CHAPTER 0

CHAPTER 0

PRELIMINARIES

In this thesis, we introduce the following concepts to make the thesis a self contained one. Here we basically use the concept of Fuzzy Cognitive Maps (FCMs), Fuzzy Relational Maps (FRMs) and supermatrices. Using these notions we build new fuzzy models like Super Fuzzy Cognitive Maps (SFCMs), Domain Super Fuzzy Relational Maps (DSFRMs), Fuzzy Linguistic Cognitive Maps (FLCMs) and Fuzzy Linguistic Relational Maps (FLRMs) models. This chapter has two sections. Section one has two subsections. Subsection one recalls definitions and properties associated with the FCMs and subsection two recalls definitions and properties of FRMs. Section two recalls the definition of supermatrices and operations on them.

0.1 INTRODUCTION TO FCMs AND FRMs MODELS

This section has two subsections. In the first subsection we just briefly recall the functioning of the FCMs from [47, 97] and in subsection two the concept of Fuzzy Relational Maps (FRMs) model is described from [94,95,97].

0.1.1 Description of FCMs model

In this section we recall the notion of Fuzzy Cognitive Maps (FCMs), which was introduced by Bart Kosko [47] in the year 1986. We also give several of its interrelated definitions. FCMs have a major role to play mainly when the data concerned is an unsupervised one. Further this method is most simple and an effective one as it can analyse the data using the directed graphs and its associated connection matrices. This model alone can give the hidden pattern of the problem.

DEFINITION 0.1.1.1: A Fuzzy Cognitive Maps (FCMs) is a directed graph with concepts like policies, events etc., as nodes and causalities as edges. It represents causal relationship between concepts.

We will just describe the FCMs by an example.

***Example 0.1.1.1**:* In Tamil Nadu (a southern state in India) in the last decade several new engineering colleges have been approved and started. The resultant increase in the number of engineering graduates in these years is disproportionate with the need of engineering graduates. This has resulted in thousands of unemployed and underemployed graduate engineers. Using an expert's opinion we study the influence of such unemployed people on the society. An expert spells out five major concepts relating to the unemployed graduated engineers as follows:

E_1	–	Frustration
E_2	–	Unemployment
E_3	–	Increase of educated criminals
E_4	–	Under employment
E_5	–	Taking up drugs etc.

The directed graph where E_1, …, E_5 are taken as the nodes and causalities as edges; as given by an expert is given in the following Figure 0.1.1.1:

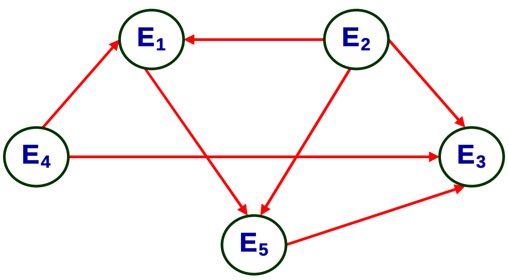

FIGURE: 0.1.1.1

According to this expert, increase in unemployment increases frustration. Increase in unemployment, increases the educated criminals.

Frustration increases the graduates to take up to evils like drugs etc. Unemployment also leads to the increase in the number of persons who take up to drugs, drinks etc., to forget their worries in their unoccupied time. Under-employment then forces to do criminal acts like theft (leading to murder) for want of more money and so on. Thus one cannot actually get data for this but can use the expert's opinion for this unsupervised data to obtain some ideas about the real plight of the situation. This is just an illustration to show how FCMs is described by a directed graph.

{If increase (or decrease) in one concept leads to increase (or decrease) in another, then we give the value 1. If there exists no relation between two concepts the value 0 is given. If increase (or decrease) in one concept decreases (or increases) another, then we give the value –1. Thus FCMs are described in this way.}

We recall the definitions related with FCMs from [47, 97].

DEFINITION 0.1.1.2: When the nodes of the FCMs are fuzzy sets then they are called as fuzzy nodes.

DEFINITION 0.1.1.3: FCMs with edge weights or causalities from the set {–1, 0, 1} are called simple FCMs.

DEFINITION 0.1.1.4: Consider the nodes / concepts $C_1, \ldots, C_n$ of the FCMs. Suppose the directed graph is drawn using edge weight; $e_{ij} \in \{0, 1, -1\}$. The matrix E be defined by $E = (e_{ij})$ where e_{ij} is the weight of the directed edge $C_i C_j$. E is called the adjacency matrix of the FCMs; also known as the connection matrix of the FCMs.

It is important to note that all matrices associated with FCMs are always square matrices with diagonal entries as zero.

