Chapter - VI

PLANE WAVE SOLUTIONS IN BIMETRIC THEORY FOR N-DIMENSIONAL SPACE-TIME*

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* Communicated

6.1 Introduction

In a previous chapters, we have studied some aspects of plane symmetric space-time in bimetric theory of relativity and obtained a vacuum cosmological model and some exact solutions for material and electromagnetic distribution in Rosen's theory of gravitation.

The plane waves in general relativity are mathematically exposed by Takeno (1961). Using Takeno's (1961) definition of plane gravitational waves in four dimension, Adhav and Karade (1994), have obtained the plane wave solutions of field equations $N_i^j = 0$ in bimetric relativity. Its extension to higher dimension was further carried out by Thengane et al. (2000). Since in the supposition of higher dimensional space-time there is nothing unphysical – the string theories are discussed in ten dimension or twenty-six dimensions of space-time. Thengane et al. (2000) have obtained N-dimensional plane wave solutions in general relativity by using the field equations $R_{ij} = 0$.

In this chapter, we have obtained the plane wave solutions in higher dimension in a framework of bimetric theory of relativity by using the field equations $N_i^j = 0$. We have observed that, the solution obtained by Adhav and Karade (1994) and Thengane (2000) will be the particular case of our solutions for $n = 4$ and for $n = 5$ and 6 respectively.

6.2. Plane Waves

Thengane et al. (2000) generalized the Takeno's (1961) definition of plane gravitational wave for N-dimensional space-time in general relativity.

We reformulate this definition of a plane gravitational wave in bimetric gravitational theory as follows :

Definition : The plane gravitational wave g_{ij} is a non-flat solution of the field equations

$$N_i^j = 0\,,\quad (i, j = 1, 2, \cdots, n) \tag{6.2.1}$$

in an empty region of the space-time such that

$$g_{ij} = g_{ij}(Z)\,,\ Z = Z(x^i)\,, \tag{6.2.2}$$

where

$$x^i = x^1, x^2, \cdots, x^{n-1}, t$$

in some suitable coordinate system such that

$$g^{ij} Z_{,i}\, Z_{,j} = 0\,,\ \ Z_{,i} = \frac{\partial Z}{\partial x^i} \tag{6.2.3}$$

$$Z = Z(x^{n-1}, t)\ ,\quad Z_{,(n-1)} \neq 0\ ,\quad Z_{,n} \neq 0\ . \tag{6.2.4}$$

The signature convention adopted is

$$g_{ll} < 0\,,\ \begin{vmatrix} g_{ll} & g_{lk} \\ g_{kl} & g_{kk} \end{vmatrix} > 0\,,\ \begin{vmatrix} g_{ll} & g_{lk} & g_{lm} \\ g_{kl} & g_{kk} & g_{km} \\ g_{ml} & g_{mk} & g_{mm} \end{vmatrix} < 0 \tag{6.2.5}$$

[not summed for $l, k, m = 1, 2, \cdots, (n-1)$].

$$\begin{vmatrix} g_{11} & g_{12} & \cdots & \cdots & g_{1(n-1)} \\ g_{21} & g_{22} & \cdots & \cdots & g_{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & \cdots & g_{(n-1)(n-1)} \end{vmatrix} < 0, \text{ when } n \text{ is even}$$

and

$$\begin{vmatrix} g_{11} & g_{12} & \cdots & \cdots & g_{1(n-1)} \\ g_{21} & g_{22} & \cdots & \cdots & g_{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & \cdots & g_{(n-1)(n-1)} \end{vmatrix} > 0, \text{ when } n \text{ is odd.}$$

$$g_{nn} > 0.$$

It is easy to show that

$$g < 0, \text{ when } n \text{ is even}$$

and

$$g > 0, \text{ when } n \text{ is odd.}$$

We have to dealt with equations (6.2.1) along with the conditions (6.2.2), (6.2.3) and (6.2.4) to obtain different forms of the plane wave solutions.

Noting some of the results obtained by Thengane et al. (2000) as follows :

$$Z_{,(n-1)} = \frac{\phi}{M}, \quad Z_{,n} = \frac{1}{M}, \quad M_{,(n-1)} = \frac{N}{M}\phi - \bar{\phi}, \quad M_{,n} = \frac{N}{M} \quad , \tag{6.2.6}$$

$$M = \overline{w} - \overline{\phi}\, x^{n-1}, \;\; N = \overline{\overline{w}} - \overline{\overline{\phi}}\, x^{n-1} \quad . \tag{6.2.7}$$

Here a bar (–) over a letter denotes derivative with respect to Z and a comma (,) denotes the partial differentiation*.

