Chapter - IV

NON EXISTENCE OF PLANE-SYMMETRIC COSMOLOGICAL MODELS IN BIMETRIC THEORY OF GRAVITATION*

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4.1 Introduction

In a previous chapters, we have investigated the non-existence of Bianchi type – I and III cosmological models in Rosen's bimetric theory when the source of gravitation is governed by either perfect fluid or mesonic perfect fluid.

In this chapter we confine ourselves to study the general cylindrically-symmetric metric given by Marder (1958).

In section [4.2], we have obtained solutions of the field equations in the presence of perfect fluid and then the special case is discussed in section [4.3]. We have observed that the plane-symmetric cosmological model does not throw any new light on the existence of perfect fluid solution.

4.2. The field equations

We consider the general cylindrically-symmetric metric given by Marder (1958)

$$ds^2 = A^2(dt^2 - dx^2) - B^2 dy^2 - C^2 dz^2, \tag{4.2.1}$$

where A, B and C are functions of time t only.

The flat space-time corresponding to (4.2.1) is

$$d\sigma^2 = dt^2 - dx^2 - dy^2 - dz^2 \ . \tag{4.2.2}$$

The energy-momentum tensor for perfect fluid distribution is given by

$$T_{ij} = (\rho + p)\, u_i\, u_j - p g_{ij} \tag{4.2.3}$$

together with

$$g_{ij}\, u^i\, u^j = 1,$$

where u^i is the four velocity vector of the fluid and p, ρ are the proper pressure and the energy density respectively.

We use co-moving coordinates so that

$$u_1 = u_2 = u_3 = 0 \quad \text{and} \quad u_4 = A.$$

Then equation (4.2.3) implies

$$T_1^1 = T_2^2 = T_3^3 = -p, \quad T_4^4 = \rho \ . \tag{4.2.4}$$

In this case we find

$$T_j^i = 0, \ \ (i \neq j).$$

The field equations (2.1.3) for the metric (4.2.1) with the help of equation (4.2.4) become

$$K_1^1 = \left(\frac{\dot{B}}{B}\right)^{\cdot} + \left(\frac{\dot{C}}{C}\right)^{\cdot} = -16\pi\kappa p \tag{4.2.5}$$

$$K_2^2 = 2\left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{\dot{B}}{B}\right)^{\cdot} + \left(\frac{\dot{C}}{C}\right)^{\cdot} = -16\pi\kappa p \tag{4.2.6}$$

$$K_3^3 = 2\left(\frac{\dot{A}}{A}\right)^{\cdot} + \left(\frac{\dot{B}}{B}\right)^{\cdot} - \left(\frac{\dot{C}}{C}\right)^{\cdot} = -16\pi\kappa p \tag{4.2.7}$$

$$K_4^4 = \left(\frac{\dot{B}}{B}\right)^{\cdot} + \left(\frac{\dot{C}}{C}\right)^{\cdot} = 16\pi\kappa\rho \quad , \tag{4.2.8}$$

where the dot (.) denotes ordinary differentiation with respect to time t.

Equations (4.2.5) – (4.2.7) reduce to

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} = \left(\frac{\dot{B}}{B}\right)^{\cdot} = \left(\frac{\dot{C}}{C}\right)^{\cdot} \tag{4.2.9}$$

with the help of equation (4.2.9), equations (4.2.5) and (4.2.8) reduce to

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} = -8\pi\kappa p \tag{4.2.10}$$

and

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} = 8\pi\kappa\rho \tag{4.2.11}$$

which gives

$$p + \rho = 0, \tag{4.2.12}$$

which is an equation of state.

In view of the reality condition $p > 0$, $\rho > 0$, equation (4.2.12) implies that

$$p = 0, \ \rho = 0, \tag{4.2.13}$$

which shows that the plane symmetric cosmological models representing perfect fluid does not exist in bimetric theory of gravitation.

when $p = 0 = \rho$ (vacuum), equations (4.2.10) and (4.2.11) give the solution

$$A = B = C = e^{k_1 t}, \tag{4.2.14}$$

where k_1 is a constant of integration.

In view of equation (4.2.14), the line element (4.2.1) take the form

$$ds^2 = e^{2k_1 t}(dt^2 - dx^2 - dy^2 - dz^2) \quad . \tag{4.2.15}$$

This can be transformed through a proper choice of coordinates and absorbing constants in the differentials to

$$ds^2 = e^{2T}(dT^2 - dX^2 - dY^2 - dZ^2) \ . \tag{4.2.16}$$

It is interesting to note that, the model is flat and free from singularity. At $T = 0$, the model reduces to flat one.

4.3 A special case

Let the metric potentials B and C are equal then the metric (4.2.1) reduces to

$$ds^2 = A^2(dt^2 - dx^2) - B^2(dy^2 + dz^2) \tag{4.3.1}$$

The flat space-time corresponding to (4.3.1) is given by (4.2.2).

The energy momentum tensor for perfect fluid distribution is given by (4.2.3).