DEFINITION 0.1.1.5: Let C_1, C_2, … , C_n be the nodes of an FCM. $A = (a_1, a_2, \ldots, a_n)$ where $a_i \in \{0, 1\}$. A is called the instantaneous state vector and it denotes the ON-OFF position of the node at an instant.

$$a_i = 0 \text{ if } a_i \text{ is OFF and}$$

$$a_i = 1 \text{ if } a_i \text{ is ON}$$

for i = 1, 2, …, n.

DEFINITION 0.1.1.6: Let C_1, C_2, … , C_n be the nodes of an FCM. Let $\overrightarrow{C_1C_2}$, $\overrightarrow{C_2C_3}$, $\overrightarrow{C_3C_4}$, … , $\overrightarrow{C_iC_j}$ be the edges of the FCM ($i \neq j$). Then the edges form a directed cycle. An FCM is said to be cyclic if it possesses a directed cycle. An FCM is said to be acyclic if it does not possess any directed cycle.

DEFINITION 0.1.1.7: An FCM with cycles is said to have a feedback.

DEFINITION 0.1.1.8: When there is a feedback in an FCM, i.e., when the causal relations flow through a cycle in a revolutionary way, the FCM is called a dynamical system.

DEFINITION 0.1.1.9: Let $\overrightarrow{C_1C_2}$, $\overrightarrow{C_2C_3}$, … ,$\overrightarrow{C_{n-1}C_n}$ be a cycle. When C_i is switched ON and if the causality flows through the edges of a cycle and if it again causes C_i , we say that the dynamical system goes round and round. This is true for any node C_i , for i = 1, 2, … , n. The equilibrium state for this dynamical system is called the hidden pattern.

DEFINITION 0.1.1.10: If the equilibrium state of a dynamical system is a unique state vector, then it is called a fixed point.

Example 0.1.1.2: Consider a FCM with C_1, C_2, …, C_n as nodes. For example let us start the dynamical system by switching on C_1. Let us assume that the FCM settles down with C_1 and C_n on i.e., the state vector remains as (1, 0, 0, …, 0, 1) this state vector (1, 0, 0, …, 0, 1) is called the fixed point.

DEFINITION 0.1.1.11: If the FCM settles down with a state vector repeating in the form

$$A_1 \to A_2 \to \ldots \to A_i \to A_1$$

then this equilibrium is called a limit cycle.

Methods of finding the hidden pattern are discussed in the following.

NOTATION: Suppose A = (a_1, … , a_n) is a vector which is passed into a dynamical system E, $a_i \in \{0, 1\}$, $1 \le i \le n$. Then AE = $(a'_1, a'_2, \ldots, a'_n)$ after thresholding and updating the vector suppose we get (b_1, … , b_n) we denote that by

$$(a'_1, a'_2, \ldots, a'_n) \hookrightarrow (b_1, b_2, \ldots, b_n).$$

Thus the symbol '$\hookrightarrow$' means the resultant vector that has been thresholded and updated. When we say thresholding we mean making the a'_i to belong to the set {0, 1}. This is carried out by the following method if $a'_i \le k$ we replace, it by 0 and if $a'_i > k$ we replace it by 1 where k is a predetermined value used by the expert.

For the dynamical system cannot realize elements other than 0 and 1 that is 'OFF' and 'ON' state respectively; that is why we need to threshold the state vector at each stage.

FCMs have several advantages as well as some disadvantages. The main advantage of this method, is simple. It functions on expert's opinion.

When the data happens to be an unsupervised one the FCM comes handy. This is the only known fuzzy technique that gives the hidden pattern of the situation.

As we have a very well known theory, which states that the strength of the data depends on, the number of experts' opinion we can use combined FCMs with several experts' opinions.

DEFINITION 0.1.1.12: Finite number of FCMs can be combined together to produce the joint effect of all the FCMs. Let E_1, E_2, … , E_p be the adjacency matrices of the FCMs with nodes C_1, C_2, …, C_n then the combined FCMs is got by adding all the adjacency matrices E_1, E_2, …, E_p .

We denote the combined FCM adjacency matrix by $E = E_1+E_2+\ldots+E_p$.

At the same time the disadvantage of the combined FCMs is when the weightages are 1 and –1 for the same C_i C_j, we have the sum adding to zero thus at all times the connection matrices E_1, …, E_k may not be conformable for addition.

Combined conflicting opinions tend to cancel out and assisted by the strong law of large numbers, a consensus emerges as the sample opinion approximates the underlying population opinion. This problem will be easily overcome if the entries of the FCMs are only 0 and 1. We have just briefly recalled the definitions. For more about FCMs please refer Kosko [47-51].

