Chapter - II

BIANCHI TYPE-I MESONIC COSMOLOGICAL MODEL IN FIVE DIMENSIONAL BIMETRIC THEORY OF RELATIVITY*

* Communicated

2.1 Introduction

There were several attempts to modify the general relativity but with little success. However, Rosen (1940, 54, 63, 66, 70, 73, 74, 75, 78, 79 80, 83, 85, 89) proposed the bimetric theory of relativity where two metric tensors at each point of the space-time are defined : a Riemannian metric g_{ij} and the background flat space-time metric f_{ij}. The tensor g_{ij} describes the geometry of the curved space-time and the f_{ij} refers to inertial forces. The metric f_{ij} enters into the field equations and interacts with g_{ij} but has no direct interaction with matter. Accordingly, at each space-time point, one has two line elements

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{2.1.1}$$

and

$$d\sigma^2 = f_{ij}\, dx^i\, dx^j \tag{2.1.2}$$

The field equations of bimetric theory of gravitation formulated by Rosen are

$$N_{ij} - \frac{1}{2} N\, g_{ij} = -8\pi\, \kappa\, T_{ij} \quad , \tag{2.1.3}$$

where

$$N^i_j = \frac{1}{2} f^{ab} (g^{hi} g_{hj|a})_{|b} \quad ,$$

that is

$$N^i_j = \frac{1}{2} f^{ab} \{ (g^{hi} g_{hj,a})_{,b} - (g^{hi} g_{mj} \Gamma^m_{ha})_{,b} - (\Gamma^i_{ja})_{,b}$$

$$+ \Gamma^i_{\lambda b} (g^{h\lambda} g_{hj,a} - g^{h\lambda} g_{mj} \Gamma^m_{ha} - \Gamma^\lambda_{ja})$$

$$- \Gamma^\lambda_{jb} (g^{hi} g_{h\lambda,a} - g^{hi} g_{m\lambda} \Gamma^m_{ha} - \Gamma^i_{\lambda a})$$

$$- \Gamma^\lambda_{ab} (g^{hi} g_{hj,\lambda} - g^{hi} g_{mj} \Gamma^m_{h\lambda} - \Gamma^i_{j\lambda}) \}$$

$$N = N^i_i \quad , \qquad f = \det(f_{ij})$$

and

$$\kappa = \left(\frac{g}{f} \right)^{1/2} ,$$

in which T_{ij} is the usual stress tensor of the matter and a vertical bar (|) denotes the covariant differentiation with respect to f_{ij}.

Though there finds considerable work exhibiting several aspects of the bimetric theory of gravitation, we hold the view that the investigation is not yet complete and there is a scope of further work which may unravel some of the hidden secrets of the theory.

In the recent year there find interest in higher dimensional space-time model when some physical ideas are involved. The idea that the work may have more than four dimension is due to Kaluza et al. (1921) who with a brilliant insight realized that a five dimension manifold could be used to unify Einstein theory of relativity with Maxwell theory of electromagnetism. After some delay, Einstein indorse the idea, but a major impetus was provided by Klein (1926). Therefore in this chapter we have taken up the idea of five dimensional space-time for homogeneous Bianchi type-I mesonic cosmological model in bimetric theory of relativity and shown that this model does not exist in case of both mesonic field and mesonic perfect fluid (with or without mass parameter). Hence only vacuum model can be constructed. This result is an extension in five dimension of the result obtained earlier by Mohanty et al. (2002) for four dimensional space-time.

2.2 Scalar meson field

We consider here the five dimensional space-time described by Bianchi type-I metric

$$ds^2 = -dt^2 + A^2 dx_1^2 + B^2 dx_2^2 + C^2 dx_3^2 + D^2 dx_4^2 \quad , \tag{2.2.1}$$

where A, B, C and D are functions of time t only.

