Chapter - V

A NON-STATIC PLANE SYMMETRIC SPACE-TIME IN GENERAL AND BIMETRIC RELATIVITY THEORY

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5.1 Introduction

In the previous chapter, we have studied some of the aspects of plane-symmetric cosmological model representing perfect fluid distribution does not exist in bimetric theory of gravitation.

To study the behaviour of singularities through the invariants of the field, the eigen values being such invariants play an important role. It is interesting to note that non-null eigen values exhibit certain characteristic features which might not have been known otherwise [Rao et al. (1978)]. This suggests that "the eigen value invariants" may explain the physical facts better as compared to the other invariants. Eigen values are also useful in obtaining the exact solutions [Tiwari, (1971)], under symmetry restrictions prescribed by the line element.

In this chapter, we have considered the plane symmetric space-time in $V_4$ and obtained some exact solutions for material distribution and electromagnetic distribution in general relativity and bimetric theory of gravitation.

Section [5.2] and [5.2](a) deals with the field equations for space-time in general relativity and its solutions, respectively. The field equations for the space-time in bimetric relativity and its solutions are discussed in section [5.3] (a) & [5.3] (b) for material and electromagnetic distribution.
5.2 General Relativity Case

Here we consider a non-static plane symmetric metric

\[ ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - d\psi^2, \quad (5.2.1) \]

when \( \psi \) is a function of \( x \) and \( t \) only.

\[ \psi = \psi(x, t) \]

\[ \Rightarrow \quad d\psi = \psi_1 dx + \psi_4 dt, \]

where

\[ \psi_1 = \frac{d\psi}{dx}, \quad \psi_4 = \frac{d\psi}{dt}, \]

then the metric (5.2.1) takes the form

\[ ds^2 = -(1 + \psi_1^2)dx^2 - dy^2 - dz^2 + (1 - \psi_4^2)dt^2 - 2\psi_1\psi_4 dx dt, \quad (5.2.2) \]

where the lower suffixes 1 and 4 after a function indicate a partial differentiation with respect to \( x \) and \( t \) respectively.

Here

\[ g_{11} = -(1 + \psi_1^2), \quad g_{22} = g_{33} = -1, \quad g_{44} = (1 - \psi_4^2), \quad g_{14} = -\psi_1\psi_4 \]

and

\[ g = \det(g_{ij}) \]

\[ = -Q, \]

where \( Q = 1 + \psi_1^2 - \psi_4^2. \)
The non-vanishing Christoffel symbols for the metric (5.2.2) are

\[ \Gamma^i_{11} = \frac{\psi_1 \psi_{11}}{Q}, \quad \Gamma^i_{44} = \frac{\psi_1 \psi_{44}}{Q}, \quad \Gamma^i_{14} = \frac{\psi_1 \psi_{14}}{Q}, \]

\[ \Gamma^4_{11} = -\frac{\psi_4 \psi_{11}}{Q}, \quad \Gamma^4_{44} = -\frac{\psi_4 \psi_{44}}{Q}, \quad \Gamma^4_{14} = -\frac{\psi_4 \psi_{14}}{Q}. \]

The Einstein's field equations are

\[ R^i_j - \frac{1}{2} \delta^i_j R = -8\pi T^i_j \quad (5.2.3) \]

The non-vanishing Ricci tensor for the metric (5.2.2) are given by

\[ R_{ij} = \Gamma^a_{bi} \Gamma^b_a - \Gamma^a_{aj} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} (\log \sqrt{-g}) + \frac{\partial^2}{\partial x^a \partial x^b} (\log \sqrt{-g}) - \frac{\partial}{\partial x^a} \Gamma^a_{ij} \quad (5.2.4) \]

