} : \alpha_i \in I_0\}$ implies $X \setminus U_{\alpha_i} \in I_0)$ implies $X = \bigcup \{X \setminus U_{\alpha_i} : \alpha_i \in I_0\}$. Thus $\{U_{\alpha_i} : \alpha_i \in I_0\}$ is a finite subcover of $X$. This shows that $X$ is $\Lambda_r$-compact. □
Theorem 4.27.

A space $X$ is $\Lambda_r$-compact if and only if every proper $\Lambda_r$-closed subset of $X$ is $\Lambda_r$-compact relative to $X$.

Proof.

Suppose $X$ is a $\Lambda_r$-compact space. Let $A$ be any proper $\Lambda_r$-closed subset of $X$. Then $X \setminus A$ is $\Lambda_r$-open in $X$. Suppose $\{G_\alpha : \alpha \in I\}$ is a $\Lambda_r$-open cover of $A$. Then $\{G_\alpha : \alpha \in I\} \cup (X \setminus A)$ is a $\Lambda_r$-open cover of $X$. Since $X$ is $\Lambda_r$-compact, by Definition 4.24, it has a finite subcover, say $\{G_1, G_2, \ldots, G_n\}$. If this subcover contains $X \setminus A$, we discard it. Otherwise leave the subcover as it is. Thus we have obtained a finite $\Lambda_r$-open subcover of $A$ and so by Definition 4.25, $A$ is $\Lambda_r$-compact relative to $X$.

Conversely, suppose every proper $\Lambda_r$-closed subset of $X$ is $\Lambda_r$-compact relative to $X$. Let $\{V_\alpha : \alpha \in I\}$ be a $\Lambda_r$-open cover of $X$. Then $X = \bigcup \{V_\alpha : \alpha \in I\}$. We fix one $\alpha_0 \in I$. Then $X \setminus V_{\alpha_0}$ is a proper $\Lambda_r$-closed subset of $X$ and $X \setminus V_{\alpha_0} \subseteq \bigcup \{V_\alpha : \alpha \in I \setminus \{\alpha_0\}\}$. Hence $\{V_\alpha : \alpha \in I \setminus \{\alpha_0\}\}$ is a $\Lambda_r$-open cover of $X \setminus V_{\alpha_0}$. By the hypothesis, there exists a finite subset $I_0$ of $I \setminus \{\alpha_0\}$ such that $X \setminus V_{\alpha_0} \subseteq \{V_\alpha : \alpha \in I_0\}$. Then $X \subseteq \bigcup \{V_\alpha : \alpha \in I_0 \cup \{\alpha_0\}\}$. This shows that $X$ is $\Lambda_r$-compact.

$\square$

Conclusion

$\Lambda_r$-connected subsets and $\Lambda_r$-compact subsets of a space $X$ relative to the given topology are characterized. For example, a space $X$ is $\Lambda_r$-connected if and only if the only subsets of $X$ which are both $\Lambda_r$-open and $\Lambda_r$-closed are the sets $X$ and $\emptyset$. Also a space $X$ is $\Lambda_r$-compact if and only if every proper $\Lambda_r$-closed subset of $X$ is $\Lambda_r$-compact relative to $X$. 