The flat metric corresponding to (2.2.1) is

$$d\sigma^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \quad . \tag{2.2.2}$$

We consider the region of the space-time consisting attractive massive scalar menson field whose energy momentum tensor is given by

$$T_{ij}^s = V_i V_j - \frac{1}{2} g_{ij} (V_k V^k - m^2 V^2) \tag{2.2.3}$$

together with the Klein-Gordon equations

$$\sigma = g^{ij} V_{;ij} + m^2 V , \tag{2.2.4}$$

where m is the mass parameter and σ is the source density of the scalar meson field V.

From equation (2.2.3), we obtain

$$T_{ij}^{s} = V_i\, V_j - \frac{1}{2} g_{ij} (V_k V_l\, g^{kl}) + \frac{1}{2} g_{ij}\, m^2 V^2$$

$$T_{00} = V_0^2 - \frac{1}{2} g_{00} (V_k\, V_l\, g^{kl}) + \frac{1}{2} g_{00} m^2 V^2$$

$$T_0^0 = V_0^2 g^{00} - \frac{1}{2} (V_k V_l g^{kl}) + \frac{1}{2} m^2 V^2$$

$$= -V_0^2 - \frac{1}{2} (V_0^2 g^{00} + V_1^2 g^{11} + V_2^2 g^{22} + V_3^2 g^{33} + V_4^2 g^{44}) + m^2 V^2$$

$$= \frac{1}{2} (m^2 V^2 - \dot{V}^2)$$

Similarly

$$T_1^1 = T_2^2 = T_3^3 = T_4^4 = \frac{1}{2} (m^2 V^2 + \dot{V}^2) \quad .$$

Hence

$$T_1^1 = T_2^2 = T_3^3 = T_4^4 = \frac{1}{2} (m^2 V^2 + \dot{V}^2), \quad T_0^0 = \frac{1}{2} (m^2 V^2 - \dot{V}^2) \qquad (2.2.5)$$

The field equations (2.1.3) for the metric (2.2.1) with the energy momentum tensor (2.2.3) can be written as

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}+\left(\frac{\dot{B}}{B}\right)^{\cdot}+\left(\frac{\dot{C}}{C}\right)^{\cdot}+\left(\frac{\dot{D}}{D}\right)^{\cdot}=-8\pi\kappa(m^2V^2-\dot{V}^2) \tag{2.2.6}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}-\left(\frac{\dot{B}}{B}\right)^{\cdot}-\left(\frac{\dot{C}}{C}\right)^{\cdot}-\left(\frac{\dot{D}}{D}\right)^{\cdot}=8\pi\kappa(m^2V^2+\dot{V}^2) \tag{2.2.7}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}-\left(\frac{\dot{B}}{B}\right)^{\cdot}+\left(\frac{\dot{C}}{C}\right)^{\cdot}+\left(\frac{\dot{D}}{D}\right)^{\cdot}=-8\pi\kappa(m^2V^2+\dot{V}^2) \tag{2.2.8}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}+\left(\frac{\dot{B}}{B}\right)^{\cdot}-\left(\frac{\dot{C}}{C}\right)^{\cdot}+\left(\frac{\dot{D}}{D}\right)^{\cdot}=-8\pi\kappa(m^2V^2+\dot{V}^2) \tag{2.2.9}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}+\left(\frac{\dot{B}}{B}\right)^{\cdot}+\left(\frac{\dot{C}}{C}\right)^{\cdot}-\left(\frac{\dot{D}}{D}\right)^{\cdot}=-8\pi\kappa(m^2V^2+\dot{V}^2)\quad, \tag{2.2.10}$$

where dot (.) denotes derivative with respect to time t only.

For the detailed calculations please refer to the Appendix [2.2.1].