For \( i = j = 1 \), equation (5.2.4) \( \Rightarrow \)

\[ R_{11} = \{ \Gamma^1_{11} \Gamma^1_{11} + \Gamma^2_{21} \Gamma^1_{11} + \Gamma^3_{31} \Gamma^3_{11} + \Gamma^4_{41} \Gamma^1_{11} \}
+ \{ \Gamma^1_{11} \Gamma^1_{12} + \Gamma^2_{21} \Gamma^2_{12} + \Gamma^3_{31} \Gamma^3_{12} + \Gamma^4_{41} \Gamma^4_{12} \}
+ \{ \Gamma^1_{11} \Gamma^3_{13} + \Gamma^3_{31} \Gamma^3_{13} + \Gamma^4_{41} \Gamma^4_{13} \}
+ \{ \Gamma^1_{11} \Gamma^1_{14} + \Gamma^2_{21} \Gamma^2_{14} + \Gamma^3_{31} \Gamma^3_{14} + \Gamma^4_{41} \Gamma^4_{14} \}
- \left\{ \Gamma^1_{11} \frac{\partial}{\partial x^1} (\log \sqrt{-g}) + \Gamma^2_{11} \frac{\partial}{\partial x^2} (\log \sqrt{-g}) 
+ \Gamma^3_{11} \frac{\partial}{\partial x^3} (\log \sqrt{-g}) + \Gamma^4_{11} \frac{\partial}{\partial x^4} (\log \sqrt{-g}) \right\}
+ \frac{\partial^2}{\partial x^1 \partial x^1} (\log \sqrt{-g}) - \frac{\partial}{\partial x^1} \Gamma^1_{11} - \frac{\partial}{\partial x^2} \Gamma^2_{11} - \frac{\partial}{\partial x^3} \Gamma^3_{11} - \frac{\partial}{\partial x^4} \Gamma^4_{11}. \]
\[= \Gamma_{11}^1 \Gamma_{11}^1 + 2 \Gamma_{11}^4 \Gamma_{14}^4 + \frac{\partial}{\partial x^1} \left( \log \sqrt{-g} \right)\]

\[-\Gamma_{11}^4 \frac{\partial}{\partial x^4} \left( \log \sqrt{-g} \right) + \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^4} \left( \log \sqrt{-g} \right) - \frac{\partial}{\partial x^1} \Gamma_{11}^1 - \frac{\partial}{\partial x^4} \Gamma_{11}^4\]

\[= \left( \frac{\psi_1 \psi_{11}}{Q} \right)^2 - 2 \left( \frac{\psi_1 \psi_{14} \psi_4 \psi_{11}}{Q^2} \right) + \left( \frac{\psi_4 \psi_{14}}{Q} \right)^2 - \left( \frac{\psi_1 \psi_{11}}{Q} \right)^2\]

\[+ \frac{\psi_1 \psi_{11} \psi_4 \psi_{41}}{Q^2} + \frac{\psi_1 \psi_{14} \psi_{11} \psi_{14}}{Q^2} - \frac{\psi_4 \psi_{11} \psi_{44}}{Q^2}\]

\[+ \frac{\psi_1 \psi_{11}}{Q} - \frac{\psi_4 \psi_{41}}{Q} - \frac{\psi_{11} \psi_{11}}{Q} + \frac{\psi_{44} \psi_{11}}{Q}\]

\[= \left( \frac{\psi_4 \psi_{14}}{Q} \right)^2 - \frac{\psi_4 \psi_1 \psi_{44}}{Q^2} \left( \frac{Q(\psi_{41} \psi_{41} + \psi_4 \psi_{411}) - \psi_4 \psi_{41} Q_1}{Q^2} \right)\]

\[+ \left\{ \frac{Q(\psi_4 \psi_{11} + \psi_{44} \psi_{11}) - \psi_4 \psi_{11} Q_4}{Q^2} \right\}\]

\[= \frac{\psi_{11} \psi_{44} - \psi_{14}^2}{Q} + \frac{\psi_4 \psi_{11} \psi_{44} - \psi_{14} \psi_{11} \psi_{44}}{Q^2}\]

\[= \frac{\psi_{11} \psi_{44} - \psi_{14} \psi_{44}}{Q} \left( 1 + \frac{\psi_4^2}{Q} \right)\]

\[= \frac{(1 + \psi_1^2)(\psi_{11} \psi_{44} - \psi_{14}^2)}{Q^2}\]
For \( i, j = 2 \), equation (5.2.4) \( \Rightarrow \)

