CHAPTER - 7

m-I-SUBMAXIMAL IDEAL MINIMAL SPACES

7.1 INTRODUCTION

The concept of submaximality of general topological spaces was introduced by Hewitt [31] in 1943. He discovered a general way of constructing maximal topologies. In [4], Alas et al proved that there can be no dense maximal subspace in a product of first countable spaces, while under Booth’s Lemma there exists a dense submaximal subspaces in $[0, 1]^\omega$. It is established that under the axiom of constructibility any submaximal Hausdorff space is $\sigma$-discrete. Any homogeneous submaximal space is strongly $\sigma$-discrete if there are no measurable cardinals. The first systematic study of submaximal spaces was undertaken in the paper of Arhangel’skii and Collins [5]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is left-separated. This led to the question whether every submaximal space is $\sigma$-discrete [5]. The notion of ideal topological spaces was studied by Kuratowski [35] and Vaidyanathaswamy [85]. In 1990, Jankovic and Hamlett [33] investigated further properties of ideal topological spaces. In [16], properties of I-submaximal ideal topological spaces is studied. Quite Recently, the notion of ideal minimal spaces is studied by Ozbakir and Yildirim [63]. In this chapter, several characterizations and properties of m-I-submaximal ideal minimal spaces are obtained.

7.2 PRELIMINARIES
Theorem 7.2.1 [63]

Let \((X, m_x)\) be a minimal space with \(I, I'\) ideals on \(X\) and \(A, B\) be subsets of \(X\). Then

(i) \(A \subseteq B \Rightarrow A^*_m \subseteq B^*_m\)

(ii) \(I \subseteq I' \Rightarrow A^*_m(I') \subseteq A^*_m(I)\)

(iii) \(A^*_m = m_x-\text{Cl}(A^*_m) \subseteq m_x-\text{Cl}(A)\),

(iv) \(A^*_m \cup B^*_m \subseteq (A \cup B)^*_m\)

(v) \((A^*_m)_m \subseteq A^*_m\).

Remark 7.2.2 [63]

If \((X, m_x)\) has property \(I\), then \(A^*_m \cup B^*_m = (A \cup B)^*_m\).

Throughout the chapter we simply write \(m^*_x\) for \(m^*_x(I, m_x)\). If \(I\) is an ideal on \(X\), then \((X, m_x, I)\) is called an ideal minimal space (or an ideal m-space).

Proposition 7.2.3 [63]

(i) If \(A \subseteq B\), then \(m-\text{Cl}^*(A) \subseteq m-\text{Cl}^*(B)\),

(ii) \(m-\text{Cl}^*(A) \cup m-\text{Cl}^*(B) \subseteq m-\text{Cl}^*(A \cup B)\).

Remark 7.2.4 [63]

If \((X, m_x)\) has property \(I\), then \(m-\text{Cl}^*(m-\text{Cl}^*(A)) = m-\text{Cl}^*(A)\) and \(m-\text{Cl}^*(A) \cup m-\text{Cl}^*(B) = m-\text{Cl}^*(A \cup B)\).

Definition 7.2.5 [63]
A subset \( A \) of an ideal minimal space \((X, m_x, I)\) is \(m^*\)-dense in itself (resp. \(m^*\)-perfect, \(m^*\)-closed) if \( A \subseteq A^*_m \) (resp. \( A^*_m = A, \ A^*_m \subseteq A \)).

**Remark 7.2.6 [63]**

A subset \( A \) of an ideal minimal space \((X, m_x, I)\) is \(m^*\)-closed if and only if \( A^*_m \subseteq A \).

**Definition 7.2.7 [63]**

A mapping \( f : (X, m_x) \rightarrow (Y, m_y) \) is said to be \(M\)-open if for each \( U \in m_x, \ f(U) \in m_y \).