Klein-Gordon equations (2.2.4) for the metric (2.2.1) can be written as

$$\sigma = g^{ij}\left(\frac{\partial V_i}{\partial x^j} - V_{,\alpha}\,\Gamma^{\alpha}_{ij}\right) + m^2 V$$

$$= g^{ij}\left(\frac{\partial^2 V}{\partial x^i \partial x^j} - V_\alpha\,\Gamma^{\alpha}_{ij}\right) + m^2 V$$

$$= g^{00}\left(\frac{\partial^2 V}{\partial x^0 \partial x^0} - V_\alpha\,\Gamma^{\alpha}_{00}\right) + g^{11}\left(\frac{\partial^2 V}{\partial x^1 \partial x^1} - V_\alpha\,\Gamma^{\alpha}_{11}\right) + g^{22}\left(\frac{\partial^2 V}{\partial x^2 \partial x^2} - V_\alpha\,\Gamma^{\alpha}_{22}\right)$$
$$+ g^{33}\left(\frac{\partial^2 V}{\partial x^3 \partial x^3} - V_\alpha\,\Gamma^{\alpha}_{33}\right) + g^{44}\left(\frac{\partial^2 V}{\partial x^4 \partial x^4} - V_\alpha\,\Gamma^{\alpha}_{44}\right) + m^2 V$$

$$= -(\ddot{V} - V_0\Gamma^0_{00}) - \frac{1}{A^2}V_0\Gamma^0_{11} - \frac{1}{B^2}V_0\Gamma^0_{22} - \frac{1}{C^2}V_0\Gamma^0_{33} - \frac{1}{D^2}V_0\Gamma^0_{44} + m^2 V$$

$$= -\ddot{V} - \frac{1}{A^2}\dot{V}(A\dot{A}) - \frac{1}{B^2}\dot{V}(B\dot{B}) - \frac{1}{C^2}\dot{V}(C\dot{C}) - \frac{1}{D^2}\dot{V}(D\dot{D}) + m^2 V$$

$$= -\ddot{V} - \dot{V}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right) + m^2 V\ .$$

Equations (2.2.7) – (2.2.10) reduce to

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} = \left(\frac{\dot{B}}{B}\right)^{\cdot} = \left(\frac{\dot{C}}{C}\right)^{\cdot} = \left(\frac{\dot{D}}{D}\right)^{\cdot}. \tag{2.2.11}$$

With the help of equation (2.2.11), equations (2.2.6) and (2.2.10) reduce to

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} = -2\pi\kappa(m^2 V^2 - \dot{V}^2) \tag{2.2.12}$$

and

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} = -4\pi\kappa(m^2V^2 + \dot{V}^2) \; . \tag{2.2.13}$$

Using equations (2.2.12) and (2.2.13), we get

$$2(m^2V^2 + \dot{V}^2) = m^2V^2 - \dot{V}^2$$

$$\Rightarrow \quad 2m^2V^2 + 2\dot{V}^2 - m^2V^2 + \dot{V}^2 = 0$$

$$\Rightarrow \quad m^2V^2 + 3\dot{V}^2 = 0 \tag{2.2.14}$$

$$\Rightarrow \quad m^2V^2 = 0 \text{ and } \dot{V}^2 = 0$$

$$\Rightarrow \quad m = 0 \text{ or } V = 0 \text{ and } V = \text{constant.}$$

Therefore equation (2.2.14) implies that either $V = 0$ or $m = 0$ and $V =$ constant.

In both the cases, the source density vanishes and mesonic scalar field do not exist.

Thus Bianchi type-I cosmological model in five dimensional bimetric theory do not exist in case of scalar meson field. In consequence of which we arrive at a theorem :

"In Rosen bimetric relativity the only possible higher dimensional solution of Bianchi type – I cosmological model representing scalar meson field – is a vacuum solution".

2.3 Five dimensional perfect fluid distribution coupled with massive scalar field

We assume the convensional form of perfect fluid distribution and energy tensor for massive meson scalar field as

$$T_{ij} = T_{ij}^{p} + T_{ij}^{s} , \tag{2.3.1}$$

where

$$T_{ij}^{p} = (\rho + p)\, u_i u_j + p g_{ij} \tag{2.3.2}$$

together with

$$g_{ij}\, u^i\, u^j = -1 ,$$

where u^i is the five velocity vector and ρ, p are energy density and proper pressure of the fluid respectively.