\[
R_{22} = \Gamma^a_{b2} \Gamma^b_{2a} - \Gamma^b_{22} \frac{\partial}{\partial x^b} (\log \sqrt{-g}) + \frac{\partial^2}{(\partial x^2)^2} (\log \sqrt{-g}) - \frac{\partial}{\partial x^a} \Gamma^a_{22}
\]

\[
= \Gamma^b_{b2} \Gamma^b_{21} + \Gamma^b_{b2} \Gamma^b_{22} + \Gamma^b_{b2} \Gamma^b_{23} + \Gamma^b_{b2} \Gamma^b_{24} - \Gamma^b_{22} \frac{\partial}{\partial x^b} (\log \sqrt{-g})
\]

\[
- \frac{\partial}{\partial x^1} \Gamma^1_{22} - \frac{\partial}{\partial x^2} \Gamma^2_{22} - \frac{\partial}{\partial x^3} \Gamma^3_{22} - \frac{\partial}{\partial x^4} \Gamma^4_{22}
\]

\[
= \Gamma^b_{b2} \Gamma^b_{21} + \Gamma^b_{b2} \Gamma^b_{24}
\]

\[
= \Gamma^1_{12} \Gamma^1_{21} + \Gamma^1_{22} \Gamma^1_{21} + \Gamma^1_{32} \Gamma^1_{21} + \Gamma^1_{42} \Gamma^1_{21} + \Gamma^4_{12} \Gamma^4_{24}
\]

\[
+ \Gamma^4_{22} \Gamma^4_{24} + \Gamma^4_{32} \Gamma^4_{24} + \Gamma^4_{42} \Gamma^4_{24}
\]

\[
= 0 .
\]

Similarly

\[
R_{33} = 0 .
\]

For \( i, j = 4 \), equation (5.2.4) \( \Rightarrow \)

\[
R_{44} = \frac{(\psi^2_4 - 1) (\psi_{11} \psi_{44} - \psi^2_{14})}{Q^2} .
\]

For \( i = 1, \ j = 4 \), equation (5.2.4) \( \Rightarrow \)

\[
R_{14} = \frac{\psi_1 \psi_4 (\psi_{11} \psi_{44} - \psi^2_{14})}{Q^2} .
\]

\[
R = -\frac{1}{2} \left( \frac{\psi_{11} \psi_{44} - \psi^2_{14}}{Q^2} \right)
\]
The field equations (5.2.3) can be written as

\[-8\pi T^1_1 = R^1_1 - \frac{1}{2} g_1^1 R\]

\[= \left(\frac{1 - \psi_4^2}{Q}\right) \left(1 + \psi_1^2\right) \left(\psi_{11} \psi_{44} - \psi_{14}^2\right) \frac{Q^2}{Q^2} - \left(\frac{\psi_1 \psi_4}{Q}\right) \left[\psi_{11} \psi_{44} \left(\psi_{11} \psi_{44} - \psi_{14}^2\right)\right] \frac{Q^2}{Q^2} + \frac{\psi_{11} \psi_{44} - \psi_{14}^2}{Q^2}\]

\[= 0\]

\[\Rightarrow T^1_1 = 0\]

Similarly

\[T^4_4 = T^4_1 = T^1_4 = 0\]

Hence

\[T^1_1 = T^4_4 = T^4_1 = T^1_4 = 0\]  \hspace{1cm} (5.2.5)

\[-8\pi T^2_2 = g^{22} R_{22} - \frac{1}{2} R\]

\[= \frac{\psi_{11} \psi_{44} - \psi_{14}^2}{Q^2}\]  \hspace{1cm} (5.2.6)

\[-8\pi T^3_3 = \frac{\psi_{11} \psi_{44} - \psi_{14}^2}{Q^2}\]  \hspace{1cm} (5.2.7)
5.2.(a) Material distribution