Presuming f_{ij} as Lorentz's metric $(-1, -1, \cdots, 1)$, the f-covariant derivative becomes the usual partial derivative and the field equation (6.2.1) assume the simple form

$$f^{\alpha\beta}(g^{hj} g_{hi,\alpha})_{,\beta} = 0$$

In view of (6.2.4), the above equation reduce to

$$f^{11}(g^{hj} g_{hi,1})_{,1} + f^{22}(g^{hj} g_{hi,2})_{,2} + \cdots + f^{(n-1)(n-1)}(g^{hj} g_{hi,(n-1)})_{,(n-1)} + f^{nn}(g^{hj} g_{hi,n})_{,n} = 0$$

$$\Rightarrow \quad -(g^{hj}\overline{g}_{hi}\, z_{,1})_{,1} - (g^{hj}\overline{g}_{hi} Z_{,2})_{,2} - \cdots - (g^{hj}\overline{g}_{hi} Z_{,(n-1)})_{,(n-1)} + (g^{hj} g_{hi} Z_{,n})_{,n} = 0$$

Using equation (6.2.6), the above equation becomes

$$-\left(g^{hj}\overline{g}_{hi}\frac{\phi}{M}\right)_{,(n-1)} + \left(g^{hj} g_{hi}\frac{1}{M}\right)_{,n} = 0$$

* The details are given in the Appendix.

$$\Rightarrow \quad -\left\{\bar{g}^{hj}\bar{g}_{hi}\, Z_{,(n-1)}\frac{\phi}{M} + g^{hj}\bar{\bar{g}}_{hi} Z_{,(n-1)}\frac{\phi}{M} + g^{hj}\bar{g}_{hi}\left(\frac{\phi}{M}\right)_{,(n-1)}\right\}$$

$$+\left\{\bar{g}^{hj}\bar{g}_{hi}\, Z_{,n}\frac{1}{M} + g^{hj}\bar{\bar{g}}_{hi} Z_{,n}\frac{1}{M} + g^{hj}\bar{g}_{hi}\left(\frac{1}{M}\right)_{,n}\right\} = 0$$

$$\Rightarrow \quad -\left\{\bar{g}^{hj}\bar{g}_{hi}\frac{\phi^2}{M^2} + g^{hj}\bar{\bar{g}}_{hi}\frac{\phi^2}{M^2} + g^{hj}\bar{g}_{hi}\left(\frac{M\bar{\phi}Z_{,(n-1)} - \phi\, M_{,(n-1)}}{M^2}\right)\right\}$$

$$+\left\{\bar{g}^{hj}\bar{g}_{hi}\frac{1}{M^2} + g^{hj}\bar{\bar{g}}_{hi}\frac{1}{M^2} - g^{hj}\bar{g}_{hi}\frac{M_{,n}}{M^2}\right\} = 0$$

$$\Rightarrow \quad -\bar{g}^{hj}\bar{g}_{hi}\left(\frac{\phi}{M}\right)^2 - g^{hj}\bar{\bar{g}}_{hi}\left(\frac{\phi}{M}\right)^2 - g^{hj}\bar{g}_{hi}\frac{\phi\bar{\phi}}{M^2} + g^{hj}\bar{g}_{hi}\frac{\phi^2 N}{M^3} - g^{hj}\bar{g}_{hi}\frac{\phi\bar{\phi}}{M^2}$$

$$+\bar{g}^{hj}\bar{g}_{hi}\frac{1}{M^2} + g^{hj}\bar{\bar{g}}_{hi}\frac{1}{M^2} - g^{hj}\bar{g}_{hi}\frac{N}{M^3} = 0$$

$$\Rightarrow \quad \frac{M}{M^3}\{(\phi^2-1)\, g^{hj}\bar{g}_{hi}\}$$

$$+\frac{1}{M^2}\{(1-\phi^2)\bar{g}^{hj}\bar{g}_{hi} + (1-\phi^2)g^{hj}\bar{\bar{g}}_{hi} - 2\phi\bar{\phi}g^{hj}\bar{g}_{hi}\} = 0$$

$$\Rightarrow \quad N\{(\phi^2-1)\, g^{hj}\bar{g}_{hi}\} + M\{(1-\phi^2)\bar{g}^{hj}\bar{g}_{hi} + (1-\phi^2)g^{hj}\bar{\bar{g}}_{hi} - 2\phi\bar{\phi}g^{hj}\bar{g}_{hi}\} = 0$$

This can be written in the form analogous to that of Takeno's (1961) as

$$N\,\rho_i^j + M\,\sigma_i^j = 0 \quad , \tag{6.2.8}$$

where

$$\rho_i^j = (\phi^2-1)g^{hj}\bar{g}_{hi} \quad ; \qquad \sigma_i^j = \frac{d}{dz}\{(1-\phi^2)g^{hj}\bar{g}_{hi}\}$$

Furthermore using (6.2.7), the field equations (6.2.8) reduce to

$$\overline{\overline{w}}\rho_i^j + \overline{w}\,\sigma_i^j = 0 = \overline{\overline{\phi}}\,\rho_i^j + \overline{\phi}\,\sigma_i^j \tag{6.2.9}$$

which is again in the corresponding Takeno's format.