0.1.2 Introduction to Fuzzy Relational Maps (FRMs) model

In this section we just recall the definition and properties of FRM given in [94, 95, 97].

Here we recall the notion of Fuzzy Relational Maps (FRMs); they are constructed analogous to FCMs described and discussed in the earlier sections of [94, 95, 97]. In FCMs we promote the correlations between causal associations among concurrently active units. But in FRMs we divide the very causal associations into two disjoint units, for example, the relation between a teacher and a student or relation between an employee and an employer or a relation between a doctor and a patient and so on. Thus for us to define a FRMs we need a domain space and a range space which are disjoint in the sense of concepts. We further assume no intermediate relation exists within the domain space elements (or node) and the range space elements. The number of elements in the range space need not in general be equal to the number of elements in the domain space.

Thus throughout this section we assume the elements of the domain space are taken from the real vector space of dimension n and that of the range space are real vectors from the vector space of dimension m (m in general need not be equal to n). We denote by R the set of nodes $R_1,\ldots, R_m$ of the range space, where $R = \{(x_1,\ldots, x_m) \mid x_j = 0 \text{ or } 1 \}$ for j = 1, 2, …, m. If $x_i = 1$ it means that the node R_i is in the ON state and if $x_i = 0$ it means that the node R_i is in the OFF state. Similarly D denotes the nodes $D_1, D_2, \ldots, D_n$ of the domain space where $D = \{(x_1,\ldots, x_n) \mid x_i = 0 \text{ or } 1\}$ for i = 1, 2,…, n. If $x_i = 1$ it means that the node D_i is in the ON state and if $x_i = 0$ it means that the node D_i is in the OFF state.

Now we proceed on to recall the definition of FRMs.

DEFINITION 0.1.2.1: A Fuzzy Relational Maps (FRMs) is a directed graph or a map from $D = \{D_1, \ldots, D_n\}$ to $R = \{R_1, R_2, \ldots, R_m\}$ with concepts like policies or events etc, as nodes and causalities as edges. It represents causal relations between spaces D and R.

Let D_i and R_j denote the two nodes of an FRM. The directed edge from D_i to R_j denotes the causality of D_i on R_j called relations. Every edge in the FRMs is weighted with a number in the set $\{0, \pm 1\}$. Let e_{ij} be the weight of the edge D_iR_j, $e_{ij} \in \{0, \pm 1\}$. The weight of the edge $D_i R_j$ is positive if increase in D_i implies increase in R_j or decrease in D_i implies decrease in R_j, i.e., causality of D_i on R_j is 1. If $e_{ij} = 0$, then D_i does not have any effect on R_j.

We do not discuss the cases when increase in D_i implies decrease in R_j or decrease in D_i implies increase in R_j.

DEFINITION 0.1.2.2: When the nodes of the FRMs are fuzzy sets then they are called fuzzy nodes. FRMs with edge weights $\{0, \pm 1\}$ are called simple FRMs.

DEFINITION 0.1.2.3: Let $D_1, \ldots, D_n$ be the nodes of the domain space D of an FRM and $R_1, \ldots, R_m$ be the nodes of the range space R of an FRM. Let the matrix E be defined as $E = (e_{ij})$ where e_{ij} is the weight of the directed edge D_iR_j (or R_jD_i), E is called the relational matrix of the FRMs.

Note: It is pertinent to mention here that unlike the FCMs, the FRMs can be associated with a rectangular matrix with rows corresponding to the domain space and columns corresponding to the range space. This is one of the marked difference between FRMs and FCMs.

DEFINITION 0.1.2.4: Let $D_1, ..., D_n$ and $R_1,\ldots, R_m$ denote the nodes of the FRMs. Let $A = (a_1,\ldots,a_n)$, $a_i \in \{0, 1\}$, $1 \leq i \leq n$. A is called the instantaneous state vector of the domain space and it denotes the ON-OFF position of the nodes at any instant. Similarly let $B = (b_1,\ldots, b_m)$, $b_i \in \{0, 1\}$. B is called instantaneous state vector of the range space and it denotes the ON-OFF position of the nodes at any instant $a_i = 0$ if a_i is OFF and $a_i = 1$ if a_i is ON for $i = 1, 2,\ldots, n$. Similarly, $b_i = 0$ if b_i is OFF and $b_i = 1$ if b_i is ON, for $i = 1, 2,\ldots, m$.