The eigen values of the energy momentum tensor are defined by the determinantal equation

\[ | T_j' - \lambda \delta_j' | = 0 \]  \quad (5.2.8)

which in view of equations (5.2.5) – (5.2.7), it reduces to

\[ \begin{vmatrix} T_1^1 - \lambda & T_2^1 & T_3^1 & T_4^1 \\ T_1^2 & T_2^2 - \lambda & T_3^2 & T_4^2 \\ T_1^3 & T_2^3 & T_3^3 - \lambda & T_4^3 \\ T_1^4 & T_2^4 & T_3^4 & T_4^4 - \lambda \end{vmatrix} = 0 \]

\[ \Rightarrow (T_2^2 - \lambda)(T_3^3 - \lambda) \{(T_1^1 - \lambda)(T_4^4 - \lambda) - T_1^4 T_4^1\} = 0 \]

\[ \Rightarrow (T_2^2 - \lambda) = 0 \text{ or } (T_3^3 - \lambda) = 0 \text{ or } (T_1^1 - \lambda)(T_4^4 - \lambda) - T_1^4 T_4^1 = 0. \]

Hence two of the eigen values are

\[ T_2^2 = T_3^3 = \lambda \]  \quad (5.2.9)

and the other two being given by

\[ (T_1^1 - \lambda)(T_4^4 - \lambda) - T_1^4 T_4^1 = 0. \]  \quad (5.2.10)

Plane-symmetry permits us only to assume two of the singularities to be equal.

Thus, we get

\[ (T_1^1 - T_2^2)(T_4^4 - T_2^2) - T_1^4 T_4^1 = 0 \]

which with the help of equations (5.2.5) – (5.2.7) it reduces to

\[ \psi_{11} \psi_{44} - \psi_{14}^2 = 0. \]  \quad (5.2.11)
The equation (5.2.11), admits the solutions

\[ \psi = f(x) + c_1 t + c_2 \quad , \tag{5.2.12} \]

where \( f \) is any twice differentiable function of \( x \) and \( c_1, c_2 \) are constants.

\[ \psi = c_3 x + g(t) + c_4 \quad , \tag{5.2.13} \]

where \( g \) is any twice differentiable function of \( t \) and \( c_3, c_4 \) are constants.

Using separation of variable method, the differential equation (5.2.11), admits the following two independent solutions

\[ \psi = a e^{c x + d t} , \quad k = 1 \tag{5.2.14} \]

and

\[ \psi = b(x + m)^{\frac{k}{k-1}} (t + n)^{\frac{1}{1-k}} , \quad k \neq 1 \tag{5.2.15} \]

where \( a, b, c, d, m \) and \( n \) are arbitrary constants of integration and \( k \) is an arbitrary constant*.

5.3. Bimetric Relativity Case

Here we consider the metric (5.2.2) in the form

\[ ds^2 = -A dx^2 - dy^2 - dz^2 + B dt^2 - 2 C \, dx \, dt , \tag{5.3.1} \]

where

\[ A = 1 + \psi_1^2, \quad B = 1 - \psi_2^2, \quad C = \psi_1 \psi_4 . \]

* The detailed calculations are given in the Appendix [5.2.1].
The background flat metric corresponding to (5.3.1) is

$$d\sigma^2 = -dx^2 - dy^2 - dz^2 + dt^2.$$  

(5.3.2)

The field equations (2.1.3) for the metric (5.3.1) can be written as

$$-8\pi\kappa T_1^1 = 8\pi\kappa T_4^4 = \frac{1}{4} \left( \frac{AB_1 - A_1 B}{Q} \right)_{,1} - \frac{1}{4} \left( \frac{AB_1 - A_1 B}{Q} \right)_{,4},$$  

(5.3.3)

$$-8\pi\kappa T_2^2 = -8\pi\kappa T_3^3 = \frac{1}{4} \left( \log Q \right)_{,1} - \frac{1}{4} \left( \log Q \right)_{,44},$$  

(5.3.4)

$$-8\pi\kappa T_4^4 = \frac{1}{2} \left( \frac{B_1 C - BC_1}{Q} \right)_{,1} - \frac{1}{2} \left( \frac{B_4 C - BC_4}{Q} \right)_{,4},$$  

(5.3.5)

$$-8\pi\kappa T_1^1 = \frac{1}{2} \left( \frac{AC_1 - A_1 C}{Q} \right)_{,1} - \frac{1}{2} \left( \frac{AC_4 - A_4 C}{Q} \right)_{,4},$$  

(5.3.6)

where

$$A = 1 + \psi_1^2, \quad B = 1 - \psi_4^2, \quad C = \psi_1 \psi_4, \quad Q = AB + C^2.$$  