Conclusion

We conclude that the plane gravitational waves g_{ij} are given by the equations (6.2.8) or (6.2.9). For $n = 4$ and for $n = 5, 6$, the work of Adhav and Karade (1994) and Thengane et al. (2000) regarding the plane gravitational waves in bimetric theory of relativity can be obtained.

Appendix

From equations (6.2.3) and (6.2.4), we obtain

$$g^{(n-1)(n-1)}Z_{,(n-1)}Z_{,(n-1)} + g^{(n-1)n}Z_{,(n-1)}Z_{,n} + g^{n(n-1)}Z_{,n}Z_{,(n-1)} + g^{nn}Z_{,n}Z_{,n} = 0$$

$$\Rightarrow \quad g^{(n-1)(n-1)}\left(\frac{Z_{,(n-1)}}{Z_{,n}}\right)^2 + g^{(n-1)n}\left(\frac{Z_{,(n-1)}}{Z_{,n}}\right) + g^{n(n-1)}\left(\frac{Z_{,(n-1)}}{Z_{,n}}\right) + g^{nn} = 0$$

$$\Rightarrow \quad g^{(n-1)(n-1)}\phi^2 + 2g^{(n-1)n}\phi + g^{nn} = 0\,,$$

where

$$\phi = \frac{Z_{,(n-1)}}{Z_{,n}} \quad . \tag{A}$$

Equation (A) further can be expressed in the form of Lagrange's partial differential equation

$$Z_{,(n-1)} - \phi\, Z_{,n} = 0$$

that is

$$\frac{\partial Z}{\partial x^{n-1}} - \phi \frac{\partial Z}{\partial x^n} = 0$$

Its auxiliary equations are

$$\frac{dx^{n-1}}{1} = \frac{dt}{-\phi(Z)} = \frac{dz}{0} \quad .$$

Comparing

$$\frac{dx^{n-1}}{1} = \frac{dz}{0} \text{ and } \frac{dx^{n-1}}{1} = \frac{dt}{-\phi(z)} \quad \text{, we get}$$

$$Z = \text{ constant and } t = -\phi(Z)x^{n-1} + w(Z),$$

where w is an arbitrary function of Z .

Therefore complete solution is given by

$$t + \phi x^{n-1} = w \tag{A$_1$}$$

Now differentiating (A_1) partially with respect to x^{n-1} and t respectively, we get

$$\phi + \bar{\phi}\, Z_{,(n-1)} x^{n-1} = \bar{w}\, Z_{,(n-1)}$$

and

$$1 + \bar{\phi}\, x^{n-1} Z_{,n} = \bar{w}\, Z_{,n}$$

which further can be rearranged in the form

$$\phi = \bar{w}\, Z_{,(n-1)} - \bar{\phi}\, Z_{,(n-1)} x^{n-1},$$

$$1 = (\bar{w} - \bar{\phi}\, x^{n-1})\, Z_{,n}$$

$$\Rightarrow \quad Z_{,(n-1)} = \frac{\phi}{(\bar{w} - \bar{\phi}\, x^{n-1})} \quad , \quad Z_{,n} = \frac{1}{(\bar{w} - \bar{\phi}\, x^{n-1})}$$

$$\Rightarrow \qquad Z_{,(n-1)} = \frac{\phi}{M} \quad , \qquad Z_{,n} = \frac{1}{M} \quad , \tag{A$_2$}$$

where $M = \overline{w} - \overline{\phi}\, x^{n-1}$. (A$_3$)

Differentiating (A$_3$) partially with respect to x^{n-1} and t respectively, we get

$$M_{,(n-1)} = (\overline{w} - \overline{\phi}\, x^{n-1})\, Z_{,(n-1)} \quad ,$$

$$M_{,n} = (\overline{w} - \overline{\phi}\, x^{n-1})\, Z_{,n}$$

$$\Rightarrow \qquad M_{,(n-1)} = \overline{\overline{w}} Z_{,(n-1)} - \overline{\overline{\phi}}\, Z_{,(n-1)} x^{n-1} - \overline{\phi} \quad ,$$

$$M_{,n} = \overline{w}\, Z_{,n} - \overline{\overline{\phi}} Z_{,n}\, x^{n-1} \quad .$$

Using equation (A$_2$), we get

$$M_{,(n-1)} = \frac{N}{M} \phi - \overline{\phi} \quad ,$$

$$M_{,n} = \frac{N}{M} \quad ,$$

where

$$N = \overline{\overline{w}} - \overline{\overline{\phi}}\, x^{n-1} .$$