The field equations (2.1.3) for the metric (4.3.1) with the help of equation (4.2.4) become

$$K_1^1 = \left(\frac{\dot{B}}{B}\right)^{\cdot} = -8\pi A^2 B^2 p \qquad (4.3.2)$$

$$K_2^2 = K_3^3 = \left(\frac{\dot{A}}{A}\right)^{\cdot} = -8\pi A^2 B^2 p \qquad (4.3.3)$$

$$K_4^4 = \left(\frac{\dot{B}}{B}\right)^{\cdot} = 8\pi A^2 B^2 \rho \quad . \qquad (4.3.4)$$

Equations (4.3.2) and (4.3.4) reduce to

$$p + \rho = 0 .$$

In view of reality condition $\rho > 0$, $p > 0$, the above equation implies that

$$p = 0 , \ \rho = 0 ,$$

which shows that, the plane symmetric space-time, the antistiff-fluid distribution does not exist in bimetric theory.

When $p = 0 = \rho$ (vacuum), equations (4.3.2) – (4.3.4) give the solution

$$A = B = e^{k_2 t} , \qquad (4.3.5)$$

where k_2 is a constant of integration.

In view of equation (4.3.5), the line element (4.3.1) reduce to

$$ds^2 = e^{2k_2 t}(dt^2 - dx^2 - dy^2 - dz^2) \qquad (4.3.6)$$

with the proper choice of coordinate transformation and constants, the line element (4.3.6) reduce to*

$$ds^2 = e^{2T}(dT^2 - dX^2 - dy^2 - dz^2) \ . \qquad (4.3.7)$$

Here the fluid distribution in the model is given by an equation of state (4.2.12) and hence using the reality condition, we get the vacuum plane-symmetric cosmological model. This shows that the plane-symmetric cosmological models representing perfect fluid distribution does not exist in bimetric theory of gravitation.

Conclusion

It is observed that the plane symmetric cosmological model does not throw any new light on the existence of perfect fluid solutions. Hence one can conclude that perfect fluid plane symmetric cosmological model does not exist in the bimetric theory of gravitation.

* The detailed calculations are given in the Appendix.

Appendix

For the line element (4.2.1), we have

$$g_{11} = -A^2,\ g_{22} = -B^2,\ g_{33} = -C^2,\ g_{44} = A^2$$

and

$$g_{ij} = 0, \quad (i \neq j).$$

Then

$$g^{ij} = \frac{1}{g_{ij}}, \qquad (i = j)$$

$$= 0 \quad , \qquad (i \neq j)\,.$$

For the line element (4.2.2) the components of a metric tensor f_{ij} are

$$f_{11} = f_{22} = f_{33} = -1,\ f_{44} = 1$$

and

$$f_{ij} = 0 \ , \ (i \neq j).$$

Then

$$f^{ij} = \frac{1}{f_{ij}},\ (i = j)$$

$$= 0, \quad (i \neq j)$$

components of N^i_j are :

$$2N^1_1 = f^{ab}(g^{h1}g_{h1|a})_{|b}$$

$$= f^{11}(g^{11}g_{11|1})_{|1} + f^{22}(g^{11}g_{11|2})_{|2} + f^{33}(g^{11}g_{11|3})_{|3} + f^{44}(g^{11}g_{11|4})_{|4}$$

$$= f^{44}(g^{11}g_{11|4})_{|4}$$

$$= \left(2\frac{1}{A^2}A\dot{A}\right)^{\cdot}$$

$$= 2\left(\frac{\dot{A}}{A}\right)^{\cdot}$$

$$\Rightarrow \quad N^1_1 = \left(\frac{\dot{A}}{A}\right)^{\cdot} .$$

Similarly

$$N^2_2 = \left(\frac{\dot{B}}{B}\right)^{\cdot}, \quad N^3_3 = \left(\frac{\dot{C}}{C}\right)^{\cdot}, \quad N^4_4 = \left(\frac{\dot{A}}{A}\right)^{\cdot}$$

and

$$N^i_j = 0 \qquad \text{for } i \neq j \ .$$

Then

$$N = N_1^1 + N_2^2 + N_3^3 + N_4^4$$

$$= 2\left(\frac{\dot{A}}{A}\right)^{\cdot} + \left(\frac{\dot{B}}{B}\right)^{\cdot} + \left(\frac{\dot{C}}{C}\right)^{\cdot} \quad .$$

For the line element (4.3.1), we have

$$g_{11} = -A^2 , \;\; g_{22} = g_{33} = -B^2 , \;\; g_{44} = A^2$$

and

$$g_{ij} = 0 \quad , \qquad (i \neq j) \, .$$

Then

$$g^{ij} = \frac{1}{g_{ij}} \quad , \quad (i = j)$$

$$= 0 \quad , \quad (i \neq j)$$

$$g = \det(g_{ij}) = -A^4 B^4 , \quad f = \det(f_{ij}) = -1 \quad .$$

Then

$$\kappa = \left(\frac{g}{f}\right)^{1/2}$$

$$= A^2 B^2 \quad . \qquad (A_1)$$

The components of N^i_j are

$$N^1_1 = N^4_4 = \left(\frac{\dot{A}}{A}\right)^{\cdot}, \quad N^2_2 = N^3_3 = \left(\frac{\dot{B}}{B}\right)^{\cdot},$$

and

$$N^i_j = 0, \qquad (i \neq j).$$

$$N = 2\left(\frac{\dot{A}}{A}\right)^{\cdot} + 2\left(\frac{\dot{B}}{B}\right)^{\cdot} .$$