### 7.3. ON SUBSETS OF IDEAL MINIMAL SPACES

**Definition 7.3.1**

A subset \( A \) of an ideal \(m\)-space \((X, m_x, I)\) is called

(i) \( \alpha\)-\(m\)-I-open if \( A \subseteq m\)-Int\((m\)-\(Cl^*(m\)-Int\((A))\)),

(ii) \( \text{pre-}m\)-I-open if \( A \subseteq m\)-Int\((m\)-\(Cl^*(A))\)),

(iii) \( \text{semi-}m\)-I-open if \( A \subseteq m\)-\(Cl^*(m\)-Int\((A))\)),

(iv) strongly \( \beta\)-\(m\)-I-open if \( A \subseteq m\)-\(Cl^*(m\)-\(Cl^*(m\)-Int\((A))\)),

(v) \(m^*\)-dense if \( m\)-\(Cl^*(A) = X\).

**Lemma 7.3.2**

(i) Every \(m^*\)-dense set is \(\text{pre-}m\)-I-open.

(ii) Every \(m^*\)-dense set is strongly \( \beta\)-\(m\)-I-open.

**Proof**

(i) Let \( A \) be \(m^*\)-dense. Then \( m\)-\(Cl^*(A) = X\). We have \( m\)-\(Int(m\)-\(Cl^*(A)) = X \supseteq A \). It shows that \( A \) is \(\text{pre-}m\)-I-open.
(ii) Let \( A \) be \( m^* \)-dense set. Then \( m\text{-Cl}^*(A) = X \). We have \( m\text{-Cl}^*(m\text{-Int}(m\text{-Cl}^*(A))) = X \supseteq A \). This shows that \( A \) is strongly \( \beta \)-m-I-open.

**Lemma 7.3.3**

Let \( A \) be a subset of an ideal m-space \((X, m_x, I)\) such that \( m\text{-Int}(m\text{-Cl}^*(A)) \in m_x \). Then if \( A \) is pre-m-I-open, then \( A = U \cap D \), where \( U \) is m-open and \( D \) is \( m^* \)-dense.

**Proof**

Let \( A \) be a pre-m-I-open, then we have \( A \subseteq m\text{-Int}(m\text{-Cl}^*(A)) = U \in m_x \). Let \( D = X - (U - A) = X - (U \cap A^c) = X \cap (U \cap A^c)^c = U^c \cup A = (X - U) \cup A \). Then \( D \) is \( m^* \)-dense since \( X = m\text{-Cl}^*(A) \cup (X - m\text{-Cl}^*(A)) \subseteq m\text{-Cl}^*(A) \cup (X - U) \subseteq m\text{-Cl}^*(A) \cup m\text{-Cl}^*(X - U) \subseteq m\text{-Cl}^*[X - U \cup A] = m\text{-Cl}^*(D) \). Also \( A = U \cap D \).

**Lemma 7.3.4**

For a subset \( A \) of an ideal m-space \((X, m_x, I)\), the following properties hold.

(i) Every \( \alpha \)-m-I-open set is semi-m-I-open.

(ii) Every \( \alpha \)-m-I-open set is pre-m-I-open.

(iii) Every semi-m-I-open set is strongly \( \beta \)-m-I-open.

(iv) Every pre-m-I-open set is strongly \( \beta \)-m-I-open.

**Proof**

(i) Let \( A \) be an \( \alpha \)-m-I-open, then we have \( A \subseteq m\text{-Int}(m\text{-Cl}^*(m\text{-Int}(A))) \subseteq m\text{-Cl}^*(m\text{-Int}(A)) \). This shows that \( A \) is semi-m-I-open.
(ii) Let $A$ be an $\alpha$-m-I-open, then we have $A \subseteq m$-$\text{Int}(m$-$\text{Cl}^*(m$-$\text{Int}(A))) \subseteq m$-$\text{Int}(m$-$\text{Cl}^*(A))$. This shows that $A$ is pre-$m$-I-open.

(iii) Let $A$ be a semi-$m$-I-open, then we have $A \subseteq m$-$\text{Cl}^*(m$-$\text{Int}(A)) \subseteq m$-$\text{Cl}^*(m$-$\text{Int}(m$-$\text{Cl}^*(A)))$. This shows that $A$ is strongly $\beta$-m-I-open.