From equations (5.3.3) and (5.3.4), we have

$$T_1^1 = -T_4^4, \quad T_2^2 = T_3^3.$$  

(5.3.7)

Here

$$T_3^3 = T_1^1 + T_2^2 + T_4^4.$$  

(5.3.8)

**The detailed calculations are given in the Appendix [5.3.1].**
5.3. (a) Material distribution

From equations (5.2.9) and (5.3.7), equation (5.2.10) reduce to

\[ \lambda^2 - (T_{1}^1)^2 - T_{1}^4 T_{4}^4 = 0 \]  \hspace{1cm} (5.3.9)

It is clear that the two roots are equal and opposite in sign. Hence we conclude that in case of plane symmetric metric, the eigen values of the energy momentum tensor are \( \alpha, \alpha, \beta \) and \( -\beta \) without considering of the type of distribution in bimetric theory of relativity.

5.3 (b) Electromagnetic distribution

As \( N = 0 \) is the necessary condition for electromagnetic radiation, it follows that for an electromagnetic field

\[ T = T_{i}^i = 0, \]

which in view of equation (5.3.7), we get

\[ T = T_{1}^1 + T_{2}^2 + T_{3}^3 + T_{4}^4 = 0 \]

\[ \Rightarrow T_{1}^1 + T_{2}^2 + T_{2}^2 - T_{1}^1 = 0 \]

\[ \Rightarrow T_{2}^2 = 0 \]  \hspace{1cm} (5.3.10)

From equations (5.3.4) and (5.3.10), we obtain

\[ (\log \mathcal{Q})_{,11} - (\log \mathcal{Q})_{,44} = 0, \]  \hspace{1cm} (5.3.11)

which gives the solution

\[ \mathcal{Q} = e^{F(x+i) - G(x-i)} \]  \hspace{1cm} (5.3.12)
That is
\[ 1 + \psi_1^2 - \psi_4^2 = e^{F-G} , \]
where \( F \) and \( G \) are arbitrary functions of \( x \) and \( t \) respectively.

Using Jacobi's method, the solution for (5.3.12) become**
\[ \psi = \int \sqrt{e^{F-G} + d} \, dx + (\sqrt{d + 1})t + d_0 \]
where \( d, d_0 \) are arbitrary constants.

5.3 (c) Special cases

Case I: When \( F(x+t) = x+t \) and \( G(x-t) = x-t \).

Then solution of (5.3.12) become
\[ \psi = d_1 x + \int \sqrt{d_1^2 + 1 - e^{2t}} \, dt + d_2 \]
where \( d_1, d_2 \) are arbitrary constants of integration.

Case II: When \( F(x+t) = \log(x+t) \) and \( G(x-t) = \log(x-t) \).

The solution of (5.3.12) become
\[ \psi = \int \sqrt{\frac{2t}{x-t}} + d_3 \, dx + \sqrt{d_3} t + d_4 \]
where \( d_3, d_4 \) are arbitrary constants of integration.

** The detailed calculations are given in the Appendix [5.3.1].
Case III: When $F(x + t) = x + t$ and $G(x - t) = -(x + t)$.

The solution of (5.3.12) become

$$\psi = d_5 t + \sqrt{d_5^2 - 1} e^{2x} dx + d_6,$$

where $d_5, d_6$ are arbitrary constants of integration.
Appendix [5.2.1]

Let \( \psi = c x^r t^r \) \((A_1)\)

be a solution of equation (5.2.11), where \( c \) and \( r \) are constant real numbers.

Differentiating \((A_1)\) twice with respect to \( x \) and \( t \) respectively, we get

\[ \psi_{11} = cr(r-1)x^{r-2}t', \quad \psi_{44} = cr(r-1)x'r^2 t^r, \quad \psi_{14} = cr^2 x^{r-1}t^{r-1}, \]

substituting these values in equation (5.2.11), we obtain

\[ c^2 r^2 x^{2r-2} t^{2r-2} \{(r-1)^2 - r^2\} = 0 \]

\[ \Rightarrow \quad r = 0 \text{ or } (r-1)^2 - r^2 = 0 \]

i.e. \( r = 0 \) or \( r = \frac{1}{2} \).