DEFINITION 0.1.2.5: Let $D_1, \ldots, D_n$ and $R_1,\ldots, R_m$ be the nodes of an FRM. Let D_iR_j (or $R_j D_i$) be the edges of an FRM, $j = 1, 2,\ldots, m$ and $i= 1, 2,\ldots, n$. Let the edges form a directed cycle. An FRM is said to be a cycle if it posses a directed cycle. An FRM is said to be acyclic if it does not posses any directed cycle.

DEFINITION 0.1.2.6: *An FRM with cycles is said to be an FRM with a feedback.*

DEFINITION 0.1.2.7: *When there is a feedback in the FRMs, i.e., when the causal relations flow through a cycle in a revolutionary manner, the FRMs is called a dynamical system.*

DEFINITION 0.1.2.8: *Let $D_i R_j$ (or $R_j D_i$), $1 \leq j \leq m$, $1 \leq i \leq n$. When R_i (or D_j) is switched ON and if causality flows through edges of the cycle and if it again* *causes R_i (or D_j), we say that the dynamical system goes round and round. This is true for any node R_i (or R_j) for $1 \leq i \leq n$, (or $1 \leq j \leq m$). The equilibrium state of this dynamical system is called the hidden pattern.*

DEFINITION 0.1.2.9: *If the equilibrium state of a dynamical system is a unique state vector, then it is called a fixed point. Consider an FRM with $R_1, R_2, \ldots, R_m$ and $D_1, D_2, \ldots, D_n$ as nodes. For example, let us start the dynamical system by switching on R_1 (or D_1). Let us assume that the FRMs settles down with R_1 and R_m (or D_1 and D_n) on, i.e., the state vector remains as (1, 0, …, 0, 1) in R and (1, 0, 0, … , 0, 1) in D, This state vector is called the fixed point.*

DEFINITION 0.1.2.10: *If the FRM settles down with a state vector repeating in the form*

$$A_1 \to A_2 \to A_3 \to \ldots \to A_i \to A_1 \text{ (or } B_1 \to B_2 \to \ldots \to B_i \to B_1)$$

then this equilibrium is called a limit cycle.

Thus the hidden pattern of the dynamical system is a pair of vectors.

We now describe the method of determining the hidden pattern.

Let $R_1, R_2, \ldots, R_m$ and $D_1, D_2, \ldots, D_n$ be the nodes of a FRM with feedback. Let E be the relational matrix. Let us find the hidden pattern when D_1 is switched ON i.e., when an input is given as vector A_1 = (1, 0, …, 0) in D_1, the data should pass through the relational matrix E. This is done by multiplying A_1 with the relational matrix E. Let $A_1E = (r_1, r_2, \ldots, r_m)$, after thresholding and updating the resultant vector we get $A_1 E \in R$. Now let $B = A_1E$ we pass on B into E^T and obtain BE^T. We update and threshold the vector BE^T so that $BE^T \in D$. This procedure is repeated till we get limit cycle or fixed point.

DEFINITION 0.1.2.11: Finite number of FRMs can be combined together to produce the joint effect of all the FRMs. Let $E_1, \ldots, E_p$ be the relational matrices of the FRMs with nodes $R_1, R_2, \ldots, R_m$ and $D_1, D_2, \ldots, D_n$, then the combined FRMs is represented by the relational matrix $E = E_1 + \ldots + E_p$.

0.2 INTRODUCTION TO SUPERMATRICES

In this section we recall the definition of supermatrices and describe only those essential operations needed in this thesis on supermatrices. A simple matrix is the usual matrix. A super matrix M is a matrix whose elements are themselves matrices with elements from R or Q or Z or [0, 1]. When the elements of a supermatrix are from [0, 1] we call it as the fuzzy supermatrix.

We will illustrate this by simple examples.

Example 0.2.1: Let $X = (0\ 1\ 2 \mid -3 \mid 7\ -8\ 5 \mid 9) = (A_1\ \ A_2\ \ A_3\ \ A_4)$ where A_i's are simple row matrices $1 \le i \le 4$, where $A_1 = (0\ 1\ 2)$, $A_2 = (-3)$, $A_3 = (7, -8, 5)$ and $A_4 = (9)$. We call X as the super row matrix.

Example 0.2.2: Let

$$Y = \left[\begin{array}{c} 9 \\ -2 \\ \hline -9 \\ 0 \\ 1 \\ \hline -1 \\ 2 \\ -3 \\ 5 \end{array}\right] = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} 9 \\ -2 \end{bmatrix},\ A_2 = \begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 5 \end{bmatrix}$$

are simple column matrices. We call Y the super column matrix.