(iv) Let $A$ be a pre-$m$-I-open, then we have $A \subseteq m$-$\text{Int}(m$-$\text{Cl}^*(A)) \subseteq m$-$\text{Cl}^*(m$-$\text{Int}(m$-$\text{Cl}^*(A)))$. This shows that $A$ is strongly-$\beta$-m-I-open.

**Lemma 7.3.5**

Let $(X, m_x, I)$ be an ideal m-space satisfying property I and $A \subseteq X$. Then $A$ is $\alpha$-m-I-open if and only if it is semi-m-I-open and pre-m-I-open.

**Proof**

Necessity. This is obvious.

Sufficiency. Since a set $A$ is both semi-m-I-open and pre-m-I-open, we have $A \subseteq m$-$\text{Int}(m$-$\text{Cl}^*(A)) \subseteq m$-$\text{Int}(m$-$\text{Cl}^*(m$-$\text{Cl}^*(m$-$\text{Int}(A)))) = m$-$\text{Int}(m$-$\text{Cl}^*(m$-$\text{Int}(A)))$. Therefore $A$ is $\alpha$-m-I-open.

**Proposition 7.3.6**

Every m-open set is semi-m-I-open.

**Proof**

Let $A$ be an m-open set in $X$. Then $A \subseteq m$-$\text{Cl}^*(A)$. Since $A = m$-$\text{Int}(A)$, $A \subseteq m$-$\text{Cl}^*(m$-$\text{Int}(A))$. This shows that $A$ is semi-m-I-open.

**Proposition 7.3.7**
Every m-open set is pre-m-I-open.

**Proof**

Let $A$ be an m-open set in $X$. Then $A \subseteq m-\text{Cl}^*(A) \Rightarrow m-\text{Int}(A) \subseteq m-\text{Int}(m-\text{Cl}^*(A))$. Since $A = m-\text{Int}(A)$, $A \subseteq m-\text{Int}(m-\text{Cl}^*(A))$. This shows that $A$ is pre-m-I-open.

### 7.4. m-I-SUBMAXIMAL IDEAL MINIMAL SPACES

**Definition 7.4.1**

An ideal m-space $(X, m_x, I)$ is called m-I-submaximal if every $m^*$-dense subset of $X$ is m-open.

**Theorem 7.4.2**

Let $A$ be a subset of $(X, m_x, I)$ such that $m-\text{Int}(m-\text{Cl}^*(A)) \in m_x$. Let $(X, m_x)$ have property I. Then the following conditions are equivalent.

(i) $(X, m_x, I)$ is m-I-submaximal.

(ii) If $A$ is pre-m-I-open set, then $A$ is m-open.

**Proof**

(i) $\Rightarrow$ (ii): Let $A$ be pre-m-I-open set. Then, by Lemma 7.3.3, $A = U \cap D$ for some $U \in m_x$ and $m^*$-dense $D \subseteq X$. Since $(X, m_x, I)$ is m-I-submaximal, $D \in m_x$. Then $A \in m_x$. 
(ii) ⇒ (i): Let $A$ be a $m^*$-dense subset of $X$. By Lemma 7.3.2, $A$ is pre-$m$-$I$-open. By hypothesis, $A$ is $m$-open and so the space is $m$-$I$-submaximal.

**Theorem 7.4.3**

Let $A$ be a subset of $(X, m_\kappa, I)$ such that $m$-$\text{Int}(m$-$\text{Cl}^*(A)) \in m_\kappa$. Let $(X, m_\kappa)$ have property $I$. Then the following properties are equivalent:

(i) $X$ is $m$-$I$-submaximal,

(ii) If $A$ is pre-$m$-$I$-open set, then $A$ is $m$-open,

(iii) If $A$ is pre-$m$-$I$-open set, then $A$ is semi-$m$-$I$-open and if $\alpha$-$m$-$I$-open set, then $A$ is $m$-open.

**Proof**

(i) $\Leftrightarrow$ (ii): It follows from Theorem 7.4.2.