Therefore, the equation (5.2.11) has non-trivial solution of the type

\( \psi = c x^{1/2} t^{1/2} \).

Thus

\( \psi = f(x) + c_1(t) + c_2 \)

is also a solution of equation (5.2.11)

OR

\( \psi = c_3(x) + g(t) + c_4 \),

is also a solution of equation (5.2.11).
Now using separation of variable method, the differential equation (5.2.11), gives the following two independent solutions as follows:

Let

\[ \psi = X(x) T(t) \]  \hspace{1cm} (A_2) 

be a solution of equation (5.2.11), where \( X \) is a function of \( x \) alone and \( T \) is a function of \( t \) alone.

Differentiating \((A_2)\) twice with respect to \( x \) and \( t \) respectively, we get

\[ \psi_{11} = X_{11}T, \quad \psi_{44} = X T_{44}, \quad \psi_{14} = X_1 T_4, \]

substituting these values in equation (5.2.11) and separating the variables, we get

\[ \frac{T_{44}}{T_4} \cdot \frac{T_4}{T} = \frac{X_{11}^2}{XX_{11}} = k = \text{constant} \]

\[ \Rightarrow \frac{T_{44}}{T_4} = k \frac{T_4}{T}, \quad \frac{X_{11}}{X_1} = \frac{1}{k} \frac{X_1}{X} \]

which on integration, gives

\[ T_4 = A_0 T^k, \quad X_1 = C_0 X^{1/k} \]

again integrating, we get

\[ T = (A_0 t + B_0)^{1-k}, \quad X = (C_0 x + D_0)^{k-1}, \quad \text{for} \quad k \neq 1, \]

and

\[ T = b_0 e^{a_0 t}, \quad X = d_0 e^{c_0 x}, \quad \text{for} \quad k = 1 \]

where \( A_0, B_0, C_0, D_0, a_0, b_0, c_0 \) are arbitrary constants of integration.
Therefore, the solutions of equation (5.2.11) are given by

$$\Rightarrow \quad \psi = ae^{cx+dt} , \quad k = 1$$

and

$$\psi = b(x+m)^{k-1}(t+n)^{1-k} , \quad k \neq 1$$

where $a, b, c, d, m$ and $n$ are arbitrary constants of integration.
Appendix [5.3.1]

For the metric (5.3.1), we have

\[ g_{11} = -A, \quad g_{22} = g_{23} = -1, \quad g_{44} = B, \quad g_{14} = -C \]

\[ g = \det(g_{ij}) \]

\[ = -(AB + C^2) \]

\[ = -Q, \]

where

\[ Q = (AB + C^2) \]

\[ = (1 + \psi_1^2 - \psi_2^2) \]

and

\[ \kappa = \left( \frac{g}{f} \right)^{1/2} \]

\[ = Q^{1/2} \]

\[ g^{11} = -\frac{B}{Q}, \quad g^{22} = g^{33} = -1, \quad g^{44} = \frac{A}{Q}, \quad g^{14} = -\frac{C}{Q}. \]

For the line element (5.3.2), we have

\[ f_{11} = f_{22} = f_{33} = -1, \quad f_{44} = 1 \]

\[ f^{ij} = \frac{1}{f_{ij}}, \quad (i = j) \]

\[ = 0, \quad (i \neq j) \]
and

\[ f = \det(f_{ij}) \]

\[ = -1 \]

The components of \( N^i_j \) are given by

\[ 2N^1_1 = f^{\alpha\beta} (g^{\alpha\beta} g_{11} g_{21})_{\alpha\beta} \]

\[ = f^{11} (g^{11} g_{11^1})_{11} + f^{22} (g^{41} g_{44^1})_{11} + f^{33} (g^{11} g_{11^4})_{14} + f^{44} (g^{41} g_{44^4})_{14} \]

\[ = \left( \frac{B A_1}{Q} \right)_{,1} \left( \frac{C C_1}{Q} \right)_{,1} + \left( \frac{B A_4}{Q} \right)_{,4} \left( \frac{C C_4}{Q} \right)_{,4} \]