Example 0.2.3: Let

$$A = \left[\begin{array}{ccc|cc} -3 & 0 & 2 & 6 & 5 \\ 1 & 2 & -6 & -7 & 2 \\ \hline 0 & 1 & 2 & 1 & 0 \\ 3 & 5 & 6 & 9 & -6 \\ -7 & 8 & 9 & & \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

be a super matrix where

$$A_1 = \begin{bmatrix} -3 & 0 & 2 \\ 1 & 2 & -6 \end{bmatrix}, A_2 = \begin{bmatrix} 6 & 5 \\ -7 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 5 & 6 \\ -7 & 8 & 9 \end{bmatrix} \text{ and } A_4 = \begin{bmatrix} 1 & 0 \\ 9 & -6 \end{bmatrix}$$

are simple matrices.

Example 0.2.4: Let

$$T = \left[\begin{array}{ccc|cc} 0 & 1 & 0.5 & & \\ 0.1 & 0.2 & 0.3 & (0) & \\ 0.7 & 0 & 1 & & \\ \hline & & & 0.3 & 0.8 \\ & (0) & & 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} A_1 & (0) \\ \hline (0) & A_2 \end{array}\right]$$

where A_1 and A_2 are two matrices; and (0) denotes the simple zero matrix.

$$\text{Here } A_1 = \begin{bmatrix} 0 & 1 & 0.5 \\ 0.1 & 0.2 & 0.3 \\ 0.7 & 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0.3 & 0.8 \\ 1 & 0 \end{bmatrix}.$$

T is a super fuzzy diagonal square matrix.

Example 0.2.5: Let

$$M = \left[\begin{array}{cc|ccc} 0.3 & 0.2 & 0.3 & 0.2 & 0.6 \\ 0.6 & 1 & 1 & 0 & 0.8 \\ \hline 0.7 & 0.8 & 0.9 & 0 & 0.6 \\ 0.1 & 0.7 & 1 & 0.6 & 0 \\ 0.5 & 1 & 0.3 & 0.7 & 0.8 \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where each A_i is a simple fuzzy matrix, $1 \le i \le 4$; M is a super fuzzy matrix with

$A_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.6 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.3 & 0.2 & 0.6 \\ 1 & 0 & 0.8 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0.7 & 0.8 \\ 0.1 & 0.7 \\ 0.5 & 1 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 0.9 & 0 & 0.6 \\ 1 & 0.6 & 0 \\ 0.3 & 0.7 & 0.8 \end{bmatrix}$.

Example 0.2.6: Let

$$S = \left[\begin{array}{ccc} 0 & 8 & -1 \\ 9 & 2 & 0 \\ 1 & 8 & -4 \\ \hline 1 & -2 & -3 \\ \hline 6 & 8 & 4 \\ -8 & 9 & -6 \\ 7 & 0 & 8 \\ -1 & 2 & 3 \\ 4 & 5 & -6 \end{array}\right] = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} 0 & 8 & -1 \\ 9 & 2 & 0 \\ 1 & 8 & -4 \end{bmatrix}, \ A_2 = [1\ {-2}\ {-3}] \text{ and } A_3 = \begin{bmatrix} 6 & 8 & 4 \\ -8 & 9 & -6 \\ 7 & 0 & 8 \\ -1 & 2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$$

are simple matrices. S is the super column vector.

Example 0.2.7: Let

$$P = \left[\begin{array}{ccccc|cc|ccc|cccc} 0 & 1 & 2 & 3 & 4 & 7 & 8 & -2 & 0 & 3 & 1 & 4 & 8 & 4 \\ 5 & 6 & 7 & 8 & 9 & -1 & 0 & 6 & 8 & 5 & 0 & -1 & 7 & 5 \\ -1 & 0 & 1 & 6 & -5 & 2 & 6 & 7 & 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right] = [A_1\ \ A_2\ \ A_3\ \ A_4]$$

where each A_i is a simple matrix, $1 \le i \le 4$. P is defined as the super row matrix (vector), here

$$A_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ -1 & 0 & 1 & 6 & -5 \end{bmatrix}, A_2 = \begin{bmatrix} 7 & 8 \\ -1 & 0 \\ 2 & 6 \end{bmatrix}, A_3 = \begin{bmatrix} -2 & 0 & 3 \\ 6 & 8 & 5 \\ 7 & 0 & 1 \end{bmatrix} \text{ and}$$

$$A_4 = \begin{bmatrix} 1 & 4 & 8 & 4 \\ 0 & -1 & 7 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

For more please refer [39, 98]. We would be using the concept of supermatrices in the construction of super fuzzy models in chapter two of this thesis.

Now we describe a few of the essential operations on supermatrices which will be used in this thesis.