(ii) ⇒ (iii): Suppose that every pre-$m$-$I$-open set is $m$-open. Then every pre-$m$-$I$-open set is semi-$m$-$I$-open by Proposition 7.3.6. Let $A \subseteq X$ be an $\alpha$-$m$-$I$-open set. Since every $\alpha$-$m$-$I$-open set is pre-$m$-$I$-open, then by (ii), $A$ is $m$-open.

(iii) ⇒ (i): Let $A$ be a $m^*$-dense subset of $X$. Then by Lemma 7.3.2, $A$ is pre-$m$-$I$-open. By (iii), $A$ is also semi-$m$-$I$-open. Since a set is $\alpha$-$m$-$I$-open if and only if it is semi-$m$-$I$-open and pre-$m$-$I$-open, then $A$ is $\alpha$-$m$-$I$-open. Thus, by (iii), $A$ is $m$-open and hence $X$ is $m$-$I$-submaximal.

**Theorem 7.4.4**
Let $A$ be a subset of $(X, m_x, I)$ such that $m\text{-Int}(m\text{-Cl}^*(A)) \in m_x$. Let $(X, m_x)$ have property I. Then the following properties are equivalent:

(i) $X$ is $m$-I-submaximal,

(ii) For all $A \subseteq X$, if $A \setminus m\text{-Int}(A) \neq \emptyset$, then $A \setminus m\text{-Int}(m\text{-Cl}^*(A)) \neq \emptyset$.

**Proof**

(i) $\Rightarrow$ (ii): Let $A \subseteq X$ and $A \setminus m\text{-Int}(A) \neq \emptyset$. Suppose that $A \setminus m\text{-Int}(m\text{-Cl}^*(A)) = \emptyset$. Then $A \subseteq m\text{-Int}(m\text{-Cl}^*(A))$. This implies that $A$ is pre-$m$-I-open. Since $X$ is $m$-I-submaximal, by Theorem 7.4.2, $A$ is $m$-open. Thus, $A \setminus m\text{-Int}(A) = A \setminus A = \emptyset$. This is a contradiction.

(ii) $\Rightarrow$ (i): Let $A$ be a pre-$m$-I-open set. Then $A \subseteq m\text{-Int}(m\text{-Cl}^*(A))$. Suppose that $A$ is not $m$-open. Then $A \not\subseteq m\text{-Int}(A)$ and hence $A \setminus m\text{-Int}(A) \neq \emptyset$. By (ii), $A \setminus m\text{-Int}(m\text{-Cl}^*(A)) \neq \emptyset$. Thus, $A \not\subseteq m\text{-Int}(m\text{-Cl}^*(A))$. This is a contradiction.

**Definition 7.4.5**

A subset $A$ of a minimal space $(X, m_x)$ is called $m$-dense if $m\text{-Cl}(A) = X$.

**Definition 7.4.6**

A minimal space $(X, m_x)$ is called $m$-submaximal space if each of its $m$-dense subset is $m$-open.

**Theorem 7.4.7**
Let $f : (X, m_x) \to (Y, m_y, I)$ be an M-open surjective mapping. If $X$ is $m$-submaximal, then $Y$ is $m$-I-submaximal.

**Proof**

Let $X$ be $m$-submaximal and $A \subseteq Y$ be a $m^*$-dense set. Since $m_y \subseteq m_y^*$, then $A$ is $m$-dense in $Y$. Since $f^1(A)$ is $m$-dense in $X$. $f^1(A)$ is $m$-open in $X$. Since $f$ is an M-open surjective function, then $A = f(f^{-1}(A))$ is $m$-open in $Y$. Hence, $Y$ is $m$-I-submaximal.

**Definition 7.4.8**

A subset $A$ of an ideal $m$-space $(X, m_x, I)$ is called $m^*$-codense if $X \setminus A$ is $m^*$-dense.

**Theorem 7.4.9**

For an ideal $m$-space $(X, m_x, I)$, the following are equivalent:

(i) $X$ is $m$-I-submaximal,

(ii) Every $m^*$-codense subset $A$ of $X$ is $m$-closed.