\[ = -\frac{1}{2} \left( \frac{2 B A_1 + 2 C C_1}{Q} \right)_{,1} + \frac{1}{2} \left( \frac{2 B A_4 + 2 C C_4}{Q} \right)_{,4} \]

\[ = -\frac{1}{2} \left( \frac{A_1 + B_1 + A_1 B - A B_1}{Q} \right)_{,1} + \frac{1}{2} \left( \frac{A_4 + B_4 + A_4 B - A B_4}{Q} \right)_{,4} \]

\[ 2N^2_2 = f^{11} (g^{22} g_{22^1})_{11} + f^{22} (g^{22} g_{22^2})_{12} + f^{33} (g^{22} g_{22^3})_{13} + f^{44} (g^{22} g_{22^4})_{14} \]

\[ = 0 \]

Similarly

\[ N^3_3 = 0 \]

\[ 2N^4_4 = -\frac{1}{2} \left( \frac{A_1 + B_1 + A_1 B - A B_1}{Q} \right)_{,1} + \frac{1}{2} \left( \frac{A_4 + B_4 + A_4 B - A B_4}{Q} \right)_{,4} \]

\[ 2N^4_4 = \left( \frac{B_1 C - B C_1}{Q} \right)_{,1} + \left( \frac{B C_4 - C B_4}{Q} \right)_{,4} \]
\[ 2N_1^4 = \left( \frac{AC_1 - A_1 C}{Q} \right)_{,1} + \left( \frac{A_4 C - AC_4}{Q} \right)_{,4}. \]

Therefore,

\[ N = N_i^l \]

\[ = N_1^1 + N_4^4 \]

\[ = -\frac{1}{2} \left( \frac{Q_1}{Q} \right)_{,1} + \frac{1}{2} \left( \frac{Q_4}{Q} \right)_{,4} \]

\[ = -\frac{1}{2} (\log Q)_{,11} + \frac{1}{2} (\log Q)_{,44}. \]

We have

\[ Q = e^{F(x+t)-G(x-t)}, \]

where

\[ Q = 1 + \psi_1^2 - \psi_4^2 = e^{F-G}. \]

By using Jacobis method:

\[ f = 1 + p^2 - q^2 - e^{F-G} = 0, \]

where

\[ u = u(x, t, \psi), \quad p = -\frac{u_1}{u_3}, \quad q = \frac{u_2}{u_3} \]

\[ \Rightarrow \quad u_x + u_t p = 0 \text{ and } u_t + u_x q = 0, \]

where \( u_x = u_1, \quad u_t = u_2, \quad u_\psi = u_3. \)
Therefore,
\[ f = u_1^2 - u_2^2 + u_3^2 - u_3^2 e^{F-G} = 0 \]  \hspace{1cm} (A_3)

\[ \Rightarrow \quad f u_1 = 2u_1, \quad f u_2 = -2u_2, \quad f u_3 = 2u_3 - 2u_3 e^{F-G} \]

\[ f_x = -u_3^2 e^{F-G} (F_x - G_x), \quad f_i = -u_3^2 e^{F-G} (F_i + G_i), \quad f_y = 0 . \]

Then Jacobi's auxiliary equations are
\[ -\frac{dx}{f u_1} = \frac{dt}{f u_2} = \frac{d\psi}{f x} = \frac{du_1}{f_1} = \frac{du_2}{f_1} = \frac{du_3}{f_1} \] \hspace{1cm} (A_4)

\[ \Rightarrow \quad \frac{dx}{-2u_1} = \frac{dt}{2u_2} = \frac{d\psi}{-2u_3 + 2u_3 e^{F-G}} \]

\[ = \frac{du_1}{-u_3^2 e^{F-G} (F_x - G_x)} = \frac{du_2}{-u_3^2 e^{F-G} (F_i + G_i)} = \frac{du_3}{0} \]

\[ \Rightarrow \quad du_3 = 0 \]

\[ \Rightarrow \quad u_3 = a = \text{constant} \] \hspace{1cm} (A_5)

Now comparing first and fourth ratio, we get
\[ a^2 e^{F-G} (F_x - G_x) dx = 2u_1 du_1 \]

which on integration give
\[ u_1^2 = a^2 e^{F-G} + d(t) \] \hspace{1cm} (A_6)