Example 0.2.8: Suppose we have

$$A = \left[\begin{array}{ccc} 0.3 & 0.2 & 1 \\ 5 & 1 & 0 \\ \hline 3 & 1 & 0 \\ 2 & 0.5 & 7 \\ 8 & 0.1 & 5 \\ \hline 0 & 9 & 0.7 \end{array}\right]$$

to be super column vector (matrix) then

$$A^t = \left[\begin{array}{ccc} 0.3 & 0.2 & 1 \\ 5 & 1 & 0 \\ \hline 3 & 1 & 0 \\ 2 & 0.5 & 7 \\ 8 & 0.1 & 5 \\ \hline 0 & 9 & 0.7 \end{array}\right]^t = \left[\begin{array}{cc|ccc|c} 0.3 & 5 & 3 & 2 & 8 & 0 \\ 0.2 & 1 & 1 & 0.5 & 0.1 & 9 \\ 1 & 0 & 0 & 7 & 5 & 0.7 \end{array}\right].$$

Clearly A^t is a super row vector or super row matrix.

Example 0.2.9: Let

$$B = \left[\begin{array}{ccccc|c|ccc|cc} 0 & 1 & 2 & 3 & 4 & 7 & 0 & 1 & 4 & 1 & 2 \\ 5 & 0 & 0 & 1 & 1 & 4 & 2 & 2 & 5 & 3 & 4 \\ 1 & 2 & 3 & 0 & 1 & 0.5 & 5 & 3 & 6 & 5 & 6 \end{array}\right]$$

be a super row matrix. Now the transpose of B denoted by

$$B^t = \left[\begin{array}{ccccc|c|ccc|cc} 0 & 1 & 2 & 3 & 4 & 7 & 0 & 1 & 4 & 1 & 2 \\ 5 & 0 & 0 & 1 & 1 & 4 & 2 & 2 & 5 & 3 & 4 \\ 1 & 2 & 3 & 0 & 1 & 0.5 & 5 & 3 & 6 & 5 & 6 \end{array}\right]^t = \left[\begin{array}{ccc} 0 & 5 & 1 \\ 1 & 0 & 2 \\ 2 & 0 & 3 \\ 3 & 1 & 0 \\ 4 & 1 & 1 \\ \hline 7 & 4 & 0.5 \\ \hline 0 & 2 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}\right].$$

Clearly B^t is a super column matrix (vector).

Example 0.2.10: Let

$$M = \left[\begin{array}{ccc|cccc} 3 & 0 & -1 & & & & \\ 0 & 3 & 4 & & & & \\ 5 & 7 & 8 & & (0) & & \\ -1 & 0 & 1 & & & & \\ \hline & & & 4 & 2 & 0 & 1 \\ & (0) & & 0 & 1 & -1 & 2 \\ & & & 3 & 0 & 1 & 1 \end{array}\right]$$

be a super diagonal matrix.

Then

$$M^t = \left[\begin{array}{ccc|cccc} 3 & 0 & -1 & & & & \\ 0 & 3 & 4 & & & & \\ 5 & 7 & 8 & & (0) & & \\ -1 & 0 & 1 & & & & \\ \hline & & & 4 & 2 & 0 & 1 \\ & (0) & & 0 & 1 & -1 & 2 \\ & & & 3 & 0 & 1 & 1 \end{array}\right]^t = \left[\begin{array}{cccc|ccc} 3 & 0 & 5 & -1 & & & \\ 0 & 3 & 7 & 0 & & (0) & \\ -1 & 4 & 8 & 1 & & & \\ \hline & & & & 4 & 0 & 3 \\ & (0) & & & 2 & 1 & 0 \\ & & & & 0 & -1 & 1 \\ & & & & 1 & 2 & 1 \end{array}\right];$$

M^t is also a super diagonal matrix, where (0) denotes the simple zero matrix.

Example 0.2.11: Let

$$A = \left[\begin{array}{cccc|cc} 0 & 2 & 1 & 4 & & \\ 9 & 1 & 0 & 2 & & \\ 0 & -1 & 3 & 0 & & (0) \\ 1 & 2 & 4 & 5 & & \\ \hline & & & & 7 & 2 \\ & & (0) & & 0 & 5 \\ & & & & -7 & 1 \\ & & & & 9 & 2 \end{array}\right]$$

be a super diagonal matrix.

$$A^t = \left[\begin{array}{cccc|cc} 0 & 2 & 1 & 4 & & \\ 9 & 1 & 0 & 2 & & \\ 0 & -1 & 3 & 0 & & (0) \\ 1 & 2 & 4 & 5 & & \\ \hline & & & & 7 & 2 \\ & & (0) & & 0 & 5 \\ & & & & -7 & 1 \\ & & & & 9 & 2 \end{array}\right]^t = \left[\begin{array}{cccc|cccc} 0 & 9 & 0 & 1 & & & & \\ 2 & 1 & -1 & 2 & & & & \\ 1 & 0 & 3 & 4 & & (0) & & \\ 4 & 2 & 0 & 5 & & & & \\ \hline & & & & 7 & 0 & -7 & 9 \\ & & (0) & & 2 & 5 & 1 & 2 \end{array}\right]$$

is also a super diagonal matrix, where (0) denotes the simple zero matrix.