**Proof**

(i) $\Rightarrow$ (ii): Let $A$ be a $m^*$-codense subset of $X$. Since $X \setminus A$ is $m^*$-dense, then $X \setminus A$ is $m$-open. Then, $A$ is $m$-closed.

(ii) $\Rightarrow$ (i): It is similar to that of (i) $\Rightarrow$ (ii).

**Definition 7.4.10**

A subset $A$ of an ideal $m$-space $(X, m_x, I)$ is called

(i) a t-m-I-set if $m\text{-Int}(A) = m\text{-Int}(m\text{-Cl}^*(A))$, 

(ii) semi-m-I-regular if A is a t-m-I-set and semi-m-I-open,

(iii) an AB\textsubscript{m-I} set if \( A = U \cap V \), where \( U \in m_x \) and \( V \) is a semi-m-I-regular set.

**Theorem 7.4.11**

Suppose m-\( \text{Cl}^* \) is a Kuratowski closure operation. Let \( V \) be a subset of \((X, m_x, I)\) such that m-\( \text{Int}(V) \in m_x \). Let \((X, m_x)\) have property I. Then the following properties are equivalent:

(i) \( A \) is m-open.

(ii) \( A \) is a pre-m-I-open set and an AB\textsubscript{m-I} set.

**Proof**

(i) \( \Rightarrow \) (ii) Obvious.

(ii) \( \Rightarrow \) (i) Let \( A \) be a pre-m-I-open set and an AB\textsubscript{m-I} set. Then, since \( A \) is a pre-m-I-open set, \( A \subseteq m-\text{Int}(m-\text{Cl}^*(A)) \). Furthermore, because \( A \) is an AB\textsubscript{m-I} set, we have \( A = U \cap V \), where \( U \in m_x \) and \( V \) is a semi-m-I-regular set. Since m-\( \text{Cl}^* \) is a Kuratowski closure operation, \( A \subseteq m-\text{Int}(m-\text{Cl}^*(A)) = m-\text{Int}(m-\text{Cl}^*(U \cap V)) \subseteq m-\text{Int}(m-\text{Cl}^*(U) \cap m-\text{Cl}^*(V)) \subseteq m-\text{Int}(m-\text{Cl}^*(U)) \cap m-\text{Int}(m-\text{Cl}^*(V)) \). Hence \( A \subseteq m-\text{Int}(m-\text{Cl}^*(U)) \cap m-\text{Int}(m-\text{Cl}^*(V)) \). Since \( V \) is a semi-m-I-regular set, \( V \) is also a t-m-I-set. Thus \( m-\text{Int}(V) = m-\text{Int}(m-\text{Cl}^*(V)) \). We have \( A \subseteq m-\text{Int}(m-\text{Cl}^*(U)) \cap m-\text{Int}(V) \). Since \( A \subseteq U \), \( A = U \cap A \subseteq U \cap [m-\text{Int}(m-\text{Cl}^*(U)) \cap m-\text{Int}(V)] = [U \cap m-\text{Int}(m-\text{Cl}^*(U)))] \cap m-\text{Int}(V) = U \cap m-\text{Int}(V) \). We
have \( A \subseteq U \cap m\text{-Int}(V) \subseteq U \cap V = A \). Hence \( A = U \cap m\text{-Int}(V) \) and \( A \in m_x \).

**Theorem 7.4.12**

Suppose \( m\text{-Cl}^* \) is a Kuratowski closure operation. Let \( V \) be a subset of \((X, m_x, I)\) such that \( m\text{-Int}(V) \in m_x \). Let \((X, m_x)\) have property \( I \). Then the following properties are equivalent:

(i) \( X \) is \( m\)-\( I \)-submaximal.
(ii) Every pre-\( m\)-\( I \)-open set is an \( AB_{m,I} \)-set.
(iii) Every \( m^* \)-dense set is an \( AB_{m,I} \)-set.

**Proof**

(i) \( \Rightarrow \) (ii): Let \( A \subseteq X \) be a pre-\( m\)-\( I \)-open set. Since \( X \) is \( m\)-\( I \)-submaximal, by Theorem 7.4.2, \( A \) is \( m \)-open. It follows from definition 7.4.10 (iii) that \( A \) is an \( AB_{m,I} \)-set.