Using equations (A_3), (A_5) and (A_6), we get
\[ u_2 = \sqrt{a^2 + d} \]
Now
\[ du = u_1 dx + u_2 dt + u_3 d\psi \]
substituting the values of \( u_1, u_2 \) and \( u_3 \) in above equation, we get
\[ du = \left[ a^2 e^{F-G} + d \right]^{1/2} dx + \sqrt{a^2 + d} dt + a d\psi \]
which on integration give
\[ u = \int \left( a^2 e^{F-G} + d \right)^{1/2} dx + \left( a^2 + d \right)^{1/2} t + a\psi + d_0, \]
put \( u = 0 \) and \( a = -1 \), we get
\[ \psi = \int \left( e^{F-G} + d \right)^{1/2} dx + \left( d - 1 \right)^{1/2} t + d_0, \]
where \( d, d_0 \) are arbitrary constants.

(i) When
\[ F(x+t) = x + t, \quad G(x-t) = x - t \]
than equation (5.3.12) become
\[ Q = e^{2t} \]
\[ \Rightarrow \quad 1 + \psi_1^2 - \psi_4^2 = e^{2t} \]
By using Jacobi's method:
\[ f \equiv 1 + p^2 - q^2 - e^{2t} = 0 \]
\[ \Rightarrow \quad f \equiv u_1^2 - u_2^2 + u_3^2 - u_5^2 e^{2t} = 0 \]
\[ \Rightarrow \quad f u_1 = 2u_1, \quad f u_2 = -2u_2, \quad f u_3 = 2u_3 - 2u_5 e^{2t}, \quad f_x = 0, \]
\[ f_t = -2u_5^2 e^{2t}, \quad f_\psi = 0 \]

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Jacobi's auxiliary equations (A4) can be expressed as:

\[
\frac{dx}{-2u_1} = \frac{dt}{2u_2} = \frac{d\psi}{-2u_3 + 2u_3e^{2it}} = \frac{du_1}{0} = \frac{du_2}{-2u_3e^{2it}} = \frac{du_3}{0}
\]

\[\Rightarrow \quad du_1 = 0, \quad du_3 = 0\]

\[\Rightarrow \quad u_1 = d_1 = \text{constant}, \quad u_3 = b = \text{constant}.
\]

Substituting the above values in equation (A7), we get

\[u_2 = d_1^2 + b^2(1 - e^{2it})\]

Now

\[du = u_1dx + u_2dt + u_3d\psi\]

\[\Rightarrow \quad du = d_1dx + \sqrt{d_1^2 + b^2(1 - e^{2it})} \, dt + b d\psi\]

which on integration give

\[u = d_1x + \int \sqrt{d_1^2 + b^2(1 - e^{2it})} \, dt + b \psi + d_2\]

put \(u = 0\) and \(b = -1\), we get

\[\psi = d_1x + \int \sqrt{d_1^2 + 1 - e^{2it}} \, dt + d_2,\]

where \(d_1, d_2\) are arbitrary constants of integration.
(ii) When \( F(x+t) = \log(x+t) \) and \( G(x-t) = \log(x-t) \)

then equation (5.3.12) gives

\[
\mathcal{Q} = \frac{x+t}{x-t}
\]

where \( \mathcal{Q} = 1 + \psi_1^2 - \psi_4^2 \).

By using Jacobi's method, we get

\[
\psi = \int \sqrt{\frac{2a^2 + 1}{x-t}} + d_3 \, dx + \sqrt{d_3} \, t + d_4,
\]

where \( d_3, d_4 \) are arbitrary constants of integration.

(iii) When \( F(x+t) = x+t \) and \( G(x+t) = -(x-t) \)

then equation (5.3.12) yield

\[
\mathcal{Q} = e^{2x}
\]

that is

\[
1 + \psi_1^2 - \psi_4^2 = e^{2x}.
\]

By using Jacobi's method, we obtain

\[
\psi = a_5 t + \int \sqrt{a_3^2 - 1 + e^{2x}} \, dx + a_6,
\]

where \( a_5, a_6 \) are arbitrary constants of integration.