Example 0.2.12: Let

$$A = \left[\begin{array}{ccc|cccc|cc} 3 & 1 & 2 & & & & & & \\ 6 & 4 & 1 & & (0) & & & & (0) \\ 0 & 1 & -3 & & & & & & \\ \hline & & & 7 & 8 & 9 & 4 & & \\ & & & -1 & 2 & 4 & 0 & & \\ & (0) & & 1 & 1 & 1 & 0 & & (0) \\ & & & 0 & 1 & 2 & 1 & & \\ & & & 0 & 6 & 1 & 5 & & \\ \hline & & & & & & & 1 & 2 \\ & (0) & & & (0) & & & 0 & 4 \end{array}\right]$$

be a super diagonal matrix.

$$A^t = \left[\begin{array}{ccc|cccc|cc} 3 & 1 & 2 & & & & & & \\ 6 & 4 & 1 & & (0) & & & & (0) \\ 0 & 1 & -3 & & & & & & \\ \hline & & & 7 & 8 & 9 & 4 & & \\ & & & -1 & 2 & 4 & 0 & & \\ & (0) & & 1 & 1 & 1 & 0 & & (0) \\ & & & 0 & 1 & 2 & 1 & & \\ & & & 0 & 6 & 1 & 5 & & \\ \hline & & & & & & & 1 & 2 \\ & (0) & & & (0) & & & 0 & 4 \end{array}\right]^t$$

$$= \left[\begin{array}{ccc|ccccc|cc} 3 & 6 & 0 & & & & & & & \\ 1 & 4 & 1 & & & (0) & & & & (0) \\ 2 & 1 & -3 & & & & & & & \\ \hline & & & 7 & -1 & 1 & 0 & 0 & & \\ & (0) & & 8 & 2 & 1 & 1 & 6 & & (0) \\ & & & 9 & 4 & 1 & 2 & 1 & & \\ & & & 4 & 0 & 0 & 1 & 5 & & \\ \hline & & & & & & & & 1 & 0 \\ & (0) & & & & (0) & & & 2 & 4 \end{array}\right]^t$$

is also super diagonal matrix, where (0) denotes the zero simple matrix. This is a super diagonal square matrix.

We give only those operations which we use in this thesis.

We define special operation of a fuzzy row matrix with a fuzzy super row vector.

Example 0.2.13: Let

$$L = \left[\begin{array}{ccc|cccc|c} 0 & 0.2 & 1 & 0 & 0.4 & 0.2 & 0.1 & 0.9 \\ 1 & 0.7 & 0.2 & 0.4 & 0.7 & 0 & 0.2 & 0.8 \\ 0.5 & 0.1 & 0.3 & 0.3 & 0 & 0.9 & 0.3 & 0 \end{array}\right]$$

be a super fuzzy row vector (matrix). Let X = (0 0.1 0.3) be a fuzzy row matrix. To find the product of X with L.

We use only the max-min operation of X on L, XL = max (min {X, L}.

= [max (min (0, 0), min (0.1, 1), min (0.3, 0.5)) max (min (0, 0.2), min (0.1, 0.7), min (0.3, 0.1)) max (min (0, 1), min (0.1, 0.2), min (0.3, 0.3)) | max (min (0, 0), min (0.1, 0.4), min (0.3, 0.3) … | min (0, 0.9), min (0.1, 0.8), min (0, 0.3)]

= [0.3 0.1 0.3 | 0.3 0.1 0.3 0.3 | 0.1] = Y.

Now in this thesis we have to find also the value

$YL^t = \max (\min \{Y, L^t\})$

$$= [0.3\ \ 0.1\ \ 0.3 \mid 0.3\ \ 0.1\ \ 0.3\ \ 0.3 \mid 0.1] \left[\begin{array}{ccc} 0 & 1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 1 & 0.2 & 0.3 \\ \hline 0 & 0.4 & 0.3 \\ 0.4 & 0.7 & 0 \\ 0.2 & 0 & 0.9 \\ 0.1 & 0.2 & 0.3 \\ \hline 0.9 & 0.8 & 0 \end{array}\right]$$

= [0.3,   0.3,   0.3].