(ii) \( \Rightarrow \) (iii): Let \( A \subseteq X \) be a \( m^* \)-dense set. Since every \( m^* \)-dense set is pre-\( m\)-\( I \)-open, then by (ii), \( A \) is an \( AB_{m,I} \)-set.

(iii) \( \Rightarrow \) (i): Let \( A \subseteq X \) be a \( m^* \)-dense set. By (iii), \( A \) is an \( AB_{m,I} \)-set. Since every \( m^* \)-dense set is pre-\( m\)-\( I \)-open, then \( A \) is pre-\( m\)-\( I \)-open. Since \( A \) is pre-\( m\)-\( I \)-open and an \( AB_{m,I} \)-set, by Theorem 7.4.11, \( A \) is \( m \)-open. Hence, \( X \) is \( m\)-\( I \)-submaximal.

**Theorem 7.4.13**

For an ideal \( m \)-space \((X, m_x, I)\) satisfying property \( \mathcal{B} \), the following are equivalent.
(i) Every m*-dense in itself subset is pre-m-I-open.

(ii) Every m*-perfect subset is m-open.

Proof

(i) ⇒ (ii) Let A ⊆ X be m*-perfect. By hypothesis, A is pre-m-I-open and hence A ⊆ m-Int(m·Cl*(A)) = m-Int(A). Thus A is m-open.

(ii) ⇒ (i) Let A ⊆ X be m*-dense in itself. Then A ⊆ A_m^* and A_m^* = m-Cl*(A). On the other hand, A_m^* ⊆ (A_m^*)_m^* ⊆ A_m^* and hence A_m^* = (A_m^*)_m^*. Consequently we have (m-Cl*(A))_m = m-Cl*(A). Then m-Cl*(A) is m*-perfect. By hypothesis, m-Cl*(A) is m-open, hence A ⊆ m-Cl*(A) = m-Int(m·Cl*(A)). Thus A is pre-m-I-open.

Definition 7.4.14

A subset A of an ideal m-space (X, m, I) is called a

(i) B_m,I-set if A = U ∩ V, where U ∈ m and V is a t-m-I-set.

(ii) I-locally m*-closed [19] if A = U ∩ V where U ∈ m and V is m*-closed.

Theorem 7.4.15 [69]

Let (X, m, I) be an ideal m-space satisfying property $\heartsuit$ and A ⊆ X is m*-dense in itself. Then the following are equivalent:

(i) A is I-locally m*-closed.

(ii) A = U ∩ A_m^* for some m-open set U.

(iii) A_m^* − A is m-closed.
Theorem 7.4.16

Let \((X, m_x, I)\) be an ideal m-space satisfying property \(\mathbb{B}\) and \(A \subseteq X\) is \(m^*\)-perfect. Then the following are equivalent:

(i) \((X, m_x, I)\) is \(m\)-I-submaximal,

(ii) \(m-\text{Cl}^*(A) - A = A^*_m - A\) is \(m\)-closed for every subset \(A\) of \(X\),

(iii) Every subset of \(X\) is \(I\)-locally \(m^*\)-closed,

(iv) Every subset of \(X\) is a \(B_{m,I}\) set,

(v) Every \(m^*\)-dense subset of \(X\) is a \(B_{m,I}\) set.

Proof

(i) \(\Rightarrow\) (ii): Let \(A \subseteq X\). Then \(\text{m-Cl}^*[X - (m-\text{Cl}^*(A) - A)] = \text{m-Cl}^*[A \cup (X - m-\text{Cl}^*(A))] = X\) and so \(X - [m-\text{Cl}^*(A) - A]\) is \(m^*\)-dense. By hypothesis, \(X - [m-\text{Cl}^*(A) - A]\) is \(m\)-open and so \(m-\text{Cl}^*(A) - A\) is \(m\)-closed.