We use co-moving coordinates so that

$$u^1 = u^2 = u^3 = u^4 = 0 \text{ and } u^0 = -1 .$$

Then equation (2.3.2) implies

$$T_1^1 = T_2^2 = T_3^3 = T_4^4 = p, \quad T_0^0 = -\rho . \tag{2.3.3}$$

In this case we find

$$T_j^i = 0 \qquad (i \neq j) .$$

The field equations (2.1.3) for the metric (2.2.1) corresponding to the energy-momentum tensor (2.3.1) can be written as

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}+\left(\frac{\dot{B}}{B}\right)^{\cdot}+\left(\frac{\dot{C}}{C}\right)^{\cdot}+\left(\frac{\dot{D}}{D}\right)^{\cdot}=16\pi\kappa\left[\rho-\frac{1}{2}(m^2V^2-\dot{V}^2)\right] \tag{2.3.4}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}-\left(\frac{\dot{B}}{B}\right)^{\cdot}-\left(\frac{\dot{C}}{C}\right)^{\cdot}-\left(\frac{\dot{D}}{D}\right)^{\cdot}=16\pi\kappa\left[p+\frac{1}{2}(m^2V^2+\dot{V}^2)\right] \tag{2.3.5}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}-\left(\frac{\dot{B}}{B}\right)^{\cdot}+\left(\frac{\dot{C}}{C}\right)^{\cdot}+\left(\frac{\dot{D}}{D}\right)^{\cdot}=-16\pi\kappa\left[p+\frac{1}{2}(m^2V^2+\dot{V}^2)\right] \tag{2.3.6}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}+\left(\frac{\dot{B}}{B}\right)^{\cdot}-\left(\frac{\dot{C}}{C}\right)^{\cdot}+\left(\frac{\dot{D}}{D}\right)^{\cdot}=-16\pi\kappa\left[p+\frac{1}{2}(m^2V^2+\dot{V}^2)\right] \tag{2.3.7}$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}+\left(\frac{\dot{B}}{B}\right)^{\cdot}+\left(\frac{\dot{C}}{C}\right)^{\cdot}-\left(\frac{\dot{D}}{D}\right)^{\cdot}=-16\pi\kappa\left[p+\frac{1}{2}(m^2V^2+\dot{V}^2)\right]. \tag{2.3.8}$$

Equations (2.3.5) – (2.3.8) reduce to

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}=\left(\frac{\dot{B}}{B}\right)^{\cdot}=\left(\frac{\dot{C}}{C}\right)^{\cdot}=\left(\frac{\dot{D}}{D}\right)^{\cdot}. \tag{2.3.9}$$

With the help of equation (2.3.9), equations (2.3.4) and (2.3.8) reduce to

$$\left(\frac{\dot{A}}{A}\right)^{\cdot}=4\pi\kappa\left[\rho-\frac{1}{2}(m^2V^2-\dot{V}^2)\right] \tag{2.3.10}$$

and

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} = -8\pi\kappa\left[p + \frac{1}{2}(m^2V^2 + \dot{V}^2)\right] . \tag{2.3.11}$$

Using equations (2.3.10) and (2.3.11), we get

$$\rho - \frac{1}{2}(m^2V^2 - \dot{V}^2) = -2\left[p + \frac{1}{2}(m^2V^2 + \dot{V}^2)\right]$$

$$\Rightarrow \quad \rho - \frac{1}{2}m^2V^2 + \frac{1}{2}\dot{V}^2 + 2p + m^2V^2 + \dot{V}^2 = 0$$

$$\Rightarrow \quad 4p + 2\rho + m^2V^2 + 3\dot{V}^2 = 0 \ . \tag{2.3.12}$$

In view of reality conditions $\rho > 0$, $p > 0$, equation (2.3.12) implies that either $p = \rho = m = 0$ and $V =$ constant or $p = \rho = V = 0$.

Thus five dimensional Bianchi type-I cosmological model in bimetric theory do not exist in case of mesonic perfect fluid.

Hence we arrive at a theorem : "In five dimensional Rosen bimetric theory of relativity, the only possible solution of Bianchi type-I cosmological model – perfect fluid distribution and scalar meson – is a vacuum solution".