This way we perform operations using fuzzy super row matrices and fuzzy row matrices.

Example 0.2.14: Let

$$M = \left[\begin{array}{ccc|cc|cccc} 0.3 & 0.4 & 0.7 & & & & & & \\ 0.1 & 0 & 0.5 & \multicolumn{2}{c|}{(0)} & \multicolumn{4}{c}{(0)} \\ 0 & 0.6 & 0 & & & & & & \\ \hline \multicolumn{3}{c|}{(0)} & 0.3 & 0.2 & \multicolumn{4}{c}{(0)} \\ & & & 0.7 & 0.8 & & & & \\ \hline & & & & & 0 & 1 & 0.3 & 0.2 \\ & & & & & 1 & 0.4 & 0 & 0.8 \\ \multicolumn{3}{c|}{(0)} & \multicolumn{2}{c|}{(0)} & 0 & 0.4 & 0.1 & 0 \\ & & & & & 0.3 & 0 & 1 & 0.2 \end{array}\right]$$

be the super fuzzy diagonal square matrix.

Let X = [0.1 1 0.2 | 0.1 0.3 | 0 1 0.3 0.2] be the super fuzzy row vector (matrix).

XM = min (max {X, M})

= [min {0.3, 1, 0.2}, min {0.4, 1, 0.6}, min {0.7, 1, 0.2} | min {0.3, 0.7}, min {0.2, 0.8} | min {0, 1, 0.3, 0.3}, min {1, 1, 0.4, 0.2}, min {0.3, 1, 0.3, 1}, min {0.2, 1, 0.3, 0.2}]

= [0.2 0.4 0.2 | 0.3 0.2 | 0 0.2 0.3 0.2].

If it is a super fuzzy square diagonal matrix we in this thesis do not go for the transpose.

Example 0.2.15: Let

$$M = \left[\begin{array}{ccc|cc|ccc} 0.7 & 0.3 & 0.4 & & & & & \\ 0.5 & 0 & 1 & & (0) & & (0) & \\ 0.6 & 0.2 & 0 & & & & & \\ 0 & 0.5 & 0.7 & & & & & \\ \hline & & & 0.3 & 0.2 & & & \\ & (0) & & 0.5 & 0.6 & & (0) & \\ & & & 0.7 & 0.8 & & & \\ \hline & & & & & 0.4 & 0 & 1 \\ & (0) & & & (0) & 0.6 & 0.7 & 0 \\ & & & & & 0 & 1 & 0.8 \end{array}\right]$$

be a super fuzzy square diagonal matrix.

Suppose X = (0 1 0.3 0.2 | 0.1 0.3 1 | 1 0.2 0.1) be a super fuzzy row vector. To find XM = max (min {X, M})

= [max (0, 0.5, 0.3, 0), max (0 0 0.2 0.2) max (0 1 0 0.2) | max (0.1 0.3 0.7) max (0.1 0.3 0.8) | max (0.4 0.2 0) max (0 0.2 0.1) max (1 0 0.1)]

= [0.5 0.2 1 | 0.7 0.8 | 0.4 0.2 1] = Y.

Y M_s^t Here M_s^t is a special transpose of M where M contains square matrices as well as rectangular matrices. While working with the special transpose we do not transpose the square matrices we only transpose the rectangular matrices. This sort of special transpose is used only while working with super combined fuzzy cognitive relational models [101].

Thus we have $M_s^t =$

$$
\left[\begin{array}{cccc|ccc|ccc}
0.7 & 0.5 & 0.6 & 0 & & & & & & \\
0.3 & 0 & 0.2 & 0.5 & & (0) & & & (0) & \\
0.4 & 1 & 0 & 0.7 & & & & & & \\
\hline
 & & & & 0.3 & 0.5 & 0.7 & & & \\
 & (0) & & & 0.2 & 0.6 & 0.8 & & (0) & \\
\hline
 & & & & & & & 0.4 & 0.6 & 0 \\
 & (0) & & & & (0) & & 0 & 0.7 & 1 \\
 & & & & & & & 1 & 0 & 0.8
\end{array}\right].
$$

Now we find YM_s^t = max {min (Y, M_s^t)} = [0.5 1 0.5 0.7 | 0.3 0.6 0.8 | 0.4 1 0.8].

This is the way we perform operations with super fuzzy diagonal matrices when some of the diagonal matrices are rectangular and some of them are square. When we perform min (max) or max (min) or min (min) or max (max) operation, use only special transpose of the diagonal super fuzzy matrix.

Please refer [101] for more about these super fuzzy matrices.