(ii) \(\iff\) (iii): It is obvious from Theorem 7.4.15

(iii) \(\Rightarrow\) (iv): Let \(A\) be \(I\)-locally \(m^*\)-closed. Then \(A = U \cap V\), where \(U \in m_x\) and \(V\) is \(m^*\)-closed set. Since \(V\) is \(m^*\)-closed, \(V^*_m \subseteq V\). Now \(m-\text{Cl}^*(V) = V \cup V^*_m = V\) It implies that \(m-\text{Int}(m-\text{Cl}^*(V)) = m-\text{Int}(V)\) which shows that \(V\) is \(t\)-\(m\)-I-set. Hence \(A\) is \(B_{m,I}\) set.

(iv) \(\Rightarrow\) (v): clear.
(v) ⇒ (i): Let A be a $m^*$-dense subset of X. By (v), A is a $B_{m^*}$-set and so $A = G \cap F$ where $G \in m_x$ and $m$-$\text{Int}(F) = m$-$\text{Int}(m$-$\text{Cl}^*(F))$. Since $A \subseteq F$, $m$-$\text{Cl}^*(A) \subseteq m$-$\text{Cl}^*(F)$ and so $X = m$-$\text{Cl}^*(F)$. Therefore $X = m$-$\text{Int}(m$-$\text{Cl}^*(F)) = m$-$\text{Int}(F)$ which implies that $F = X$. Hence $A = G \cap F = G \cap X = G \in m_x$.

**Theorem 7.4.17**

Let $(X, m_x, I)$ be an ideal m-space satisfying property $\mathcal{B}$ and $A \subseteq X$ is $m^*$-perfect. Then the following are equivalent:

(i) X is $m$-I-submaximal,

(ii) Every subset of X is a $B_{m^*}$-set,

(iii) Every strongly $\beta$-$m$-I-open set is a $B_{m^*}$-set,

(iv) Every $m^*$-dense subset of X is a $B_{m^*}$-set.

**Proof**

(i) ⇒ (ii): It follows from Theorem 7.4.16.

(ii) ⇒ (iii): Obvious.

(iii) ⇒ (iv): It follows from the fact that every $m^*$-dense subset of X is a strongly $\beta$-$m$-I-open set.

(iv) ⇒ (i): It follows from Theorem 7.4.16.

**Proposition 7.4.18**

Every m-submaximal space is m-I-submaximal space.

**Proof**
Let $A \subseteq X$ be $m^*$-dense. Since $m_x \subseteq m^*_x$, $A$ is $m$-dense. Since $X$ is $m$-submaximal, then $A$ is $m$-open in $X$. Then $X$ is $m$-I-submaximal space.

**Theorem 7.4.19**

Let $(X, m_x, I)$ be an ideal $m$-space satisfying property $\mathcal{B}$ and $A \subseteq X$ is $m^*$-perfect. Then the following properties are equivalent:

(i) $X$ is $m$-I-submaximal,

(ii) Every subset of $X$ is $I$-locally $m^*$-closed,

(iii) Every subset of $X$ is a union of an $m^*$-open subset and a $m$-closed subset of $X$,

(iv) Every $m^*$-dense subset of $X$ is an intersection of a $m^*$-closed subset and an $m$-open subset of $X$.

**Proof**

(i) $\iff$ (ii): It follows from Theorem 7.4.16

(ii) $\iff$ (iii): Let $A \subseteq X$. By (ii), we have $X \setminus A = U \cap K$, where $U$ is $m$-open and $K$ is $m^*$-closed in $X$. This implies that $A = (X \setminus U) \cup (X \setminus K)$, where $X \setminus U$ is $m$-closed and $X \setminus K$ is $m^*$-open in $X$. The converse is similar.

(ii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (i): Let $A \subseteq X$ be a $m^*$-dense set. Then $A = U \cap B$, where $U$ is $m$-open and $B$ is $m^*$-closed. Since $A \subseteq B$ and so $B$ is $m^*$-dense, then $m$-
\[ \text{Int}(B) = m-\text{Int}(m-\text{Cl}^*(B)) = m-\text{Int}(X) = X. \text{ Hence } B = X \text{ and } A = U \text{ is } m\text{-open. Thus, } X \text{ is } m\text{-I-submaximal.} \]