Conclusion

In view of the recent higher dimensional infrastationary scenario, we have shown that five dimensional bimetric space-time does not admit scalar meson field and mesonic perfect fluid sources. It may be noted that the introduction of the 5^{th} dimension does not throw any new light on the existence of scalar meson field and mesonic perfect fluid solution in the bimetric theory. Hence one can conclude that higher dimensional Bianchi type-I cosmological model do not exist in case of scalar meson field and mesonic perfect fluid in bimetric relativity.

Appendix [2.2.1]

For the line element (2.2.1), the components of a metric tensor g_{ij} are

$$g_{00} = -1,\ g_{11} = A^2,\ g_{22} = B^2,\ g_{33} = C^2,\ g_{44} = D^2$$

and

$$g_{ij} = 0, \quad (i \neq j)$$

then

$$g^{ij} = \frac{1}{g_{ij}}, \qquad (i = j)$$

$$= 0, \qquad (i \neq j).$$

The non-vanishing Christoffel symbols for the metric (2.2.1) are

$$\Gamma^0_{11} = A\dot{A},\ \Gamma^0_{22} = B\dot{B},\ \Gamma^0_{33} = C\dot{C},\ \Gamma^0_{44} = D\dot{D}$$

$$\Gamma^1_{10} = \frac{\dot{A}}{A},\ \Gamma^2_{20} = \frac{\dot{B}}{B},\ \Gamma^3_{30} = \frac{\dot{C}}{C},\ \Gamma^4_{40} = \frac{\dot{D}}{D}.$$

For the line element (2.2.2), the components of a metric tensor f_{ij} are

$$f_{11} = f_{22} = f_{33} = f_{44} = 1,\ f_{00} = -1$$

and

$$f_{ij} = 0, \quad (i \neq j).$$

Then

$$f^{ij} = \frac{1}{f_{ij}} \quad , \quad (i = j)$$

$$= 0 \quad , \quad (i \neq j)$$

From equation

$$N^i_j = \frac{1}{2} f^{ab} (g^{hi} g_{hj|a})_{|b} \quad ,$$

we write

$$2N^0_0 = f^{ab}(g^{h0} g_{h0|a})_{|b}$$

$$= f^{00}(g^{00} g_{00|0})_{|0} + f^{11}(g^{00} g_{00|1})_{|1} + f^{22}(g^{00} g_{00|2})_{|2} + f^{33}(g^{00} g_{00|3})_{|3} + f^{44}(g^{00} g_{00|4})_{|4}$$

$$= f^{00}(g^{00} g_{00|0})_{|0}$$

$$= 0$$

$$\Rightarrow \quad N^0_0 = 0$$

$$2N^1_1 = f^{ab}(g^{h1} g_{h1|a})_{|b}$$

$$= f^{00}(g^{11} g_{11|0})_{|0} + f^{11}(g^{11} g_{11|1})_{|1} + f^{22}(g^{11} g_{11|2})_{|2} + f^{33}(g^{11} g_{11|3})_{|3} + f^{44}(g^{11} g_{11|4})_{|4}$$

$$= f^{00}(g^{11} g_{11|0})_{|0}$$

$$= -2\left(\frac{\dot{A}}{A}\right)^{\cdot}$$

$$\Rightarrow \quad N_1^1 = -\left(\frac{\dot{A}}{A}\right)^{\cdot} \ .$$

Similarly,

$$N_2^2 = -\left(\frac{\dot{B}}{B}\right)^{\cdot}, \quad N_3^3 = -\left(\frac{\dot{C}}{C}\right)^{\cdot}, \quad N_4^4 = -\left(\frac{\dot{D}}{D}\right)^{\cdot}$$

and

$$N_j^i = 0 \qquad \text{for } i \neq j \,.$$

Then

$$N = N_0^0 + N_1^1 + N_2^2 + N_3^3 + N_4^4$$

$$= -\left[\left(\frac{\dot{A}}{A}\right)^{\cdot} + \left(\frac{\dot{B}}{B}\right)^{\cdot} + \left(\frac{\dot{C}}{C}\right)^{\cdot} + \left(\frac{\dot{D}}{D}\right)^{\cdot}\right]$$